Area

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Theorem 1. *R is a region on the xy-plane whose projection to x-axis is the interval $[a,b]$. If for each $x \in [a,b]$, the vertical section of R at x has length $L(x)$, then the area of R is*

$$\int_a^b L(x)\,dx$$

Proof. We will divide $[a,b]$ into n intervals of length $\Delta x = \frac{b-a}{n}$, and construct rectangles with width Δx, and length $L(x_i)$. The sum of the area of these rectangles is

$$\sum_{i=1}^{n} L(x_i)\Delta x$$

When $n \to \infty$, the total area of these rectangles converges to the area of the region R, so the area of R is

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} L(x_i)\Delta x = \int_a^b L(x)\,dx$$

□

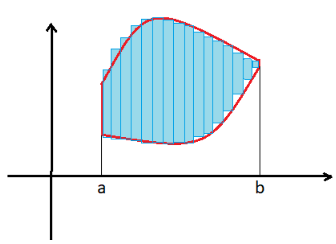


Figure 1: $\int_a^{+\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$

Similarly, if we consider the y-axis instead, we have:

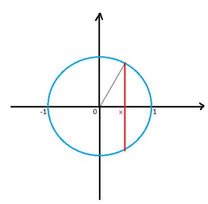
Theorem 2. *R is a region on the xy-plane whose projection to y-axis is the interval $[c, d]$. If for each $y \in [c, d]$, the vertical section of R at y has length $L(y)$, then the area of R is*

$$\int_c^d L(y)\, dy$$

Example 3. *Find the area enclosed by a circle of radius R.*

We can put the circle into xy-coordinate such that the centre of the circle is at $(0, 0)$. Then the projection of the circle to the x-axis is $[-1, 1]$, and for each $x \in [-1, 1]$, by Pythagorean Theorem, the vertical segment has length

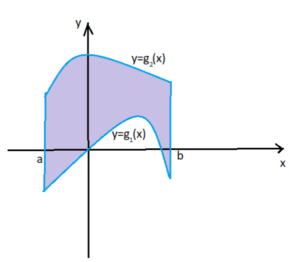
$$L(x) = 2\sqrt{1 - x^2}$$

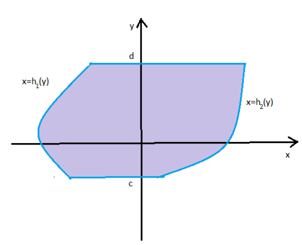


So the area enclosed by the circle is

$$\int_{-1}^1 2\sqrt{1 - x^2}\, dx = 2 \int_{-1}^1 \sqrt{1 - x^2}\, dx = \pi R^2$$

A special case of the above theorem is that when the area is bounded by the graph of a pair of functions:





Corollary 4. *If $y = g_1(x)$ and $y = g_2(x)$ are integrable functions on $x \in [a, b]$ such that $g_1(x) \le g_2(x)$, then the area enclosed by these two functions between $[a, b]$ is*

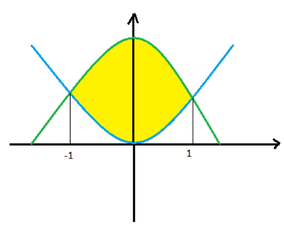
$$\int_a^b g_2(x) - g_1(x)\, dx$$

Corollary 5. *If $x = h_1(y)$ and $x = h_2(y)$ are integrable functions on $y \in [c, d]$ such that $h_1(y) \le h_2(y)$, then the area enclosed by these two functions between $[c, d]$ is*

$$\int_c^d h_2(y) - h_1(y)\, dy$$

Example 6. *Find the area enclosed by the parabolas $y = x^2$ and $y = 2 - x^2$*

Solving for $\begin{cases} y = x^2 \\ y = 2 - x^2 \end{cases}$, *we obtain $x = 1, y = 1$ or $x = -1, y = 1$, so the two curves intersect at $(-1, 1)$ and $(1, 1)$. By the graph of the two functions, we get the area is*

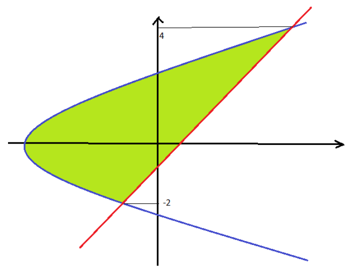


$$\int_{-1}^{1} (2 - x^2) - x^2\, dx = \int_{-1}^{1} 2 - 2x^2\, dx = \frac{8}{3}$$

Example 7. *Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$*

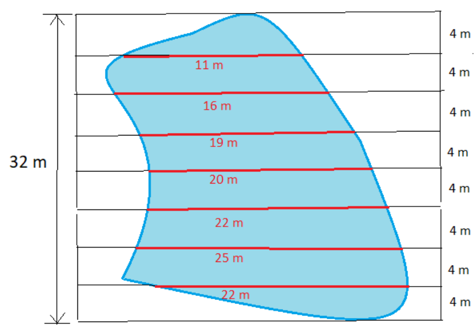
Solving for $\begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases}$, *we obtain $x = -1, y = -2$ or $x = 5, y = 4$, so the two curves intersect at $(-1, -2)$ and $(5, 4)$. By the graph of the two functions, we get the area is*

$$\int_{-2}^{4} (y + 1) - \frac{1}{2}(y^2 - 6)\, dy = \int_{-2}^{4} -\frac{y^2}{2} + y + 4\, dy = \frac{62}{3}$$



Because we can compute the area as an integral, we may make use of the approximation technique for integrals to estimate the area of a region.

Example 8. *A pool of irregular shape is shown as follows. Try to use the Simpson's Rule to approximate the area of the pool.*



Let $L(y)$ be the width of the pool. Based on the given information, we take $n = 8$ and $\Delta x = 4$. The area is approximated as:

$$\begin{aligned}\int_0^{32} L(y)\,dy &\approx \frac{4}{3}(0 + 4 \times 22 + 2 \times 25 + 4 \times 22 + 2 \times 20 + 4 \times 19 + 2 \times 16 + 4 \times 11 + 0)\\ &\approx 557.3\end{aligned}$$

Sometimes, instead of using Rectangular approximations of an area, we can also use other shapes based on the given question.

Example 9. *We are going to compute the area formula for circles in another way.*

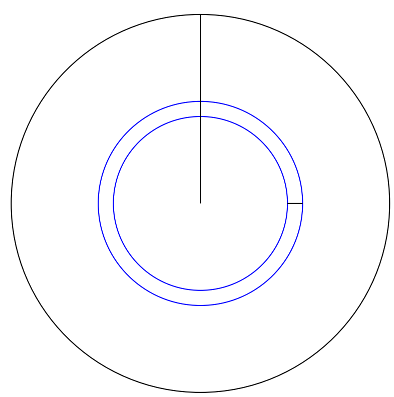
First, recall the definition of the number π: π is the ratio of the circumference and diameter of a circle. By this definition, we know that the circumference of a circle of radius R is $2\pi R$, since $2R$ is the diameter.

Now given a circle of radius R, we are going to find its area. We divide $[0, R]$ into n subintervals of equal length $\Delta r = \frac{R}{n}$, with endpoints $r_0 = 0, r_1, ..., r_{n-1}, r_n = R$, and by the following picture, we see that when n is getting big, the are of the circle can be approximated by the following:

$$R_n = \sum_{i=1}^{n} (2\pi r_i)\Delta r$$

Taking the limit, we get the area of the circle is

$$\lim_{n\to\infty} R_n = \int_0^R 2\pi r \, dr = \pi r^2 \Big|_0^R = \pi R^2$$



In other words, for a circle, the circumference is the rate of change of the area with respect to its radius.