Theorem 1. \( R \) is a region on the \( xy \)-plane whose projection to \( x \)-axis is the interval \([a, b]\). If for each \( x \in [a, b] \), the vertical section of \( R \) at \( x \) has length \( L(x) \), then the area of \( R \) is

\[
\int_{a}^{b} L(x) \, dx
\]

Proof. We will divide \([a, b]\) into \( n \) intervals of length \( \Delta x = \frac{b-a}{n} \), and construct rectangles with width \( \Delta x \), and length \( L(x_i) \). The sum of the area of these rectangles is

\[
\sum_{i=1}^{n} L(x_i) \Delta x
\]

When \( n \to \infty \), the total area of these rectangles converges to the area of the region \( R \), so the area of \( R \) is

\[
\lim_{\Delta x \to 0} \sum_{i=1}^{n} L(x_i) \Delta x = \int_{a}^{b} L(x) \, dx
\]

\[\quad \square\]

Similarly, if we consider the \( y \)-axis instead, we have:
Theorem 2. \( R \) is a region on the \( xy \)-plane whose projection to \( y \)-axis is the interval \( [c,d] \). If for each \( y \in [c,d] \), the vertical section of \( R \) at \( y \) has length \( L(y) \), then the area of \( R \) is

\[
\int_{c}^{d} L(y) \, dy
\]

Example 3. Find the area enclosed by a circle of radius \( R \).

We can put the circle into \( xy \)-coordinate such that the centre of the circle is at \( (0,0) \). Then the projection of the circle to the \( x \)-axis is \([-1,1]\), and for each \( x \in [-1,1] \), by Pythagorean Theorem, the vertical segment has length

\[
L(x) = 2\sqrt{1-x^2}
\]

So the area enclosed by the circle is

\[
\int_{-1}^{1} 2\sqrt{1-x^2} \, dx = 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx = \pi R^2
\]

A special case of the above theorem is that when the area is bounded by the graph of a pair of functions:
Corollary 4. If \( y = g_1(x) \) and \( y = g_2(x) \) are integrable functions on \( x \in [a, b] \) such that \( g_1(x) \leq g_2(x) \), then the area enclosed by these two functions between \([a, b] \) is

\[
\int_a^b g_2(x) - g_1(x) \, dx
\]

Corollary 5. If \( x = h_1(y) \) and \( x = h_2(y) \) are integrable functions on \( y \in [c, d] \) such that \( h_1(y) \leq h_2(y) \), then the area enclosed by these two functions between \([c, d] \) is

\[
\int_c^d h_2(y) - h_1(y) \, dy
\]

Example 6. Find the area enclosed by the parabolas \( y = x^2 \) and \( y = 2 - x^2 \)

Solving for \( \begin{cases} y = x^2 \\ y = 2 - x^2 \end{cases} \), we obtain \( x = 1, y = 1 \) or \( x = -1, y = 1 \), so the two curves intersect at \((-1, 1)\) and \((1, 1)\). By the graph of the two functions, we get the area is

\[
\int_{-1}^{1} (2 - x^2) - x^2 \, dx = \int_{-1}^{1} 2 - 2x^2 \, dx = \frac{8}{3}
\]

Example 7. Find the area enclosed by the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \)

Solving for \( \begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases} \), we obtain \( x = -1, y = -2 \) or \( x = 5, y = 4 \), so the two curves intersect at \((-1, -2)\) and \((5, 4)\). By the graph of the two functions, we get the area is

\[
\int_{-2}^{4} (y + 1) - \frac{1}{2}(y^2 - 6) \, dy = \int_{-2}^{4} -\frac{y^2}{2} + y + 4 \, dy = \frac{62}{3}
\]
Because we can compute the area as an integral, we may make use of the approximation technique for integrals to estimate the area of a region.

**Example 8.** A pool of irregular shape is shown as follows. Try to use the Simpson’s Rule to approximate the area of the pool.

Let $L(y)$ be the width of the pool. Based on the given information, we take $n = 8$ and $\Delta x = 4$. The area is approximated as:

$$
\int_{0}^{32} L(y) \, dy \approx \frac{4}{3} (0 + 4 \times 22 + 2 \times 25 + 4 \times 22 + 2 \times 20 + 4 \times 19 + 2 \times 16 + 4 \times 11 + 0)
$$

$$
\approx 557.3
$$

Sometimes, instead of using Rectangular approximations of an area, we can also use other shapes based on the given question.

**Example 9.** We are going to compute the area formula for circles in another way.
First, recall the definition of the number \( \pi \): \( \pi \) is the ratio of the circumference and diameter of a circle. By this definition, we know that the circumference of a circle of radius \( R \) is \( 2\pi R \), since \( 2R \) is the diameter.

Now given a circle of radius \( R \), we are going to find its area. We divide \([0, R]\) into \( n \) subintervals of equal length \( \Delta r = \frac{R}{n} \), with endpoints \( r_0 = 0, r_1, ..., r_{n-1}, r_n = R \), and by the following picture, we see that when \( n \) is getting big, the area of the circle can be approximated by the following:

\[
R_n = \sum_{i=1}^{n} (2\pi r_i) \Delta r
\]

Taking the limit, we get the area of the circle is

\[
\lim_{n \to \infty} R_n = \int_{0}^{R} 2\pi r \, dr = \pi r^2 \bigg|_{0}^{R} = \pi R^2
\]

In other words, for a circle, the circumference is the rate of change of the area with respect to its radius.