1. Find the critical numbers of the function \( f(x) = |3x - 4| \).

**Solution:** When \( x > \frac{4}{3} \), \( f'(x) = 3 \). When \( x < \frac{4}{3} \), \( f'(x) = -3 \). \( f'(x) \) doesn’t exist at \( x = \frac{4}{3} \), so the critical number is \(-\frac{4}{3}\).

2. Find the absolute maximum and absolute minimum values of the function \( f(x) = 5 + 54x - 2x^3 \) on the interval \([0, 4]\).

**Solution:** \( f'(x) = 54 - 6x^2 \). Let \( f'(x) = 0 \), we get \( x = 3 \) or \( x = -3 \). So the critical numbers are 3 and \(-3\). Since \(-3\) is not in the interval \([0, 4]\), we only need to consider \( x = 3 \).

\( f(3) = 149 \), \( f(0) = 5 \) and \( f(4) = 93 \), so the absolute maximum on \([0, 4]\) is \( f(3) = 149 \) and the absolute minimum is \( f(0) = 5 \).

3. Sketch a graph of a function \( f \) that is continuous on \([1, 5]\) and has absolute maximum at \( x = 2 \), absolute minimum at \( x = 3 \), and local minimum at \( x = 4 \).

**Solution:**

![Graph](image)

4. Suppose \( 3 \leq f'(x) \leq 5 \) for all values of \( x \). Show that \( 18 \leq f(8) - f(2) \leq 30 \).

**Solution:** By the Mean Value Theorem, there is \( c \) in \((2, 8)\) such that \( f(8) - f(2) = f'(c)(8 - 2) = 6f'(c) \). \( 3 \leq f'(c) \leq 5 \), so \( 18 \leq 6f'(x) \leq 30 \), i.e. \( 18 \leq f(8) - f(2) \leq 30 \).
5. Show that the equation $x^4 + 4x + c = 0$ ($c$ is a constant) has at most two real roots.

**Solution:** Let $f(x) = x^4 + 4x + c$. Suppose it has at least three roots $x_1 < x_2 < x_3$, then by the Rolle’s Theorem, there is a number $c$ on $(x_1, x_2)$ such that $f'(c) = 0$ and there is a number $d$ on $(x_2, x_3)$ such that $f'(d) = 0$.

But $f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)(x^2 - x + 1) = 4(x + 1)((x - \frac{1}{2})^2 + \frac{3}{4}) = 0$ has only one solution, so it’s impossible that $f'(c) = f'(d) = 0$.

We conclude $f$ cannot have more than two solutions.

6. Suppose that $f$ and $g$ are continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for all $a < x < b$. Prove that $f(b) < g(b)$.

**Solution:** Let $F(x) = f(x) - g(x)$. $F(a) = f(a) - g(a) = 0$ and $F'(x) = f'(x) - g'(x) < 0$ for all $a < x < b$. By the Mean Value Theorem, there is $c$ in $(a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

Since $F(a) = 0$ and $F'(c) < 0$, we get

$$F(b) = F'(c)(b - a) < 0$$

So $f(b) - g(b) = F(b) < 0$, i.e. $f(b) < g(b)$.