1. Use logarithmic differentiation to find the derivative of $y = x^{\cos x}$

   Solution: $\ln y = \ln x^{\cos x} = (\cos x)\ln x$. Differentiation both sides with respect to x:

   $\frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}$

   $y' = y(-\sin x \ln x + \frac{\cos x}{x}) = x^{\cos x}(-\sin x \ln x + \frac{\cos x}{x})$

2. Strontium-90 has half life of 28 days. A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.

   Solution: Let $m(t) = Ce^{kt}$. Its half life is 28 days, so $\frac{C}{2} = m(28) = Ce^{28k}$, i.e. $\frac{1}{2} = e^{28k}$. We get $k = -\frac{\ln 2}{28}$.

   $m(t) = Ce^{kt} = Ce^{-\frac{\ln 2}{28}t} = C2^{-\frac{t}{28}}$.

   $C = m(0) = 50$, so $m(t) = 50 \times 2^{-\frac{t}{28}}$

3. Prove that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

   Solution: Let $y = \sin^{-1} x$, then $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, and $x = \sin y$. Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$, we get

   $$\cos(\sin^{-1} x) = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

4. Find the derivative of the following functions:

   (i). $y = \tan^{-1}(x^2)$

   Solution: $y' = \frac{1}{1+(x^2)^2}(2x) = \frac{2x}{1+x^4}$

   (ii). $y = \sin^{-1}(\cos x)$

   Solution: $y' = \frac{1}{\sqrt{1-\cos^2 x}}(-\sin x) = -\frac{\sin x}{|\sin x|}$

5. Compute the following limits:

   (i). $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

   Solution: $\lim\limits_{x\to 0}(e^x - 1 - x) = 0$ and $\lim\limits_{x\to 0} x^2 = 0$, so we apply the L'Hospital's Rule:

$\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2} = \lim\limits_{x\to 0} \dfrac{e^x - 1}{2x}$

$\lim\limits_{x\to 0}(e^x - 1) = 0$ and $\lim\limits_{x\to 0} 2x = 0$, so apply the L'Hospital's Rule again:

$\lim\limits_{x\to 0} \dfrac{e^x - 1}{2x} = \lim\limits_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

(ii). $\lim\limits_{x\to\infty} \dfrac{2^x}{x}$

Solution: $\lim\limits_{x\to\infty} 2^x = \infty$ and $\lim\limits_{x\to\infty} x = \infty$, so we apply the L'Hospital's Rule:

$\lim\limits_{x\to\infty} \dfrac{2^x}{x} = \lim\limits_{x\to\infty} \dfrac{2^x \ln 2}{1} = \infty$

(iii). $\lim\limits_{x\to 0^+} (\sin x)(\ln x)$

Solution: $\lim\limits_{x\to 0^+} \sin x \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{\sin x}}$, and $\lim\limits_{x\to 0^+} \ln x = \infty$, $\lim\limits_{x\to 0^+} \dfrac{1}{\sin x} = \infty$, so apply the L'Hospital's Rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = -\lim_{x\to 0^+} \frac{\sin x}{x} \lim_{x\to 0^+} \tan x = 1 \times 0 = 0$$

(iv). $\lim\limits_{x\to 0^+} (\tan 2x)^x$

Solution: $\ln(\tan 2x)^x = x \ln \tan 2x = \frac{\ln(\tan 2x)}{x^{-1}}$.

$\lim\limits_{x\to 0^+} \ln(\tan 2x) = -\infty$ and $\lim\limits_{x\to 0^+} x^{-1} = \infty$, so apply the L'Hospital's Rule:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(\tan 2x)}{x^{-1}} &= \lim_{x\to 0^+} \frac{\frac{1}{\tan 2x}\frac{2}{\cos^2 2x}}{-x^{-2}} \\
&= -\lim_{x\to 0^+} \frac{2x^2}{\sin 2x \cos 2x} \\
&= -\lim_{x\to 0^+} \frac{2x}{\sin 2x} \lim_{x\to 0^+} \frac{x}{\cos 2x} \\
&= -1 \times 0 \\
&= 0
\end{aligned}$$

So $\lim\limits_{x\to 0^+} (\tan 2x)^x = \lim\limits_{x\to 0^+} e^{\ln(\tan 2x)^x} = e^0 = e^{\lim\limits_{x\to 0^+} \ln(\tan 2x)^x} = e^0 = 1$

(v). $\lim_{x\to\infty}(1+\dfrac{a}{x})^{bx}$

Solution: $\ln(1+\frac{a}{x})^{bx} = bx\ln(1+\frac{a}{x}) = \frac{b\ln(1+\frac{a}{x})}{x^{-1}}$

$\lim_{x\to\infty} b\ln(1+\dfrac{a}{x}) = 0$ and $\lim_{x\to\infty} x^{-1} = 0$, so we apply the L'Hospital's Rule:

$$\begin{aligned}
\lim_{x\to\infty}\frac{b\ln(1+\frac{a}{x})}{x^{-1}} &= \lim_{x\to\infty}\frac{b\ln(\frac{x+a}{x})}{x^{-1}} \\
&= \lim_{x\to\infty}\frac{b(\ln(x+a)-\ln x)}{x^{-1}} \\
&= \lim_{x\to\infty}\frac{b(\frac{1}{x+a}-\frac{1}{x})}{-x^{-2}} \\
&= \lim_{x\to\infty}\frac{abx}{x+a} \\
&= ab
\end{aligned}$$

$\lim_{x\to\infty}(1+\dfrac{a}{x})^{bx} = \lim_{x\to\infty} e^{\ln(1+\frac{a}{x})^{bx}} = e^{ab}$