

1. Find the limit

(i). $\lim_{x \rightarrow \infty} 2^{-x^2}$

Solution: If $x \rightarrow \infty$, $-x^2 \rightarrow -\infty$, so $2^{-x^2} \rightarrow 0$

(ii). $\lim_{x \rightarrow \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} &= \lim_{x \rightarrow \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} \times \frac{e^{-3x}}{e^{-3x}} \\ &= \lim_{x \rightarrow \infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} \\ &= \frac{1 - 0}{1 + 0} \\ &= 1\end{aligned}$$

(iii). $\lim_{x \rightarrow 3^+} \log_{\frac{1}{2}}(x^2 - 9)$

Solution: $x \rightarrow 3^+$, $x^2 - 9 \rightarrow 0^+$, so $\log_{\frac{1}{2}}(x^2 - 9) \rightarrow +\infty$

(iv). $\lim_{x \rightarrow \infty} [\ln(1 + x^2) - \ln(1 + x)]$

Solution:

$$\begin{aligned}\lim_{x \rightarrow \infty} [\ln(1 + x^2) - \ln(1 + x)] &= \lim_{x \rightarrow \infty} \ln \frac{1 + x^2}{1 + x} \\ &= \lim_{x \rightarrow \infty} \ln \frac{\frac{1}{x} + x}{\frac{1}{x} + 1} \\ &= +\infty\end{aligned}$$

2. Prove that $f(x) = \frac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}$ is an odd function.

Solution:

$$\begin{aligned}
 f(-x) &= \frac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} \\
 &= \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}} \\
 &= \frac{\frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}}}}{\frac{e^{\frac{1}{x}} + 1}{e^{\frac{1}{x}}}} \\
 &= -\frac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}} \\
 &= -f(x)
 \end{aligned}$$

3. Find the inverse function of $f(x) = \frac{4x-1}{2x+3}$

Solution: Let $y = \frac{4x-1}{2x+3}$, then $(2x+3)y = 4x-1$, we get $x = \frac{3y+1}{4-2y}$, so interchanging the role of x and y , we get

$$f^{-1}(x) = \frac{3x+1}{4-2x}$$

4. If $f(x) = x^5 + x^3 + x$, find $(f^{-1})'(3)$

Solution: $f'(x) = 5x^4 + 3x^2 + 1$, and $f(1) = 3$, so $f^{-1}(3) = 1$.

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{9}$$

5. Compute the value of $2 \log_3 10 - \log_3 18 - \log_3 50$

Solution:

$$\begin{aligned}
 2 \log_3 10 - \log_3 18 - \log_3 50 &= \log_3 10^2 - \log_3 18 - \log_3 50 \\
 &= \log_3 \frac{10^2}{18 \times 50} \\
 &= \log_3 \frac{1}{9} \\
 &= -2
 \end{aligned}$$

6. Differentiate the following functions:

(i). $f(x) = 10^{1-x^2}$

Solution:

$$f'(x) = (10^{1-x^2} \ln 10)(-2x) = -2x10^{1-x^2} \ln 10$$

(ii). $f(x) = x^2 e^{-\frac{1}{x}}$

Solution:

$$f'(x) = 2x e^{-\frac{1}{x}} + x^2 \left(\frac{1}{x^2} e^{-\frac{1}{x}} \right) = (2x + 1) e^{-\frac{1}{x}}$$

(iii). $f(x) = \ln(\ln x)$

Solution:

$$f'(x) = \frac{1}{\ln x} \frac{1}{x} = \frac{1}{x \ln x}$$

(iv). $f(x) = \log_2(\sin x^2)$

Solution:

$$f'(x) = \frac{1}{\sin x^2 \ln 2} (\cos x^2)(2x) = \frac{2x \cos x^2}{\sin x^2 \ln 2}$$