1. Show that the function is continuous on $(-\infty, +\infty)$:

$$f(x) = \begin{cases} x^2, \text{ if } x < 1 \\ \sqrt{x}, \text{ if } x \geq 1 \end{cases}$$

Solution: When $x > 1$, $f(x) = \sqrt{x}$ is continuous at x. When $x < 1$, $f(x) = x^2$ is continuous at x. When $x = 1$, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x} = \sqrt{1} = 1$, and $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$, so we see $\lim_{x \to 1} f(x) = 1 = f(1)$, hence f is also continuous at $x = 1$. We conclude x is continuous on $(-\infty, +\infty)$.

2. Find the values a, b that makes f continuous everywhere:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, \text{ if } x < 2 \\ ax^2 - bx + 3, \text{ if } 2 \leq x < 3 \\ 2x - a + b, \text{ if } x \geq 3 \end{cases}$$

Solution: We see when $x \neq 2$ and $x \neq 3$, the function is continuous, so we need to choose a, b to make the function continuous at $x = 2$ and $x = 3$.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \dfrac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} (x + 2) = 4$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3 = f(2)$

If f is continuous at $x = 2$, we need $4 = 4a - 2b + 3$.

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 9a - 3b + 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b = 6 - a + b = f(3)$

If f is continuous at $x = 3$, we need $9a - 3b + 3 = 6 - a + b$

Solving

$$\begin{cases} 4 = 4a - 2b + 3 \\ 9a - 3b + 3 = 6 - a + b \end{cases}$$

we get $a = b = \frac{1}{2}$.

3. Show that the equation $x = \cos x$ has a solution on $(0, \frac{\pi}{2})$

   Solution: Let $f(x) = x - \cos x$. $f(0) = 0 - \cos 0 = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0$, so by the Intermediate Value Theorem, there is x in the interval $(0, \frac{\pi}{2})$ such that $f(x) = x - \cos x = 0$, i.e. $x = \cos x$.

4. Compute the limits:

   (i).

$$\lim_{x\to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4}$$

   Solution:

$$\begin{aligned}\lim_{x\to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4} &= \lim_{x\to 2^-} \frac{x(x-2)}{(x-2)^2}\\ &= \lim_{x\to 2^-} \frac{x}{(x-2)}\\ &= -\infty\end{aligned}$$

   (ii).

$$\lim_{x\to -\infty} \frac{1 + x^6}{x^4 + 1}$$

   Solution:

$$\begin{aligned}\lim_{x\to -\infty} \frac{1 + x^6}{x^4 + 1} &= \lim_{x\to -\infty} \frac{(1 + x^6)\frac{1}{x^4}}{(x^4 + 1)\frac{1}{x^4}}\\ &= \lim_{x\to -\infty} \frac{x^2 + \frac{1}{x^4}}{\frac{1}{x^4} + 1}\\ &= \lim_{x\to -\infty} \frac{+\infty + 0}{0 + 1}\\ &= +\infty\end{aligned}$$

   (iii).

$$\lim_{x\to\infty} (x - \sqrt{x})$$

Solution:

$$\begin{aligned}\lim_{x\to\infty}(x-\sqrt{x}) &= \lim_{x\to\infty}\sqrt{x}(\sqrt{x}-1)\\ &= (+\infty)(+\infty)\\ &= +\infty\end{aligned}$$

(iv).

$$\lim_{x\to 1}\frac{2-x}{(x-1)^2}$$

Solution: $(x-1)^2 \to 0^+$ as $x \to 1$, so $\lim_{x\to 1}\frac{1}{(x-1)^2} = +\infty$, hence $\lim_{x\to 1}\frac{2-x}{(x-1)^2} = \lim_{x\to 1}(2-x)\lim_{x\to 1}\frac{1}{(x-1)^2} = 2\times(+\infty) = +\infty$

(v).

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x)$$

Solution:

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) &= \lim_{x\to\infty}\frac{(\sqrt{9x^2+x}-3x)(\sqrt{9x^2+x}+3x)}{\sqrt{9x^2+x}+3x}\\ &= \lim_{x\to\infty}\frac{9x^2+x-(3x)^2}{\sqrt{9x^2+x}+3x}\\ &= \lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{1}{x}}+3}\\ &= \frac{1}{\sqrt{9+0}+3}\\ &= \frac{1}{6}\end{aligned}$$

(vi).

$$\lim_{x\to\infty}\frac{\sin^2 x}{x^2}$$

Solution:

Observe that $0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$, and

$$\lim_{x\to\infty} 0 = 0, \lim_{x\to\infty} \frac{1}{x^2} = 0$$

So by the Squeeze Theorem, $\displaystyle\lim_{x\to\infty} \frac{\sin^2 x}{x^2} = 0$