Definition. A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F(x) = f(x)$ for all $x$ in $I$.

Example. If $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f$.

Also note that for any constant $C$, $F(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of $f$, since $\frac{d}{dx} \left( \frac{1}{3}x^3 + C \right) = x^2$.

In general, we can do this for the antiderivative of any function:

Theorem. If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$F(x) + C$$

where $C$ is arbitrary constant.

Example. Find the most general antiderivative of $f(x) = \sin x$.

If $F(x) = -\cos x$, $F(x) = 5\sin x$.

So the most general antiderivative of $f$ is

$$F(x) = -\cos x + C$$

Proposition. If $F(x)$ is an antiderivative of $f(x)$, $G(x)$ is an antiderivative of $g(x)$.

(i) $CF(x)$ is an antiderivative of $c f(x)$.

(ii) $F(x) \pm G(x)$ is an antiderivative of $f(x) \pm g(x)$.

Example. Find the most general antiderivative of $f(x) = \frac{1 - \cos^2 x}{1 - \sin^2 x}$.

$$f(x) = \frac{1 - \cos^2 x}{1 - \sin^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - \cos x$$

$$F(x) = \tan x - \sin x + C.$$
We've seen that for a given function \( f \), there're many antiderivatives: \( F(x) + C \) for each constant \( C \).

But sometimes we are looking for an antiderivative with some extra condition, in this case, the constant \( C \) is often determined by the extra condition.

**Example.** Find \( f \) if \( f'(x) = e^x + 20(1+x^2)^{-1} \) and \( f(10) = -2 \).

\[
f'(x) = e^x + \frac{20}{1+x^2}.
\]

so \( f(x) = e^x + 20 \tan^{-1} x + C \).

\[-2 = f(10) = e^0 + 20 \tan^{-1} 0 + C \Rightarrow -2 = 1 + 0 + C \Rightarrow C = -3 \]

so \( f(x) = e^x + 20 \tan^{-1} x - 3 \)

**Example.** Find \( f(x) \) if \( f''(x) = 12x^2 + 6x - 4 \), \( f'(10) = 4 \), \( f'(1) = 1 \).

\[
f''(x) = 12x^2 + 6x - 4, \quad so \quad f'(x) = 4x^3 + 3x^2 - 4x + C_1
\]

\[
f(x) = x^4 + x^3 - 2x^2 + C_1 x + C_2
\]

\[
\begin{cases}
4 = f(10) = C_2 \\
1 = f(1) = 1 + 1 - 2 + C_1 + C_2
\end{cases} \Rightarrow \begin{cases}
C_1 = -3 \\
C_2 = 4
\end{cases}
\]

so \( f(x) = x^4 + x^3 - 2x^2 - 3x + 4 \)

**Example.** A particle moves in a straight line and has acceleration given by \( a(t) = 6t + 4 \). Its initial velocity is \( v(0) = -6 \) m/s and its initial displacement is \( s(0) = 9 \) m. Find its position function \( s(t) \).

Recall that \( v(t) = s'(t) \) and \( a(t) = v'(t) \).
\( a(t) = 6t + 4 \), so \( v(t) = 3t^2 + 4t + C_1 \).

-6 = v(0) = C_1, so \( v(t) = 3t^2 + 4t - 6 \)

Then \( s(t) = t^3 + 2t^2 - 6t + C_2 \).

9 = s(0) = C_2, so \( s(t) = t^3 + 2t^2 - 6t + 9 \).

**Example.** A ball is thrown upward with speed 4.9 m/s from the edge of a cliff 9.8 m. above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground? (From Newtonian physics, the acceleration of the ball is 9.8 m/s² pointing downward).

\( a(t) = -9.8 \), so \( v(t) = -9.8t + C \).

4.9 = v(0) = C, so \( v(t) = -9.8t + 4.9 \).

It reaches maximum height when \( v(t) = s'(t) = 0 \).

Let \( v(t) = -9.8t + 4.9 = 0 \) we get \( t = 0.5 \).

So it reaches maximum height at 0.5 second.

\( s(t) = -4.9t^2 + 4.9t + C_2 \).

9.8 = s(0) = C_2 \Rightarrow \ s(t) = -4.9t^2 + 4.9t + 9.8 \)

Let \( s(t) = -4.9t^2 + 4.9t + 9.8 = 0 \).

\[-4.9t^2 - t - 2 = 0\]

\[-4.9(t+1)(t-2) = 0\]

So \( t = 2 \) or \( t = -1 \) (drop).

We see it hit the ground at 2 second.