**Theorem.** (Increasing/Decreasing Test)

If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on it.
If \( f'(x) < 0 \) on an interval, then \( f \) is decreasing on it.

Geometrically, the intuition is clear: if \( f'(x) > 0 \), then the slope is positive, which indicates the graph of \( f \) is going "upward" as \( x \) increases.

**Proof.**

If \( f'(x) > 0 \) on an interval and \( x_1 < x_2 \) are two numbers in this interval, then by the Mean Value Theorem, there exists \( c \) in \((x_1, x_2)\) such that

\[
f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0. \quad \Rightarrow \quad f(x_2) > f(x_1)
\]

Similarly we can prove the case for \( f'(x) < 0 \).

**Example.** Find where the function \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \) is increasing and where it's decreasing.

\[
f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)
\]

Let \( f'(x) > 0 \), we get \(-1 < x < 0 \) or \( x > 2 \).
Let \( f'(x) < 0 \), we get \(- \infty < x < -1 \) or \( 0 < x < 2 \).

So \( f \) is increasing on \((-1, 0)\) and \((2, +\infty)\) and decreasing on \((-\infty, -1)\) and \((0, 2)\).
The Increasing/Decreasing Test can be applied to determine if a critical number gives a local extreme.

**Theorem** (The First Derivative Test)
Suppose that \( c \) is a critical number of a continuous function \( f \).

(a) If \( f'(x) \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \).
(b) If \( f'(x) \) changes from negative to positive at \( c \), then \( f \) has a local minimum.
(c) If \( f'(x) \) doesn't change sign at \( c \), then \( f \) has no local maximum or minimum at \( c \).

**Example.** \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \)

We've computed in the previous example that
\[ f'(x) = 12x(x-2)(x+1), \] the critical numbers are \(-1, 0, 2\).

The sign of \( f'(x) \) is:

<table>
<thead>
<tr>
<th>( x )</th>
<th>((-\infty, -1))</th>
<th>((-1, 0))</th>
<th>((0, 2))</th>
<th>((2, +\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
</tr>
</tbody>
</table>

So \( f(x) \) obtains local maximum \( f(0) = 5 \)
local minimum \( f(-1) = 0 \) and \( f(2) = -2 \)
Example \( f(x) = x^3 \).

\( f'(x) = 3x^2 \). The critical number is \( x = 0 \).

But \( f'(x) > 0 \) for both \( x < 0 \) and \( x > 0 \), i.e. \( f' \) doesn't change sign at \( x = 0 \), we conclude \( f'(0) \) is not a local extreme.

**Definition** If the graph of \( f \) lies above all of its tangents on an interval \( I \), then it's called **concave upward** on \( I \).

If the graph of \( f \) lies below all of its tangents on an interval \( I \), then it's called **concave downward** on \( I \).

**Remark.** In some other books, people use the terms "**convex**" for concave upward and "**concave**" for concave downward.

**Definition.** A point \( P \) on a curve \( y = f(x) \) is called an **inflection point** if \( f \) is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward.

**Theorem (Concavity Test)**
(a). If \( f''(x) > 0 \) for all \( x \) in \( I \), then the graph of \( f \) is concave upward on \( I \).

(b). If \( f''(x) < 0 \) for all \( x \) in \( I \), then the graph of \( f \) is concave downward on \( I \).
Corollary. There is a point of inflection at any point where the second derivative changes sign.

Example. Find the inflection points of \( f(x) = x^3 - 12x + 2 \) and the intervals on which \( f \) is concave upward/downward.

\[
\begin{align*}
 f'(x) &= 3x^2 - 12, \\
 f''(x) &= 6x
\end{align*}
\]

We see \( f''(0) = 0 \), \( f''(x) < 0 \) for \( x < 0 \), \( f''(x) > 0 \) for \( x > 0 \).

So \( x = 0 \) is an inflection point, \( f \) is concave upward on \((0, +\infty)\), concave downward on \((-\infty, 0)\).

Example. Sketch a possible graph of a function \( f \) that satisfies the following conditions:

1. \( f'(x) > 0 \) on \((-\infty, 1)\), \( f'(x) < 0 \) on \((1, +\infty)\)
2. \( f''(x) > 0 \) on \((-\infty, -2)\) and \((2, +\infty)\), \( f''(x) < 0 \) on \((-2, 2)\)
3. \( \lim_{x \to -\infty} f(x) = -2 \), \( \lim_{x \to +\infty} f(x) = 0 \)
Theorem \emph{(The Second Derivative Test)}

Suppose $f''$ is continuous near $c$.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

Example. Find the local extrema of $f(x) = x^4 - 4x^2$.

$f'(x) = 4x^3 - 12x$. Let $f'(x) = 0$, we get the critical numbers are $0$ and $3$.

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

So $f''(0) = 0$, $f''(3) = 36$

$f(3) = 0$ and $f''(3) > 0 \Rightarrow f(3) = -27$ is a local minimum.

$f'(0) = 0$ and $f''(0) = 0$, The Second Derivative Test doesn't work. But observe $f'(x) < 0$ on both $(-\infty, 0)$ and $(0, 3)$, so $f'(x)$ doesn't change sign at $x = 0$, we conclude $f(0)$ is not a local extreme.