DEFINITE INTEGRAL AND ANTIDERIVATIVE

Theorem. If $f$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a),$$

where $F$ is an antiderivative of $f$ (i.e. $F' = f$).

Proof. We divide $[a, b]$ into $n$ subintervals, with endpoints $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. The length of the interval $[x_{i-1}, x_i]$ is $\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}$.

On the interval $[x_{i-1}, x_i]$, we apply the Mean Value Theorem to the function $F$:

$$F(x_i) - F(x_{i-1}) = F'(x_i^*) (x_i - x_{i-1}) = f(x_i^*) (x_i - x_{i-1})$$

for some $x_i^*$ in $(x_{i-1}, x_i)$.

$$\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(x_i^*) (x_i - x_{i-1})$$

$$\Rightarrow \quad F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n}$$

Taking the limit above as $n \to +\infty$,

$$F(b) - F(a) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = \int_a^b f(x) \, dx.$$ 

Example. Evaluate $\int_1^3 e^x \, dx$.

An antiderivative of $f(x) = e^x$ is $F(x) = e^x$.

$$\int_1^3 e^x \, dx = F(3) - F(1) = e^3 - e.$$
Example. Find the area under the cosine curve from 0 to \(b\), where \(0 < b < \frac{\pi}{2}\).

An antiderivative of \(f(x) = \cos x\) is \(F(x) = \sin x\).

So \(A = \int_0^b \cos x \, dx = \sin x \bigg|_0^b = \sin b - \sin 0 = \sin b\).

Definition. Define the indefinite integral of \(f\) to be \(\int f(x) \, dx = F(x)\), where \(F(x) = f(x)\), i.e., \(F(x)\) is an antiderivative of \(f(x)\).

The relation between indefinite integral and definite integral of \(f\) is indicated by the previous theorem:

\[ \int_a^b f(x) \, dx = \int_a^b f(x) \, dx \bigg|_a^b \]

Example. Evaluate \(\int_0^3 (x^3 - 6x) \, dx\)

\(\int x^3 - 6x \, dx = \frac{1}{4}x^4 - 3x^2\), so

\(\int_0^3 (x^3 - 6x) \, dx = \left[ \frac{1}{4}x^4 - 3x^2 \right]_0^3 = \left( \frac{3^4}{4} - 3 \times 3^2 \right) - 0 = -\frac{27}{4}\)

Example. Evaluate \(\int_1^9 \frac{2t^2 + t^2(4t - 1)}{t^2} \, dt\)

\(\int \frac{2t^2 + t^2(4t - 1)}{t^2} \, dt = \int \left[ 2 + t^2 - \frac{1}{t} \right] \, dt = 2t + \frac{2}{3}t^3 + \frac{1}{t}\).

So \(\int_1^9 \frac{2t^2 + t^2(4t - 1)}{t^2} \, dt = \left[ 2t + \frac{2}{3}t^3 + \frac{1}{t} \right]_1^9 = \frac{292}{9}\).

Another way to view the theorem we introduced today is to replace \(f(x)\) by \(F'(x)\):

\(\int_a^b F'(x) \, dx = F(b) - F(a)\)
Theorem (Net Change Theorem)

The integral of a rate of change is the net change:

\[ \int_{a}^{b} F'(x) \, dx = F(b) - F(a). \]

Example. A particle moves along a line so that its velocity at time \( t \) is \( v(t) = t^2 - t - 6 \).

(a) Find the displacement of the particle during \( 1 \leq t \leq 4 \).

(b) Find the distance travelled during \( 1 \leq t \leq 4 \).

(a) The displacement is

\[ \int_{1}^{4} v(t) \, dt = \int_{1}^{4} t^2 - t - 6 \, dt = \left[ \frac{1}{3} t^3 - \frac{1}{2} t^2 - 6t \right]_{1}^{4} = -\frac{9}{2} \]

(b) The distance travelled is

\[ \int_{1}^{4} |v(t)| \, dt = \int_{1}^{4} |t^2 - t - 6| \, dt \]

We know \( t^2 - t - 6 = (t - 3)(t + 2) \).

So \( v(t) < 0 \) when \(-2 < t < 3\), \( v(t) > 0 \) when \( t < -2 \) or \( t > 3 \)

\[ \int_{1}^{4} |t^2 - t - 6| \, dt = \int_{1}^{3} (-t^2 + t + 6) \, dt + \int_{3}^{4} (t^2 - t - 6) \, dt \]

\[ = \left[ -\frac{1}{3} t^3 + \frac{1}{2} t^2 + 6t \right]_{1}^{3} + \left[ \frac{1}{3} t^3 - \frac{1}{2} t^2 - 6t \right]_{3}^{4} \]

\[ = 6 \]

We use \( |v(t)| \) in the integrand because \( |v(t)| \) is the rate of change of the distance.