1. $p > 3$ is a prime and $p \equiv 2 \pmod{3}$. $G$ is a group of order $3p$. Prove $G \cong \mathbb{Z}/3p\mathbb{Z}$.

**Solution:** $|G| = 3p$, by Theorem, the number of Sylow $p$-subgroups divides $3$ and is congruent to $1$ modulo $p$, so it is $1$. Denote this unique Sylow $p$-subgroup by $H$, $H$ is a normal subgroup of $G$. The number of Sylow 3-subgroups divide $p$ and is congruent to $1$ modulo $3$, so it is $1$, since $p \equiv 2 \pmod{3}$). Denote this unique Sylow 3-subgroup by $K$, $K$ is a normal subgroup.

$p$ and $3$ are relatively prime, so $H \cap K = \{1\}$.

$|HK| = \frac{|H| \cdot |K|}{|H \cap K|} = 3p = |G|$, so $G = HK$.

We thus conclude

$$G = H \times K \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3p\mathbb{Z}$$

2. Prove $\text{Aut}(S_3) \cong S_3$

**Solution:** Consider the homomorphism

$$\phi : S_3 \longrightarrow \text{Aut}(S_3)$$

$$g \mapsto \phi_g$$

where $\phi_g(x) = gxg^{-1}$ for any $x \in S_3$. $\ker \phi = Z(S_3) = \{id\}$, so $\phi$ is injective.

Every $f \in \text{Aut}(S_3)$ sends $2$-cycles to $2$-cycles since it needs to preserve the order of elements. Every element in $S_3$ is a product of the $3$-cycles, so the image of the three $2$-cycles determine the automorphism $f$. There are at most $3! = 6$ ways to assign the values for the three cycles under $f$, so $|\text{Aut}(S_3)| \leq 6$. But $\phi : S_3 \longrightarrow \text{Aut}(S_3)$ is injective homomorphism with $|S_3| = 6$, so $|\text{Aut}(S_3)| = 6$ and $\phi$ is an automorphism, $S_3 \cong \text{Aut}(S_3)$.

3. Prove the unit quaternion group $Q_8$ is not a semidirect product of its proper subgroups.

**Solution:** If $G$ is a semidirect product of its proper subgroups $H$ and $K$, then it follows $H \cap K = \{1\}$.

If $|H| = 2$, since $-1$ is the unique element of order $2$, $H = \{1, -1\}$. If $|H| = 4$, then Cauchy’s Theorem implies $H$ contains an element of order $2$, which again has to be $-1$, so $-1 \in H$ in any case. Similarly, $-1 \in K$ in any case. This implies $-1 \in H \cap K$, contradiction, so $Q_8$ is not a semidirect product of its proper subgroups.

Solution: let $G$ be a group of order 28. By Sylow Theorem, we know $G$ has a Sylow 7-subgroup $H$ and a Sylow 2-subgroup $K$, and the number of Sylow 7-subgroup is 1, so $H$ is a normal subgroup of $G$.

$|H| = 7$ and $|K| = 4$ are relatively prime, so $H \cap K = \{1\}$.

$|HK| = \frac{|H| \times |K|}{|H \cap K|} = 28 = |G|$, so $G = HK$.

We conclude $G = H \rtimes K$.

$|H| = 7$ implies $H \cong \mathbb{Z}/7\mathbb{Z}$, $|K| = 4$ implies $K \cong \mathbb{Z}/4\mathbb{Z}$ or $K \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Case 1. If $K \cong \mathbb{Z}/4\mathbb{Z}$, $G \cong \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ for some $\phi : \mathbb{Z}/4\mathbb{Z} \to \text{Aut}(\mathbb{Z}/7\mathbb{Z}) \cong (\mathbb{Z}/7\mathbb{Z})^* = \{1, 2, 3, 4, 5, 6\}$. Note that $6 = \bar{1}$ is the only element of order 2 in $(\mathbb{Z}/7\mathbb{Z})^*$.

$\phi$ is determined by $\phi(\bar{1})$, and $|\bar{1}| = 4$, so $|\phi(\bar{1})|$ divides 4. Also $|\phi(\bar{1})|$ divides $|\text{Aut}(\mathbb{Z}/4\mathbb{Z})| = 6$ implies $|\phi(\bar{1})|$ is 1 or 2.

Case 1.(a): If $|\phi(\bar{1})| = 1$, then $\phi$ is the trivial map, $G \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Case 1.(b): If $|\phi(\bar{1})| = 2$, then $\phi(\bar{1}) = \bar{6} = \bar{1}$. $\phi(\bar{m}) : \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$ is then defined by $\phi(\bar{m})(\bar{k}) = (-1)^m \bar{k}$. In this case we obtain a semidirect product different from direct product.

Case 2. If $K \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $G \cong \mathbb{Z}/7\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ for some $\phi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{Z}/7\mathbb{Z}) \cong (\mathbb{Z}/7\mathbb{Z})^* = \{1, 2, 3, 4, 5, 6\}$.

Note $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong K_4 = \{1, a, b, c\}$, so we can regard $\phi$ as a homomorphism $K_4 \to (\mathbb{Z}/7\mathbb{Z})^*$. Note $|a| = |b| = |c| = 2$, so $|\phi(a)|, |\phi(b)|, |\phi(c)| \in \{1, 2\}$. By the relations $ab = c, \phi(a)\phi(b) = \phi(c)$, it has to be the case $\phi(a) = \phi(b) = \phi(c) = \bar{1}$ or two of $\phi(a), \phi(b), \phi(c)$ is $\bar{1}$ and the remaining is $\bar{1}$.

Case 2.(a): If $\phi(a) = \phi(b) = \phi(c) = \bar{1}$, we get $\phi$ is the trivial group, so $G \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Case 2.(b): If two of $|\phi(a)|, |\phi(b)|, |\phi(c)|$ is $\bar{1}$ and the remaining is $\bar{1}$, $\phi$ is not trivial and we will get a semidirect product that is not isomorphic to direct product.

5. A short exact sequence of groups is a sequence of groups and homomorphisms:

$$\{1\} \to A \xrightarrow{f} B \xrightarrow{g} C \to \{1\}$$
such that the image of each map equals to the kernel of the next map. For example, \( \text{Im}(f) = \ker(g) \). Given the above short exact sequence, prove:

(i). \( f : A \rightarrow B \) is injective
(ii). \( g : B \rightarrow C \) is surjective
(iii). \( B/f(A) \cong C \)
(iv). Given two groups \( G \) and \( G' \) and a homomorphism \( \phi : G' \rightarrow \text{Aut}(G) \), prove the following is a short exact sequence:

\[
\{1\} \rightarrow G \xrightarrow{i_1} G \rtimes_G G' \xrightarrow{\pi_2} G' \rightarrow \{1\}
\]

where \( i_1(g) = (g, 1') \) and \( \pi_2(g, g') = g' \) for any \( g \in G, g' \in G' \)

Solution:

(i). \( \ker f \) is the image of the map \( \{1\} \rightarrow A \), which is \( \{1\} \), so \( f \) is injective.
(ii). \( \text{Im}(g) \) is the kernel of \( C \rightarrow \{1\} \), which is \( C \), so \( g \) is surjective.
(iii). \( f(A) = \text{Im}(A) = \ker(g) \), and \( g \) is surjective, so by the First Isomorphism Theorem:

\[
B/f(A) = B/\ker(g) \cong C
\]

(iv). The image of \( \{1\} \rightarrow G \) is \( \{1\} \), and \( \ker(i_1) = \{(g, 1') \in G \rtimes_G G' | i_1(g, 1') = (1, 1')\} = \{1\} \).

\( \text{Im}(i_1) = \{(g, 1') \in G \rtimes_G G' | g \in G \} \) and \( \ker(\pi_2) = \{(g, g') \in G \rtimes_G G' | \pi_2(G \rtimes_G G') = 1'\} = \{(g, 1') \in G \rtimes_G G' | g \in G \} \)

\( \text{Im}(\pi_2) = G' \) since \( \pi_2 \) is surjective, and \( \ker(G' \rightarrow \{1\}) = G' \)