1. Prove

\[ \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \]

\[ (k, n) \mapsto k \cdot n = (-1)^k n \]

defines a group action of the group \( \mathbb{Z} \) on itself, and find the stabilizer of \( n \in \mathbb{Z} \).

**Solution:**

(1). For any \( n \in \mathbb{Z} \), \( 0 \cdot n = (-1)^0 n = n \).

(2). For any \( k_1, k_2 \in \mathbb{Z} \), any \( n \in \mathbb{Z} \):

\[ (k_1)(k_2 \cdot n) = k_1((-1)^{k_2} n) = (-1)^{k_1}((-1)^{k_2} n) = (-1)^{k_1+k_2} n = (k_1 + k_2) \cdot n \]

2. \( G \) is a group acting on a set \( S \). \( g \in G \) and \( s \in S \). Prove

\[ G_{g, s} = gG_s g^{-1} \]

**Solution:** \( h \in G_{g, s} \iff h.(g.s) = g.s \iff (hg).s = g.s \iff (g^{-1}hg).s = s \iff g^{-1}hg \in G_s \iff h \in gG_s g^{-1} \]

3. \( G \) is a finite group acting on a finite set \( S \). For each \( g \in G \), define the set

\[ S^g = \{ s \in S | g.s = s \} \]

(i). Prove \( \sum_{s \in S} |G_s| = \sum_{g \in G} |S^g| \).

(ii). Prove \( \sum_{s \in S} |G_s| = |G| \times n \), where \( n \) is the number of orbits in \( S \).

**Solution:**

(i). Define a function (We can call it the characteristic function of the group action) \( \chi : G \times S \rightarrow \{0, 1\} \) by:

\[ \chi(g, s) = \begin{cases} 1, & \text{if } g.s = s \\ 0, & \text{if } g.s \neq s \end{cases} \]

Then \( |G_s| = \sum_{g \in G} \chi(g, s) \) and \( |S^g| = \sum_{s \in S} \chi(g, s) \).

\[ \sum_{s \in S} |G_s| = \sum_{s \in S} \sum_{g \in G} \chi(g, s) = \sum_{g \in G} \sum_{s \in S} \chi(g, s) = \sum_{g \in G} |S^g| \]
(ii). Let \( \mathcal{O} \) be the set of orbits in this action.

\[
\sum_{s \in S} |G_s| = |G| \sum_{s \in S} \frac{1}{|O_s|} = |G| \sum_{O \in \mathcal{O}} \sum_{s \in O} \frac{1}{|O|} = |G| \sum_{O \in \mathcal{O}} 1 = |G| |\mathcal{O}| = |G| \times n
\]

4. \( G \) is a group, \( H \) and \( K \) are normal subgroups of \( G \), \( G = HK \).

(i). Prove \( G \times (G/H \times G/K) \rightarrow G/H \times G/K \) given by

\[
g.(xH, yK) = (gxH, gyK)
\]
is a group action.

(ii). Compute the stabilizer \( G_{(xH,yK)} \)

(iii). If \( G \) is a finite group, express the order of \( G/H \times G/K \) in terms of \(|H|, |K|, |H \cap K|\).

(iv). Use the Counting Formula to prove: if \( G \) is a finite group, the above action is transitive.

(v). Let \( f : G \rightarrow G/H \times G/K \) be the map \( f(g) = (gH, gK) \). Prove \( f \) is surjective if and only if the action in (i) is transitive.

(vi). Prove \( f \) is surjective **without** the assumption that \( G \) is finite.

(2) Prove (v) and (vi) together imply that the action in (i) is transitive.

**Solution:**

(i). This is a group action since

1. \( 1.(xH, yK) = ((1x)H, (1y)K) = (xH, yK) \) for any \((xH, yK) \in G/H \times G/K\)

2. \( g_1.(g_2.(xH, yK)) = g_1.(g_2xH, g_2yK) = (g_1(g_2)xH, g_1(g_2)yK) = ((g_1g_2)xH, (g_1g_2)yK) = (g_1g_2).(xH, yK) \) for any \( g_1, g_2 \in G \), any \((xH, yK) \in G/H \times G/K\)

(ii).

\[
G_{(xH,yK)} = \{ g \in G | g.(xH, yK) = (xH, yK) \}
= \{ g \in G | (gxH, gyK) = (xH, yK) \}
= \{ g \in G | x^{-1}gx \in H, y^{-1}gy \in K \}
= \{ g \in G | g \in xHx^{-1}, g \in yKy^{-1} \}
= \{ g \in G | g \in H, g \in K \}
= H \cap K
\]
(iii). \[ |G/H \times G/K| = |G/H| \times |G/K| \]
\[ = \frac{|G|}{|H|} \times \frac{|G|}{|K|} \]
\[ = \frac{|HK|^2}{|H| \times |K|} \]
\[ = \frac{1}{|H| \times |K|} \left( \frac{|H| \times |K|}{|H \cap K|} \right)^2 \]
\[ = \frac{|H| \times |K|}{|H \cap K|^2} \]

(iv). If \( G \) is a finite group, by the Counting Formula,
\[ O(xH, yH) = \frac{|G|}{|G_{xH,yH}|} = \frac{|HK|}{|H \cap K|} = \frac{1}{|H| \times |K|} \cdot \frac{|H \times |K|}{|H \cap K|^2} = |G/H \times G/K| \]
So \( O(xH, yH) = G/H \times G/K \), the action is transitive.

(v).
\[ O(H, K) = \{ \,(H, K) \in G/H \times G/K \mid g \in G \} \]
\[ = \{ (gH, gK) \in G/H \times G/K \mid g \in G \} \]
\[ = \{ f(g) \in G/H \times G/K \mid g \in G \} \]
\[ = Im(f) \]

\( f \) is surjective if and only if \( G/H \times G/K = Im(f) \) if and only if \( G/H \times G/K = O(H, K) \) if and only if the action is transitive.

(vi). For any \((xH, yK) \in G/H \times G/K\), \( x^{-1}y \in G = HK \), so there exists \( h \in H, k \in K \) such that \( x^{-1}y = hk \), so \( xH = yK^{-1} \). Let \( g = xH = yK^{-1} \in G \), then \( gH = xH = xH, gK = yK^{-1}K = yK \), we get \((xH, yK) = f(g)\), so the function is surjective.

5. \( G \) is a group with \( |G| = p^2 \) for some prime \( p \). Prove either \( G \cong \mathbb{Z}/p^2\mathbb{Z} \) or \( G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \)

**Solution:** \( |G| = p^2 \) and \( p \) is a prime, so \( G \) is an abelian group.

If \( G \) is a cyclic group of order \( p^2 \), then \( G \cong \mathbb{Z}/p^2\mathbb{Z} \).
If $G$ is not a cyclic group, then all the non-identity elements of $G$ have order $p$. Choose $x \in G \setminus \{1\}$, and choose $y \in G \setminus <x>$. In particular, $|x| = |y| = p$. $G$ is a normal subgroup, so $<x>$ and $<y>$ are normal subgroups of $G$. $y \notin <x>$ implies $<x> \cap <y> = \{1\}$ since $p$ is a prime. $|<x><y>| = \frac{|<x>|\|y|}{\|<x>\cap<y>\|} = p^2$, so $<x><y> = G$. We conclude

$$G \cong <x> \times <y> \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$$