1. Write \((1 \, 3 \, 7)(2 \, 4 \, 5 \, 6) \in S_7\) as a product of 2-cycles.

   **Solution:**
   \[(1 \, 3 \, 7)(2 \, 4 \, 5 \, 6) = (1 \, 7)(1 \, 3)(2 \, 6)(2 \, 5)(2 \, 4)\]

2. Compute the signature of the element \((1 \, 2 \, 3)(6 \, 7)(4 \, 5 \, 9) \in S_9\).

   **Solution:**
   \[
sgn((1 \, 2 \, 3)(6 \, 7)(4 \, 5 \, 9)) = sgn((1 \, 2 \, 3))sgn((6 \, 7))sgn((4 \, 5 \, 9))
   = (+1) \times (-1) \times (+1)
   = -1
   \]

3. \(H\) is a subgroup of \(S_n\) with \(|H|\) an odd number. Prove \(H \subseteq A_n\).

   **Hint:** Consider \([H : H \cap A_n]\)

   **Solution:** Consider the restriction of the signature function to the subgroup \(H\):
   \[
f : H \longrightarrow \{\pm 1\} \text{ is defined by } f(\sigma) = sgn(\sigma).
   \]
   By the definition of \(A_n\), \(\ker(f) = \{\sigma \in H | sgn(\sigma) = +1\} = H \cap A_n\).
   Suppose \(H \not\subseteq A_n\), then there exists \(\sigma \in H\) with \(sgn(\sigma) = -1\). It follows \(f\) is surjective, and then by First Isomorphism Theorem, \(H/(H \cap A_n) \cong \{\pm 1\}\).
   This means \([H : H \cap A_n] = |\{\pm 1\}| = 2\), so \([H : H \cap A_n]\) is even. By Lagrange Theorem, \(|H| = [H : H \cap A_n]|H \cap A_n|\), contradict to \(|H|\) is odd.

4. \(\tau, \sigma \in S_n\). Prove that \(\tau\sigma\) and \(\sigma\tau\) have the same cycle type.

   **Solution:** \(\sigma(\tau\sigma)\sigma^{-1} = \sigma\tau\), so \(\tau\sigma\) and \(\sigma\tau\) are conjugate, hence they have the same cycle type.

5. If \(H\) is a normal subgroup of \(G\) and \(K\) is a normal subgroup of \(H\), is it always true that \(K\) is a normal subgroup of \(G\)? If yes, prove it; if no, provide a counter-example.

   **Solution:** The answer is NO. A possible example is \(\{id, (1 \, 2)(3 \, 4)\}\) is a normal subgroup of \(K = \{id, (1 \, 2)(3 \, 4), (1 \, 3)(2 \, 4), (1 \, 4)(2 \, 3)\}\), and \(K\) is a normal subgroup of \(A_4\), but \(\{id, (1 \, 2)(3 \, 4)\}\) is not a normal subgroup of \(A_4\).
6. If \( n \geq 5 \), prove the only proper normal subgroup of \( S_n \) is \( A_n \).

**Solution:** Let \( N \) be a normal subgroup of \( S_n \). Then \( N \cap A_n \) is a normal subgroup of \( A_n \). Because \( A_n \) is simple when \( n \geq 5 \), \( N \cap A_n = \{id\} \) or \( N \cap A_n = A_n \).

Case (1). \( N \cap A_n = \{id\} \). Consider the restriction of the signature function to \( N \)

\[
sgn|_N : N \longrightarrow \{\pm 1\}
\]

\[
\sigma \mapsto sgn(\sigma)
\]

\( \ker(sgn|_N) = N \cap A_n = \{id\} \), so \( sgn|_N \) is injective, we see \( |N| = |\text{Im}(sgn|_N)| \).

If \( \text{Im}(sgn|_N) = \{+1\} \), then \( N = \{id\} \).

If \( \text{Im}(sgn|_N) = \{\pm 1\} \), then \( |N| = 2 \), so we can write \( N = \{id, \sigma\} \) for some odd permutation \( \sigma \). Then \( N \) cannot be normal since it is not hard to see that for non-identity \( \sigma \in S_n \), its cycle type consists of more than just \( \sigma \), but those elements who are conjugate to \( \sigma \) are not in \( N \).

Case (2). \( N \cap A_n = A_n \), then \( A_n \subset N \subset S_n \), we get

\[
2 = [S_n : A_n] = [S_n : N][N : A_n]
\]

So \([N : A_n] = 1 \) or \([N : A_n] = 2 \).

If \([N : A_n] = 1 \), then \( N = A_n \).

If \([N : A_n] = 2 \), then \([S_n : N] = 1 \), so \( N = S_n \).

By Case (1) and (2), we conclude \( N \) can \( \{id\} \), \( A_n \) or \( S_n \), so the only choice for a proper normal subgroup is \( A_n \).