1. How many different equivalence relations can we define on a set of four elements?

Solution: We know equivalence relations are in one-to-one correspondence with partition of a set, so we only need to find all the partitions of a set of four elements.

Denote this set by \{a, b, c, d\}, we see the possible partitions are as follows:

\{a\} \sqcup \{b\} \sqcup \{c\} \sqcup \{d\},
\{a\} \sqcup \{b\} \sqcup \{c, d\}, \{a\} \sqcup \{b, c\}, \{b\} \sqcup \{c\} \sqcup \{a, d\},
\{b\} \sqcup \{d\} \sqcup \{a, c\}, \{c\} \sqcup \{d\} \sqcup \{a, b\},
\{a, b\} \sqcup \{c, d\}, \{a, c\} \sqcup \{b, d\}, \{a, d\} \sqcup \{b, c\},
\{a\} \sqcup \{b, c, d\}, \{b\} \sqcup \{a, c, d\}, \{c\} \sqcup \{a, b, d\}, \{d\} \sqcup \{a, b, c\}
\{a, b, c, d\}

So there are in total 15 of them.

Remark: In general, the number of partitions of a set of \(n\) elements is called the Bell Number. You may read this Wikipedia Page for more story on that: https://en.wikipedia.org/wiki/Bell_number

2. Define a relation on \(\mathbb{R}^\times = \mathbb{R} \setminus \{0\}\) by \(a \sim b\) if \(\frac{a}{b} \in \mathbb{Q}\). Prove this is an equivalence relation.

Solution:

(i). For any \(r \in \mathbb{R}^\times\), \(\frac{r}{r} = 1 \in \mathbb{Q}\), so \(r \sim r\).

(ii). If \(a \sim b\), then \(\frac{a}{b} \in \mathbb{Q}\), \(\frac{b}{a} \in \mathbb{Q}\), so \(b \sim a\).

(iii). If \(a \sim b\) and \(b \sim c\), then \(\frac{a}{b} \in \mathbb{Q}\) and \(\frac{b}{c} \in \mathbb{Q}\), \(\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} \in \mathbb{Q}\), so \(a \sim c\).

3. \(G\) is a group, \(H_1\) and \(H_2\) are finite subgroups of \(G\). If \(|H_1|\) and \(|H_2|\) are relatively prime, prove \(H_1 \cap H_2 = \{1\}\).

Solution: \(H_1 \cap H_2\) is a subgroup of \(H_1\), so \(|H_1 \cap H_2|\) divides \(|H_1|\). \(H_1 \cap H_2\) is a subgroup of \(H_2\), so \(|H_1 \cap H_2|\) divides \(|H_2|\). \(|H_1|\) and \(|H_2|\) are relatively prime, so \(|H_1 \cap H_2| = 1\), \(H_1 \cap H_2 = \{1\}\).
4. If $G$ has five subgroups of order 7, prove $G$ has at least 35 elements.

Solution: If $H$ and $K$ are two subgroups of order 7 in $G$ such that $H \neq K$, then $H \cap K = \{1\}$: Suppose there is non-identity $x \in H \cap K$, then in particular, $x \in H$, so $|x|$ divides $|H| = 7$, and $x$ is not the identity, it follows $|x| = 7$, so $< x > = H$. Similarly, $< x > = K$, we get $H = K$, contradict to the assumption $H \neq K$, therefore such $x$ does not exist. We conclude $H \cap K = \{1\}$.

If there are five different subgroups of order 7, then by the above paragraph, except the identity element, any two of those five subgroups share no element.

It follows there are $1 + 5 \times (7 - 1) = 31$ elements in the union of these five subsets, so $|G| \geq 31$.

$G$ contains subgroups of order 7 implies 7 divides $|G|$, and 35 is the smallest multiple of 7 that is no smaller than 31, so we conclude $|G| \geq 35$.

5. $G$ is a group. $H$ and $K$ are subgroups of $G$.

(i). For any $x, y \in G$, prove either $xH \cap yK = \emptyset$ or $xH \cap yK = g(H \cap K)$ for some $g \in G$.


Solution:

(i). If $xH \cap yK \neq \emptyset$, let $g \in xH \cap yK$, then $g \in xH$ implies $gH = xH$, and $g \in yK$ implies $gK = yK$. So $xH \cap yK = gH \cap gK = g(H \cap K)$.

(ii). $g(H \cap K) = gH \cap gK$, so each left coset of $H \cap K$ in $G$ is the intersection of some left coset $H$ in $G$ with some left coset of $K$ in $G$, and $[G : H] < \infty$, $[G : K] < \infty$ implies there are finitely many left cosets of $H$ and $K$ in $G$ respectively, so the number of their intersection is also finite, which is greater than or equal to $[G : H \cap K]$.

6. $\mathbb{R}$ is the group of rational numbers with addition. Prove that $r + \mathbb{Z}$ is an element of finite order in $\mathbb{R}/\mathbb{Z}$ if and only if $r \in \mathbb{Q}$.

Solution:

If $r + \mathbb{Z}$ is of order $k < \infty$, then $k(r + \mathbb{Z}) = 0 + \mathbb{Z}$, i.e., $rk + \mathbb{Z} = 0 + \mathbb{Z}$, we get $rk \in \mathbb{Z}$, so $r \in \mathbb{Q}$.

Conversely, for any rational number $\frac{a}{b}$ ($a, b \in \mathbb{Z}$, $b > 0$), we see

$$b(\frac{a}{b} + \mathbb{Z}) = a + \mathbb{Z} = 0 + \mathbb{Z}$$

which implies the order of $\frac{a}{b} + \mathbb{Z}$ is finite.
7. Let $H$ and $K$ be subgroups of $G$. Let $g \in G$, the set

$$HgK = \{hgk \in G| h \in H, k \in K\}$$

is called a double coset. Prove the double cosets form a partition of $G$.

**Solution:** Define a relation on $G$ by $x \sim y$ if $x \in HyK$. This is an equivalence relation:

(i). For any $x \in G$, $x = 1x1 \in HxK$

(ii). If $x \sim y$, then $x \in HyK$, there exists $h \in H$ and $k \in K$ such that $x = hyk$, so $y = h^{-1}xk^{-1} \in HxK$, $y \sim x$.

(iii). If $x \sim y$ and $y \sim z$, then $x \in HyK$ and $y \in HzK$, there exists $h_1 \in H, k_1 \in K$ such that $x = h_1yk_1$, and there exists $h_2 \in H, k_2 \in K$ such that $y = h_2zk_2$, so $x = h_1yk_1 = h_1(h_2zk_2)k_1 = (h_1h_2)zk_2k_1 \in HzK$, $x \sim z$.

By the definition of this equivalence relation, the double cosets are exactly the equivalence classes, so they form a partition of $G$.

8. $S$ is a subset of a group $G$ and $1 \in S$. For any $g \in G$, we define

$$gS = \{gs \in G| s \in S\}$$

If for any $a, b \in G$, either $aS = bS$ or $aS \cap bS = \emptyset$, prove $S$ is a subgroup of $G$.

**Solution:** For any $a, b \in S$, $a = a1 \in aS$, so $a \in S \cap aS$, $a \cap aS \neq \emptyset$, $S = aS$. Then $ab \in aS = S$.

For any $a \in S$, $1 = a^{-1}a \in a^{-1}S$, so $1 \in a^{-1}S \cap S$, $a^{-1}S \cap S \neq \emptyset$, $a^{-1}S = S$. $a^{-1} = a^{-1}1 \in a^{-1}S = S$. 

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