1. Use strong induction to prove that any integer \( n \geq 2 \) can be written as a product of one or more prime numbers.

**Solution:**

(1). When \( n = 2 \), we know 2 is a prime, so 2 is a product of one prime.

(2). Suppose the statement holds for all \( 2 \leq k \leq n \). If \( n + 1 \) is a prime, then we are done, since it is a product of one prime. If \( n + 1 \) is not a prime, then we can write \( n + 1 = ab \) where \( 2 \leq a \leq n \), \( 2 \leq b \leq n \). By induction hypothesis, both \( a \) and \( b \) can be written as a product of primes, so the product \( ab = n + 1 \) is also a product of primes.

We finish the proof.

2. Let \( G \) be a group. Define an opposite group \( G^o \) with law of composition \( a \ast b \) as follows:

The underlying set is the same as \( G \), but the law of composition is \( a \ast b = ba \).

Prove \( G^o \) is a group.

**Solution:**

(1). \( \forall a, b, c \in G^o: \)

\([a \ast b] \ast c = c(a \ast b) = c(ba) = (cb)a = (b \ast c)a = a \ast (b \ast c)\]

(2). The original identity \( 1 \in G \) is also the identity for this new composition:

\( \forall g \in G^o \)

\( g \ast 1 = 1g = g, 1 \ast g = g1 = g \)

(3). \( \forall g \in G^o \), the original inverse \( g^{-1} \) in \( G \) is also the inverse for the new composition:

\( g \ast g^{-1} = g^{-1}g = 1, g^{-1} \ast g = gg^{-1} = 1 \)

So we conclude \( G^o \) is a group.

3. Prove the set of all \( n \times n \) matrices with real entries, \( M_n(R) \), is an abelian group if we define the law of composition to be addition of matrices.

**Solution:**
(1). Addition of matrices is associative: 
\((A + B) + C = A + (B + C)\) for any 
\(A, B, C \in M_n(\mathbb{R})\).

(2). The identity element is the \(n \times n\) zero matrix \(0_n\): 
\(A + 0_n = 0_n + A = A\) for any \(A \in M_n(\mathbb{R})\).

(3). The inverse of \(A\) is \(-A\): 
\(A + (-A) = (-A) + A = 0_n\).

So \(M_n(\mathbb{R})\) with matrix addition is a group. It is an abelian group since the matrix addition is commutative: 
\(A + B = B + A\) for any \(A, B \in M_n(\mathbb{R})\).

4. If \(G\) is a group such that \(|G|\) is an even number, prove there exists \(g \in G\) such that 
\(g = g^{-1}\) and \(g \neq 1\).

Solution:
Suppose there is no such \(g\), then 1 is the only element that equals to its inverse. This implies all the non-identity elements can be put in pairs \(\{g, g^{-1}\}\), and only 1 is left without pairing. So there are odd number of elements in total, contradicts to the condition \(|G|\) is even. We conclude there has to exists such non-identity \(g\) with \(g = g^{-1}\).

5. \(G\) is a group and \(g \in G\). \(k\) is a positive integer. Prove \(g^{-k} = (g^k)^{-1}\).

Solution:
We can prove by induction.
(1). when \(k = 1\), \((g^1)^{-1} = g^{-1} = (g^{-1})^1\) is obviously true.
(2). Assume \((g^{k})^{-1} = (g^{-1})^k\), i.e. \(g^k(g^{-1})^k = 1\), then for \(k + 1\):

\[
g^{k+1}(g^{-1})^{k+1} = g^k gg^{-1}(g^{-1})^k = g^k (gg^{-1})(g^{-1})^k = g^k(g^{-1})^k = 1
\]

So \((g^{k+1})^{-1} = (g^{-1})^{k+1}\) is also true.

We finish the proof.

6. \(G\) is a group. If \(g^2 = 1\) for any \(g \in G\), prove that \(G\) is an abelian group.

Solution:
\(g^2 = 1\) for any \(g \in G\) is equivalent to \(g = g^{-1}\) for any \(g \in G\).

For any \(a, b \in G\), \(ab\) is also an element in \(G\), so \((ab)^{-1} = b^{-1}a^{-1} = ba\). So the group is abelian.
7. Make a multiplication table for the group $S_3$.

**Solution:**

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<th>(2 3)</th>
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</tbody>
</table>

8. If $G$ is a nonempty set with a law of composition $G \times G \rightarrow G$ satisfying:

(i). Associativity: $(ab)c = a(bc)$ for all $a, b, c \in G$

(ii). Existence of left identity: there is $e \in G$ such that $eg = g$ for all $g \in G$

(iii). Existence of left inverse: for any $g \in G$, there exists $h \in G$ such that $hg = e$

Prove $G$ with this composition is a group.

**Solution:**

For any $g \in G$, it has left inverse $h \in G$ such that $hg = e$. For this $h \in G$, it has left inverse $k \in G$ such that $kh = e$. We thus see

$$g = eg = (kh)g = k(hg) = ke$$

which further implies

$$gh = (ke)h = k(eh) = kh = e$$

So $h$ is also the right inverse of $g$.

Next, we show $e$ is also the right identity by:

$$ge = g(hg) = (gh)g = eg = g$$

We conclude $G$ is a group.