Lemma. Compositions of isometries of $\mathbb{R}^n$ is an isometry of $\mathbb{R}^n$.

Proof. If $\varphi_1, \varphi_2$ are isometries on $\mathbb{R}^n$, then for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\varphi_1 \varphi_2 (\mathbf{u}) - \varphi_1 \varphi_2 (\mathbf{v})| = |\varphi_2 (\mathbf{u}) - \varphi_2 (\mathbf{v})| = |\mathbf{u} - \mathbf{v}|.$$

Corollary. Every isometry $f: \mathbb{R}^n \to \mathbb{R}^n$ is the composition of an orthogonal linear operator and a translation.

Proof. $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry. Define the translation

$$t_{-f(\mathbf{o})} : \mathbb{R}^n \to \mathbb{R}^n$$

$$\mathbf{u} \mapsto \mathbf{u} - f(\mathbf{o}).$$

Then $t_{-f(\mathbf{o})} \circ f$ is an isometry such that

$$t_{-f(\mathbf{o})} \circ f (\mathbf{o}) = f(\mathbf{o}) - f(\mathbf{o}) = \mathbf{0}.$$

so by the previous theorem, $t_{-f(\mathbf{o})} \circ f = \varphi$ is an orthogonal linear operator, then

$$f = t_{-f(\mathbf{o})} \circ \varphi = t_{-f(\mathbf{o})} \circ t_{f(\mathbf{o})} \circ \varphi$$

is a composition of an orthogonal linear operator and a translation.

Corollary. An isometry $f: \mathbb{R}^n \to \mathbb{R}^n$ is a bijective map.

Proof. This is because $f$ is a composition of an orthogonal linear operator and a translation, both of which are bijective maps.

By all the above discussions, we can verify that the set of all isometries of $\mathbb{R}^n$ form a group with composition of maps. We denote this group by $M_n$. 

\(\text{Page } 56\)
Lemma. If $t_{\mathbf{a}}$ is a translation and $\varphi$ is an orthogonal linear operator, then $\varphi t_{\mathbf{a}} = t_{\varphi(\mathbf{a})} \varphi$

Proof. For any $\mathbf{u} \in \mathbb{R}^n$, $\varphi t_{\mathbf{a}} (\mathbf{u}) = \varphi (\mathbf{u} + \mathbf{a}) = \varphi (\mathbf{u}) + \varphi (\mathbf{a})$

$= t_{\varphi(\mathbf{a})} (\varphi (\mathbf{u}))$

$= t_{\varphi(\mathbf{a})} \varphi (\mathbf{u})$

Let $T_n$ be the set of translations of $\mathbb{R}^n$. $O_n$ be the set of orthogonal linear operators on $\mathbb{R}^n$. It's easy to verify both of them are subgroups of $M_n$

Proposition. The map defined by $M_n \xrightarrow{\pi} O_n$ is a surjective homomorphism.

$f \mapsto t_{\varphi(\mathbf{a})} \varphi$

Proof. It's easy to see $\pi$ is surjective.

We need to show it's a homomorphism

For any $f, g \in M_n$, we can write $f = t_{\mathbf{a}} \varphi$ and $g = t_{\mathbf{b}} \varphi$. where $t_{\mathbf{a}}, t_{\mathbf{b}} \in T_n$ and $\varphi, \varphi \in O_n$

So $\pi(f) = \varphi$ and $\pi(g) = \varphi$

$\pi(fg) = \pi(t_{\mathbf{a}} \varphi t_{\mathbf{b}} \varphi) = \pi(t_{\mathbf{a}} (t_{\varphi(\mathbf{b})} \varphi) \varphi) = \varphi \varphi = \varphi = \pi(f) \pi(g)$

We see the kernel of this projection map is

$\ker(\pi) = \{ t_{\mathbf{a}} \varphi \in M_n \mid \pi(t_{\mathbf{a}} \varphi) = \text{id} \} = \{ t_{\mathbf{a}} \varphi \in M_n \mid \varphi = \text{id} \} = T_n$

So $T_n$ is a normal subgroup of $M_n$

By First Isomorphism Theorem, $M_n / T_n \cong O_n$
ISOMETRIES ON R^2

Based on the general discussion in the previous section, we know that each isometry of R^2 is a composition of an orthogonal linear operator with a translation.

Also, we know orthogonal linear operators are in one-to-one correspondence with the orthogonal matrices O_n(R).

In O_2(R), there is a subgroup SO_2(R) = \{A \in O_2(R) \mid \det(A) = 1\}. SO_2(R) is the kernel of the restriction of the determinant map to O_2(R), and in fact, the determinant of an orthogonal matrix is always 1 or -1. So by First Isomorphism Theorem, \( O_2(R) / SO_2(R) \cong \{\pm 1\} \).

We conclude \( [O_2(R) : SO_2(R)] = 2 \).

Let \( R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in O_2(R) \), then \( O_2(R) = SO_2(R) \sqcup R \cdot SO_2(R) \).

From now on, when there's no confusion, we shall write O_2 and SO_2 instead of O_2(R) and SO_2(R). Note this is not a lecture notes in chemistry, so here O_2 is not Oxygen and SO_2 is not sulfur dioxide...

Recall that in the homework we've shown that every element of SO_2 is of form

\[
A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \theta \in \mathbb{R}.
\]
Proposition. The map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by
\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
is the rotation of angle $\theta$ centered at origin.

Proof. We can write $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R \cos \alpha \\ R \sin \alpha \end{pmatrix}$ by polar coordinates.

\[
\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} R \cos \alpha \\ R \sin \alpha \end{pmatrix} = \begin{pmatrix} R \cos (\alpha + \theta) \\ R \sin (\alpha + \theta) \end{pmatrix}
\]

Proposition. The map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by
\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}
\]
is the reflection with respect to $x$-axis.

Combining all the above discussion, we see:

Theorem. Let $m$ be an isometry of the plane, then $m = t_{\alpha} p_{\theta}$ or else $m = t_{\alpha} p_{\theta} r$, where $\alpha \in \mathbb{R}^2$, and $p_{\theta}$ is the rotation of angle $\theta$ with respect to origin; $r$ is reflection with respect to $x$-axis.

Lemma. $\bullet \quad p_{\theta} t_{\alpha} = t_{p_{\theta}(\alpha)} p_{\theta}$

$\bullet \quad r t_{\alpha} = t_{r(\alpha)} r$

$\bullet \quad r p_{\theta} = p_{\theta} r$

$\bullet \quad t_{\alpha} \cdot t_{\beta} = t_{\alpha + \beta}$, $p_{\theta} \cdot p_{\eta} = p_{\theta + \eta}$, $r^2 = \text{id}$. 