1 Classification of Isometry on the Plane

Based on the general discussion in the previous chapter, we know that each isometry of $\mathbb{R}^2$ is a composition of an orthogonal linear operator with a translation.

Also we know orthogonal linear operators are in one-to-one correspondence with the orthogonal matrices $O_n(\mathbb{R})$ once a standard basis is given.

**Exercise 1.** A is a real $n \times n$ matrix. $A \in O_n(\mathbb{R})$ if and only if the rows (columns) of $A$ are unit vectors that are pairwisely perpendicular.

In $O_n(\mathbb{R})$, there is a subgroup $SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) | \det(A) = 1\}$. $SO_n(\mathbb{R})$ is the kernel of the restriction of the determinant map to $O_n(\mathbb{R})$.

**Exercise 2.** The determinant of an orthogonal matrix can only be $\pm 1$.

The First Isomorphosm Theorem implies

$$O_n(\mathbb{R})/SO_n(\mathbb{R}) \cong \{\pm 1\}$$

and it follows $[O_n(\mathbb{R}) : SO_n(\mathbb{R})] = 2$, which tells us $SO_n(\mathbb{R})$ is a normal subgroup of $O_n(\mathbb{R})$. Let $r$ be the $n \times n$ matrix with entries

$$r_{ij} = \begin{cases} 1, & \text{if } 1 \leq i = j \leq n - 1 \\ -1, & \text{if } i = j = n \\ 0, & \text{if } i \neq j \end{cases}$$

then

$$O_n(\mathbb{R}) = SO_n(\mathbb{R}) \sqcup (SO_n(\mathbb{R}))r$$

From now on we will concentrate on the case $n = 2$. 

Lemma 3. Every element in $\text{SO}_2(\mathbb{R})$ can be written as
\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]
for some $\theta \in \mathbb{R}$.

Proof. Assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SO}_2$, then $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc = 1$, so
\[
\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}^{-1} = \begin{bmatrix} a & b \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}
\]
So $a = d$ and $b = -c$, the matrix becomes $\begin{bmatrix} a & -c \\ c & a \end{bmatrix}$, with $a^2 + c^2 = 1$. We know for a pair of real numbers $a, c$ satisfying $a^2 + c^2 = 1$, the angle $\theta$ whose terminal edge passing through $(a, c)$ has $\cos \theta = a$ and $\sin \theta = c$, hence the matrix can be written as $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

Proposition 4. The map $\rho_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by
\[
\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]
is the rotation of angle $\theta$ around origin.

Proof. We can write
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} R \cos \alpha \\ R \sin \alpha \end{bmatrix}
\]
by polar coordinates.
\[
\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} R \cos \alpha \\ R \sin \alpha \end{bmatrix} = \begin{bmatrix} R \cos \alpha \cos \theta - R \sin \alpha \sin \theta \\ R \cos \alpha \sin \theta + R \sin \alpha \cos \theta \end{bmatrix} = \begin{bmatrix} R \cos(\alpha + \theta) \\ R \sin(\alpha + \theta) \end{bmatrix}
\]

Proposition 5. The map $r : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by
\[
\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}
\]
is the reflection with respect to $x$-axis.
Combining the above discussion we conclude:

**Theorem 6.** Let \( m \) be an isometry of the plane, then \( m = t_\bar{a} \rho_\theta \) or \( m = t_\bar{a} \rho_\theta r \), where \( \bar{a} \in \mathbb{R}^2 \), \( \rho_\theta \) and \( r \) are defined as above.

**Lemma 7.** The following identities hold:

1. \( \rho_\theta t_\bar{a} = t_{\rho_\theta(\bar{a})} \rho_\theta \)
2. \( r t_\bar{a} = t_{r(\bar{a})} r \)
3. \( r \rho_\theta = \rho_{-\theta} r \)
4. \( t_{\bar{a}} t_{\bar{b}} = t_{\bar{a}+\bar{b}} \), \( \rho_\theta \rho_\eta = \rho_{\theta+\eta} \), \( r^2 = \text{id} \)

Geometrically, there is a simpler description of isometries of the plane.

**Theorem 8.** Every isometry of the plane has one of the following forms:

1. Translation along \( \bar{a} \in \mathbb{R}^2 \)
2. Rotation through a nonzero angle about a point
3. Reflection along a line \( l \)
4. Glide Reflection: reflection along a line \( l \), followed by a translation along a nonzero vector parallel to \( l \)

The first two are orientation preserving and the last two are orientation reversing.

**Proof.** If the isometry is of form \( t_\bar{a} \), then it is a translation.

If the isometry is of form \( t_\bar{a} \rho_\theta \), we are going to show it is a rotation through an angle \( \theta \) about some point \( \bar{p} \): by assumption, \( \theta \neq 0 \), so the matrix

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} - 
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix} = 
\begin{bmatrix}
1 - \cos \theta & \sin \theta \\
-\sin \theta & 1 - \cos \theta \\
\end{bmatrix}
\]

is invertible. Let \( \bar{p} = (id - \rho_\theta)^{-1} \bar{a} \), then \( \bar{a} = (id - \rho_\theta) \bar{p} = \bar{p} - \rho_\theta(\bar{p}) \), we get

\[
t_\bar{a} \rho_\theta = t_{\bar{p} - \rho_\theta(\bar{p})} \rho_\theta = t_{\bar{p}} t_{\rho_\theta(-\bar{p})} \rho_\theta = t_{\bar{p}} \rho_\theta t_{-\bar{p}}
\]

and it is geometrically clear that \( t_{\bar{p}} \rho_\theta t_{-\bar{p}} \) is the rotation about \( \bar{p} \) of angle \( \theta \).
If the isometry is of form $\rho_\theta r$, then
$$\rho_\theta r = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$
and
$$\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} R \cos \alpha \\ R \sin \alpha \end{bmatrix} = \begin{bmatrix} R \cos(\theta - \alpha) \\ R \sin(\theta - \alpha) \end{bmatrix}$$
So $\rho_\theta r$ is the reflection along the line through origin with angle $\frac{\theta}{2}$ to $x$-axis.

If the isometry is of form $t_{\vec{a}} \rho_\theta r$, then it is a reflection along a line $l$ passing through origin followed by a translation along $\vec{a}$. Decompose $\vec{a} = \vec{a}_1 + \vec{a}_2$ such that $\vec{a}_1$ is parallel to $l$ and $\vec{a}_2$ is orthogonal to $l$. Then
$$t_{\vec{a}} \rho_\theta r = t_{\vec{a}_1} (t_{\vec{a}_2} \rho_\theta r)$$
and geometrically, $t_{\vec{a}_2} \rho_\theta r$ is the reflection along the line $l'$, which is obtained by translating $l$ along $\frac{1}{2} \vec{a}_2$. Note $\vec{a}_1$ is parallel to $l'$, we conclude this corresponds to the last case in the statement of the theorem.

2 Dihedral Group

Definition 9. $n$ is a positive integer. Let $\rho = \rho_{\frac{2\pi}{n}}$. The dihedral group is the subgroup of $O_2$ defined by
$$D_n = \{ \rho^i r^j \in O_2 | 0 \leq i \leq n-1, 0 \leq j \leq 1 \}$$

Exercise 10. $\rho^n = 1$, $r^2 = 1$, $r \rho = \rho^{-1} r$.

Proof. By direct computation. $\square$

Corollary 11. $r \rho^i r = \rho^{-i}$ for any $i \in \mathbb{Z}$

Exercise 12. Show that $D_n$ is a subgroup of $O_2$.

Geometrically, $D_n$ is the group of symmetries of a regular $n$-gon. Each element of $D_n$ permutes the $n$ vertices of the $n$-gon, we can view $D_n$ as a subgroup of $S_n$.

Example 13. $D_3$ has 6 elements, same as $S_3$, and $D_3$ can be viewed as a subgroup of $S_3$, we get $D_3 \cong S_3$.

Example 14. $D_1 = \{1, r\} \cong C_2$, $D_2 = \{1, \rho_\pi, r, \rho_\pi r\} \cong C_2 \times C_2 \cong K_4$. 

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