Homomorphisms and Isomorphisms

Liming Pang

1 Homomorphisms and Normal Subgroups

Definition 1. $G$ and $G'$ are groups. A homomorphism $\phi : G \rightarrow G'$ is a function satisfying $\forall a, b \in G \implies \phi(ab) = \phi(a)\phi(b)$. That is, the function is compatible with the group structures.

Example 2. Determinant function is a homomorphism:

$$GL_n(\mathbb{R}) \xrightarrow{\text{det}} \mathbb{R}^\times$$

$A \mapsto \text{det}(A)$

Example 3. If $G$ is a group, and $x \in G$, then there is a homomorphism:

$$\mathbb{Z} \rightarrow G$$

$k \mapsto x^k$

Proposition 4. A homomorphism $\phi : G \rightarrow G'$ maps identity to identity, and inverse to inverse:

1. $\phi(1_G) = 1_{G'}$
2. $\forall g \in G \implies \phi(g)^{-1} = \phi(g^{-1})$

Proof. 1. $\phi(1_G) = \phi(1_G 1_G) = \phi(1_G)\phi(1_G)$, so $\phi(1_G) = 1_{G'}$

2. $\phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = \phi(1_G) = 1_{G'}$, so $\phi(g)^{-1} = \phi(g^{-1})$
Definition 5. $\phi : G \rightarrow G'$ is a homomorphism. Define the **kernel** of $\phi$ to be
$$\ker \phi = \{ g \in G | \phi(g) = 1_{G'} \}$$
Define the **image** of $\phi$ to be
$$\text{Im} \phi = \{ \phi(g) \in G' | g \in G \}$$

Proposition 6. $\phi : G \rightarrow G'$ is a homomorphism, then $\ker \phi$ is a subgroup of $G$ and $\text{Im} \phi$ is a subgroup of $G'$.

*Proof.* If $a, b \in \ker \phi$, $\phi(a) = \phi(b) = 1_{G'}$.
$$\phi(a^{-1}b) = \phi(a)^{-1}\phi(b) = 1_{G'}$$
So $a^{-1}b \in \ker \phi$, $\ker \phi$ is a subgroup of $G$.

If $x, y \in \text{Im} \phi$, then there exists $a, b \in G$ such that $x = \phi(a)$ and $y = \phi(b)$. Then $x^{-1}y = \phi(a)^{-1}\phi(b) = \phi(a^{-1}b) \in \text{Im} \phi$. So $\text{Im} \phi$ is a subgroup of $G'$. \qed

Proposition 7. $\phi : G \rightarrow G'$ is a homomorphism, then $\phi$ is injective if and only if $\ker \phi = \{ 1_G \}$.

*Proof.* If $\phi$ is injective, then $\ker \phi = \phi^{-1}(1_{G'})$ consists of at most one element, and we know $1_G \in \ker \phi$, we get $\ker \phi = \{ 1_G \}$.

Conversely, if $\ker \phi = \{ 1_G \}$, for any $a, b \in G$ such that $\phi(a) = \phi(b)$, we have $\phi(a^{-1}b) \in \ker \phi = \{ 1_G \}$ so $a^{-1}b = 1_G$, $a = b$, hence $\phi$ is injective. \qed

Example 8. $\det : GL_n(\mathbb{R}) \rightarrow \mathbb{R}^\times$ is a homomorphism, and
$$\ker \det = \{ A \in GL_n(\mathbb{R}) | \det A = 1 \} = SL_n(\mathbb{R})$$

An important observation regarding the kernel of a homomorphism $\phi : G \rightarrow G'$ is that if $g \in \ker \phi$, then for any $x \in G$, $xgx^{-1} \in \ker \phi$:
$$\phi(xgx^{-1}) = \phi(x)\phi(g)\phi(x)^{-1} = \phi(x).1_{G'}.\phi(x)^{-1} = 1_{G'}$$

This observation leads to the following concept:

Definition 9. $G$ is a group. The **conjugate** of $g \in G$ by $x \in G$ is the element $xgx^{-1} \in G$. We say $g$ and $xgx^{-1}$ are conjugate elements.

Definition 10. A subgroup $N$ of $G$ is called a **normal subgroup** if $\forall n \in N, \forall x \in G \Rightarrow xgx^{-1} \in N$.  

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Example 11. If $G$ is an abelian group, then any subgroup of $G$ is a normal subgroup of $G$.

This definition describes our previous observation of kernels:

Proposition 12. The kernel of a homomorphism is a normal subgroup of the domain group.

Example 13. We have shown that $SL_n(\mathbb{R}) = \ker \det$, so by the above proposition, $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

Definition 14. The centre of a group $G$ is the subset

\[ Z(G) = \{ g \in G | gx = xg \text{ for any } x \in G \} \]

Exercise 15. Prove $G$ is abelian if and only if $Z(G) = G$.

Exercise 16. Prove the centre of $G$, $Z(G)$, is a normal subgroup of $G$.

Definition 17. If $G$ is a group whose only normal subgroups are \{1\} and $G$, we say $G$ is a simple group.

Corollary 18. If $G$ is a simple group, then either $G$ is abelian or $G$ has trivial centre.

## 2 Isomorphisms

Definition 19. An isomorphism is a bijective homomorphism.

Example 20. If $G = \langle x \rangle$ in an infinite cyclic group, then

\[ \phi : \mathbb{Z} \rightarrow G \]

\[ k \mapsto x^k \]

is an isomorphism.

Proposition 21. If $\phi : G \rightarrow G'$ is an isomorphism, then $\phi^{-1}$ is also an isomorphism.
Proof. We need to show the inverse function \( \phi^{-1} : G' \rightarrow G \) is also a homomorphism.

For any \( x, y \in G' \), let \( a = \phi^{-1}(x) \) and \( b = \phi^{-1}(y) \). This means \( x = \phi(a) \) and \( y = \phi(b) \). Since \( \phi \) is a homomorphism, \( xy = \phi(a)\phi(b) = \phi(ab) \), we get

\[
\phi^{-1}(xy) = ab = \phi^{-1}(x)\phi^{-1}(y)
\]

\( \square \)

Definition 22. Two groups \( G \) and \( G' \) are called isomorphic if there exists an isomorphism \( \phi : G \rightarrow G' \), and we write \( G \cong G' \).

Intuitively, two groups are isomorphic means they have the same algebraic structures, that is, they will share all the algebraic properties. We can interpret an isomorphism as ”a change of name” for the elements in the group.

Definition 23. All the groups isomorphic to a given group \( G \) form the isomorphic class of \( G \). When we classify groups we will classify them up to isomorphism classes.

Definition 24. An isomorphism \( \phi : G \rightarrow G \) of a group to itself is called an automorphism of \( G \).

Example 25.

\[
\phi : \mathbb{Z} \rightarrow \mathbb{Z}
\]

\[
k \mapsto -k
\]

Example 26. \( G \) is a group. \( g \in G \). Then there is an automorphism of \( G \) given by conjugation:

\[
\phi_g : G \rightarrow G
\]

\[
x \mapsto gxg^{-1}
\]

Exercise 27. Verify \( \phi_g \) in the above example is an automorphism.

Definition 28. The set of all automorphisms of \( G \) with the law of composition to be composition of functions form a group, called the group of automorphisms of \( G \), denoted by \( \text{Aut}(G) \).

Example 29. We will prove in homework that \( \text{Aut}(\mathbb{Z}) \) is isomorphic to a cyclic group of order 2.
Definition 30. The **inner automorphism group** of a group $G$ is the subgroup

$$\text{Inn}(G) = \{\phi_g \in \text{Aut}(G) | g \in G\}$$

where $\phi_g$ is defined in Example 26.

Exercise 31. Verify $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$.

Exercise 32. Prove $f : G \to \text{Aut}(G)$ defined by $f(g) = \phi_g$ defines a group homomorphism, and the kernel of $f$ is $Z(G)$. 