1. $G = GL_2(\mathbb{R})$. $G$ acts on itself by conjugation. Compute the orbit and the stabilizer of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

2. $M_2$ is the isometry group of the plane. $X$ is the set of all the squares with edge length 2 on the plane. $M_2$ acts on $X$ by sending each square to the image of the isometry. Determine the stabilizer of the square with vertices $(\pm 1, \pm 1)$.

3. $G$ is a group and $H$ is a subgroup of $G$. $G$ acts on the set of left cosets $G/H$ by $g.xH = (gx)H$.
   (i). Prove this is a well-defined group action.
   (ii). Is this action transitive?
   (iii). What is the stabilizer of $xH$?
   (iv). Give a proof of the Lagrange Theorem using Counting Formula.

4. If a group $G$ acts on a set $X$, we define the kernel of the group action to be $K = \{ g \in G | \forall x \in X, g.x = x \}$.
   (i). Prove that $K$ is a normal subgroup of $G$.
   (ii). We say an action is faithful if for any $g \neq g'$ in $G$, there exists $x \in X$ such that $g.x \neq g'.x$. Prove an action is faithful if and only if its kernel is trivial.
   (iii). Prove that there is a well-defined faithful induced group action of $G/K$ on $S$ by $(gK).s = g.s$.

5. $G$ is a finite group acting on a finite set $S$. For each $g \in G$, define the set $S^g = \{ s \in S | g.s = s \}$.
   (i). Prove $\sum_{s \in S} |G_s| = \sum_{g \in G} |S^g|$.
   (ii). Prove $\sum_{s \in S} |G_s| = |G| \times n$, where $n$ is the number of orbits in $S$.

6. $G$ is a group acting on a set $X$. $S$ is a set. Let $M(X, S)$ to be the set of all functions $X \rightarrow S$. Prove $(g.f)(x) = f(g^{-1}.x)$ defines a group action of $G$ on $M(X, S)$.

7. $H$ is a subgroup of $G$. Define $N = N(H) = \{ g \in G | gHg^{-1} = H \}$ to be the normalizer of $H$.
   (i). Prove $N$ is a subgroup of $G$.
   (ii). Prove $H$ is a normal subgroup of $N$.
   (iii). Prove $H$ is a normal subgroup of $G$ if and only if $N = G$. 

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