1. Prove
\[ \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \]
\[(k, n) \mapsto k \cdot n = (-1)^k \cdot n \]
defines a group action of the group \( \mathbb{Z} \) on itself, and find the stabilizer of \( n \in \mathbb{Z} \).

2. \( G \) is a group acting on a set \( S \). \( g \in G \) and \( s \in S \). Prove
\[ G_{g,s} = gG_s g^{-1} \]

3. \( G \) is a finite group acting on a finite set \( S \). For each \( g \in G \), define the set \( S^g = \{ s \in S | g \cdot s = s \} \).
   (i). Prove \( \sum_{s \in S} |G_s| = \sum_{g \in G} |S^g| \).
   (ii). Prove \( \sum_{s \in S} |G_s| = |G| \times n \), where \( n \) is the number of orbits in \( S \).

4. \( G \) is a group, \( H \) and \( K \) are normal subgroups of \( G \), \( G = HK \).
   (i). Prove \( G \times (G/H \times G/K) \to G/H \times G/K \) given by
   \[ g.(xH, yK) = (gxH, gyK) \]
is a group action.
   (ii). Compute the stabilizer \( G_{(xH, yK)} \)
   (iii). If \( G \) is a finite group, express the order of \( G/H \times G/K \) in terms of \( |H|, |K|, |H \cap K| \).
   (iv). Use the Counting Formula to prove: if \( G \) is a finite group, the above action is transitive.
   (v). Let \( f : G \to G/H \times G/K \) be the map \( f(g) = (gH, gK) \). Prove \( f \) is surjective if and only if the action in (i) is transitive.
   (vi). Prove \( f \) is surjective without the assumption that \( G \) is finite.
   (So (v) and (vi) together imply that the action in (i) is transitive)

5. \( G \) is a group with \( |G| = p^2 \) for some prime \( p \). Prove either \( G \cong \mathbb{Z}/p^2\mathbb{Z} \) or \( G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \)