1. Define a map
\[ \Psi : M_2 \rightarrow \{ \pm 1 \} \]
\[ t_{\vec{a} \rho \theta}^k \mapsto (-1)^k \]

Prove \( \Psi \) is a well-defined homomorphism.

(Remark: This provides an algebraic way to define the orientation of an isometry. Those corresponding to \(+1\) are called orientation preserving, and those corresponding to \(-1\) are called orientation reversing.)

2. Let \( s \) be the rotation of the plane with angle \( \frac{\pi}{2} \) about the point \((1,1)\). Write the formula for \( s \) as a product \( t_{\vec{a} \rho \theta} \).

3. Let \( s \) be the reflection along the line \( y = x + 1 \) followed by a translation along the vector \( \vec{v} = (1,1) \). Write \( s \) in the form \( t_{\vec{a} \rho \theta} \).

4. Describe in geometry what operation we will get if we first rotate the plane around \((2,3)\) by angle \( \pi \) and then do a reflection along the \( y \)-axis.

5. Find subgroups \( H \) and \( K \) of \( D_4 \) satisfying: \( H \) is a normal subgroup of \( D_4 \), \( K \) is a normal subgroup of \( H \), but \( K \) is not a normal subgroup of \( D_4 \).

(Remark. This exercise shows that a normal subgroup of a normal subgroup of a group \( G \) may not be a normal subgroup of \( G \))

6. \( G \) is a group acting on a set \( X \). \( S \) is a set. Let \( M(X, S) \) to be the set of all functions \( X \rightarrow S \). Prove \((g.f)(x) = f(g^{-1}.x)\) defines a group action of \( G \) on \( M(X, S) \).

7. Given a group action of \( G \) on a set \( X \), we define the kernel of the group action to be \( K = \{ g \in G | \forall x \in X, g.x = x \} \).

(i). Prove that \( K \) is a normal subgroup of \( G \).

(ii). We say an action is **faithful** if for any \( g \neq g' \) in \( G \), there exists \( x \in X \) such that \( g.x \neq g'.x \). Prove an action is faithful if and only if its kernel is trivial.

(iii). Prove that there is a well-defined faithful induced group action of \( G/K \) on \( X \) by \((gK).x = g.x\)