Topologically protected states in one-dimensional continuous systems and Dirac points

Charles L. Fefferman[a,1], **James P. Lee-Thorp**[b], **and Michael I. Weinstein**[b]

[a]Department of Mathematics, Princeton University, Princeton, NJ 08544; and [b]Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027

Contributed by Charles L. Fefferman, April 28, 2014 (sent for review March 22, 2014)

We study a class of periodic Schrödinger operators on $\mathbb{R}$ that have Dirac points. The introduction of an "edge" via adiabatic modulation of a periodic potential by a domain wall results in the bifurcation of spatially localized "edge states," associated with the topologically protected zero-energy mode of an asymptotic one-dimensional Dirac operator. The bound states we construct can be realized as highly robust transverse-magnetic electromagnetic modes for a class of photonic waveguides with a phase defect. Our model captures many aspects of the phenomenon of topologically protected edge states for 2D bulk structures such as the honeycomb structure of graphene.

Floquet–Bloch theory | Hill's equation | surface states | multiple scale analysis | wave-packets

Energy localization in surface modes or edge states at the interface between dissimilar media has been explored, going back to the 1930s, as a vehicle for localization and transport of energy (1–8). These phenomena can be exploited in, for example, quantum, electronic, or photonic device design. An essential property for applications is robustness; the localization properties of such surface states needs to be stable with respect to distortions of or imperfections along the interface.

A class of structures, which has attracted great interest since about 2005, is topological insulators (9, 10). In certain energy ranges, such structures behave as insulators in their bulk (this is associated with an energy gap in the spectrum of the bulk Hamiltonian), but have boundary conducting states with energies in the bulk energy gap; these are states that propagate along the boundary and are localized transverse to the boundary. Some of these states may be topologically protected; they persist under deformations of the interface that preserve the bulk spectral gap, e.g., localized perturbations of the interface. In honeycomb structures, e.g., graphene, where a bulk gap is opened at a "Dirac point" by breaking time-reversal symmetry (10–14), protected edge states are unidirectional. Furthermore, these edge states do not backscatter in the presence of interface perturbations (3–5, 8). Chiral edge states, observed in the quantum Hall effect, are a well known instance of topological protected states in condensed matter physics. In tight-binding models, this property can be understood in terms of topological invariants associated with the band structure of the bulk periodic structure (15–20).

In this article we introduce a one-dimensional continuum model, a Schrödinger equation with periodic potential modulated by a domain wall, in which we rigorously study the bifurcation of topologically protected edge states as a parameter lifts a Dirac point degeneracy (symmetry-protected linear band crossing). This model, which has many of the features of the above examples, is motivated by the study of photonic edge states in honeycomb structures in ref. 3. The bifurcation we study is governed by the existence of a topologically protected zero-energy eigenstate of a one-dimensional Dirac operator, $\mathcal{D}$ (Eq. **5**). The zero mode of this operator plays a role in electronic excitations in coupled scalar–spinor fields (21) and polymer chains (22). There are numerous studies of edge states for tight-binding models (for example, refs. 7, 9, and 18). The present work considers the far less-explored setting of edge states in the underlying partial differential equations (13, 14). A version of this article with detailed rigorous proofs can be found in ref. 23.

Finally, we remark that the topologically protected states we construct can be realized as highly robust transverse-magnetic (TM) electromagnetic guided modes of Maxwell's equations for a class of photonic waveguides.

Floquet–Bloch Theory

Let $Q\in C^\infty$ denote a one-periodic real-valued potential, i.e., $Q(x+1)=Q(x)$, $x\in\mathbb{R}$, and consider the Schrödinger operator

$$H_Q=-\partial_x^2+Q(x).$$

Definition 1: The space of $k-$ pseudoperiodic L^2 functions is given by

$$L^2_k=\{f\in L^2_{\text{loc}}:f(x+1;k)=e^{ik}f(x;k)\}.$$

The Sobolev spaces H^N_k, $N=0,1,\ \ldots$ are analogously defined.

Because the $k-$ pseudoperiodic boundary condition is invariant under $k\mapsto k+2\pi$, it is natural to work with a fundamental dual-period cell or Brillouin zone, which we take to be $\mathcal{B}\equiv[0,2\pi]$.

We next consider a one-parameter family of Floquet–Bloch eigenvalue problems, parameterized by $k\in\mathcal{B}$:

$$H_Q\Phi=E\Phi,\quad \Phi(x+1;k)=e^{ik}\Phi(x;k). \qquad [1]$$

The eigenvalue problem [**1**] is self-adjoint on L^2_k and has a discrete set of eigenvalues, listed with repetitions

$$E_1(k)\le E_2(k)\le\cdots\le E_j(k)\le\cdots$$

with corresponding L^2_k eigenfunctions $\Phi_1(x;k),\ \Phi_2(x;k),\ldots$, which, for fixed k, can be taken to be a complete orthonormal set in

Significance

Topological insulators (TIs) have been a topic of intense study in recent years. When appropriately interfaced with other structures, TIs possess robust edge states, which persist in the presence of localized interface perturbations. Therefore, TIs are ideal for the transfer or storage of energy or information. The prevalent analyses of TIs involve idealized discrete tight-binding models. We present a rigorous study of a class of continuum models, for which we prove the emergence of topologically protected edge states. These states are bifurcations at linear band crossings (Dirac points) of localized modes. The bifurcation is induced by the 0-energy eigenmode of a class of one-dimensional Dirac equations.

Author contributions: C.L.F., J.P.L.-T., and M.I.W. performed research and wrote the paper.

The authors declare no conflict of interest.

[1]To whom correspondence should be addressed. E-mail: cf@math.princeton.edu.

MATHEMATICS

$L^2[0,1]$. Furthermore, the states $\Phi_j(x;k),\ j\geq 1,\ k\in\mathcal{B}$ are complete in $L^2(\mathbb{R})$,

$$f(x)\in L^2(\mathbb{R})\Rightarrow f(x)=\sum_{j\geq 1}\int_{\mathcal{B}}\tilde{f}_j(k)\Phi_j(x;k)dk,$$

where $\tilde{f}_j(k)=\langle\Phi_j(\cdot\,;k),f\rangle_{L^2(\mathbb{R})}$.

We make use of the decompositions $L^2_k=L^2_{k,\mathrm{e}}\oplus L^2_{k,\mathrm{o}}$ and $H^s_k=H^s_{k,\mathrm{e}}\oplus H^s_{k,\mathrm{o}}$, into subspaces defined in terms of even- and odd-index Fourier projections, introduced as follows:

Definition 2:

1) $L^2_{k,\mathrm{e}}$ is the subspace of L^2_k consisting of functions of the form

$$e^{ikx}P_{\mathrm{e}}(x)=e^{ikx}\quad\sum\nolimits_{m\in 2\mathbb{Z}}p(m)e^{2\pi imx},\quad \sum\nolimits_{m\in 2\mathbb{Z}}|p(m)|^2<\infty;$$

i.e., $P_{\mathrm{e}}(x)$ is an even-index $1-$ periodic Fourier series.

2) $L^2_{k,\mathrm{o}}$ is the subspace of L^2_k consisting of functions of the form

$$e^{ikx}P_{\mathrm{o}}(x)=e^{ikx}\quad\sum\nolimits_{m\in 2\mathbb{Z}+1}p(m)e^{2\pi imx},\quad \sum\nolimits_{m\in 2\mathbb{Z}+1}|p(m)|^2<\infty;$$

i.e., $P_{\mathrm{o}}(x)$ is an odd-index $1-$ periodic Fourier series.

3) Sobolev spaces, $H^M_{k,\mathrm{e}}$ and $H^M_{k,\mathrm{o}}$, $M=0,\ 1,\ 2,\ldots,$ are defined in the natural way.

Motivating Example

Start with a smooth, real-valued, even, and $1-$ periodic function, $Q(x)$. Define the one-parameter family of potentials $\mathcal{Q}(x;s)$, a sum of translates of $Q(x)$:

$$\mathcal{Q}(x;s)=Q\left(x+\frac{s}{2}\right)+Q\left(x-\frac{s}{2}\right),\quad 0\leq s\leq 1.$$

Clearly, $\mathcal{Q}(x;s)$ is $1-$ periodic. Moreover, $\mathcal{Q}(x;s=1/2)$ has minimal period equal to $1/2$. That is, $\mathcal{Q}(x;s)$ has an additional translation symmetry at $s=1/2$ (Fig. 1). The function $\mathcal{Q}(x;s)$ may be expressed as a Fourier series

$$\mathcal{Q}(x;s)=\sum_{m\in\mathbb{Z}_+}\mathcal{Q}_m\cos(\pi ms)\cos(2\pi mx),$$

which for $s=1/2$ reduces to an even-index cosine series:

$$V_{\mathrm{e}}(x)\equiv\mathcal{Q}(x;1/2)=\sum_{m\in 2\mathbb{Z}_+}\tilde{\mathcal{Q}}_m\cos(2\pi mx).\qquad\textbf{[2]}$$

Dirac Points

The operator $-\partial_x^2+V_{\mathrm{e}}(x)$ can be shown to have distinguished ("symmetry-protected") quasi-momentum/energy pairs, called Dirac points, at which neighboring spectral bands touch and at which dispersion loci cross linearly (Fig. 2).

Theorem 1. *Consider the Schrödinger operator $H=-\partial_x^2+V_{\mathrm{e}}(x)$, where $V_{\mathrm{e}}(x)$ is a generic smooth even-index cosine series. Then, H has Dirac points $(k_*=\pi,E_*)$ in the following sense:*

1) *There exists $b_*\geq 1$ such that $E_*=E_{b_*}(\pi)=E_{b_*+1}(\pi)$.*
2) *E_* is an $L^2_{k_*}$ eigenvalue of multiplicity 2.*
3) *$H^2_{k_*}=H^2_{k_*,\mathrm{e}}\oplus H^2_{k_*,\mathrm{o}}$, where $H:H^2_{k_*,\mathrm{e}}\to L^2_{k_*,\mathrm{e}}$ and $H:H^2_{k_*,\mathrm{o}}\to L^2_{k_*,\mathrm{o}}$.*
4) *The inversion $\mathcal{S}[f](x)=f(-x)$, where $\mathcal{S}:H^2_{k_*,\mathrm{e}}\to H^2_{k_*,\mathrm{o}}$ and $\mathcal{S}:H^2_{k_*,\mathrm{o}}\to H^2_{k_*,\mathrm{e}}$, is such that $\mathcal{S}$ commutes with $H(k_*)=e^{-ik_*x}He^{ik_*x}$; i.e., $[H(k_*),\mathcal{S}]\equiv H(k_*)\mathcal{S}-\mathcal{S}H(k_*)$ vanishes.*
5) *The $L^2_{k_*}-$ nullspace of $H-E_*I$ is spanned by functions $\Phi_1\in L^2_{k_*,\mathrm{e}}$, $\Phi_2\equiv\mathcal{S}[\Phi_1](x)\in L^2_{k_*,\mathrm{o}}$, $\langle\Phi_a,\Phi_b\rangle_{L^2[0,1]}=\delta_{ab},\ \ a,b=1,2.$*
6) *The "Fermi velocity" satisfies*

$$\lambda_\sharp=2i\langle\Phi_1,\partial_x\Phi_2\rangle_{L^2[0,1]}\neq 0\qquad\textbf{[3]}$$

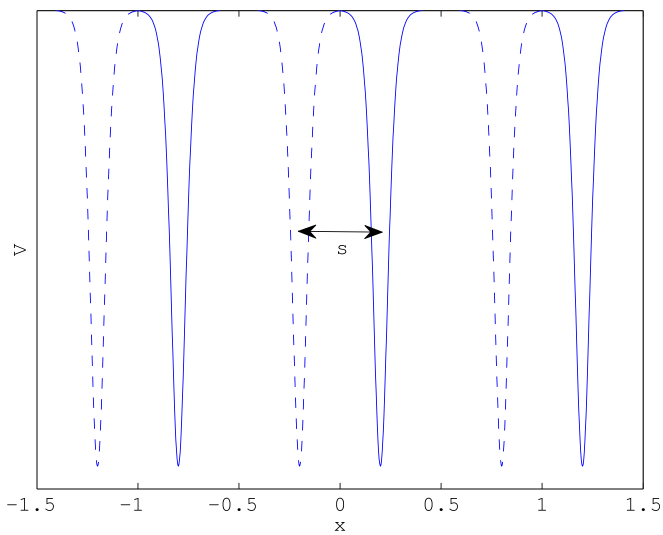

Fig. 1. Dimer periodic potential: $\mathcal{Q}(x;s)=Q(x+s/2)+Q(x-s/2)$. For $0<s<1/2$, the potential $\mathcal{Q}(x;s)$ has minimal period 1 and the period cell contains a double well. For $s=1/2$, $\mathcal{Q}(x;1/2)$ has minimal period $1/2$.

(for generic V_{e}), and there exist $\zeta_0>0$ and Floquet–Bloch eigenpairs

$$(\Phi_+(x;k),E_+(k))\quad\text{and}\quad(\Phi_-(x;k),E_-(k))$$

and smooth functions $\eta_\pm(k)$, with $\eta_\pm(0)=0$, defined for $|k-k_*|<\zeta_0$ and such that $\Phi_\pm(x;k)$ have the same span as the functions $\Phi_j(x;k),j=1,2$ and

$$E_\pm(k)-E_*=\pm\lambda_\sharp(k-k_*)\left(1+\eta_\pm(k-k_*)\right).$$

Fig. 2, *Left* illustrates the existence of Dirac points for a periodic potential, V_{e} of the type plotted in Fig. 3, *Top*. Superimposed on V_{e} is the mode $\Phi_1\in L^2_{k_*,\mathrm{e}}$.

In ref. 24, Dirac points play a central role in the construction of one-dimensional almost periodic potentials for which the Schrödinger operator has nowhere a dense spectrum.

Dimers

For $s\neq 1/2$, $\mathcal{Q}(x;s)$ is a periodic potential consisting of dimers, double-well potentials in each period cell (Fig. 1). Set $s^\delta=1/2+\delta\kappa_0$, with $0<\delta\ll 1$ and $0\neq\kappa_0\in\mathbb{R}$ fixed. Then, $\mathcal{Q}(x;s^\delta)$ is of the form

$$\mathcal{Q}(x;s^\delta)=\sum_{m\in 2\mathbb{Z}_+}\mathcal{Q}^\delta_m\cos(2\pi mx)+\delta\kappa_0\sum_{m\in 2\mathbb{Z}_++1}\mathcal{W}^\delta_m\cos(2\pi mx),$$

where $\mathcal{Q}^\delta_m,\mathcal{W}^\delta_m=\mathcal{O}(1)$ as $\delta\to 0$. The operator

$$H\left(\frac{1}{2}+\delta\kappa_0\right)=-\partial_x^2+\mathcal{Q}(x;s^\delta)$$

is a Hill's operator (25). The character of its spectrum is well known (Fig. 4, *Upper*). For δ fixed, the gaps that open at the Dirac points have widths of order $\mathcal{O}(\delta)$.

Now fix a constant $\kappa_\infty>0$ and let $s^\delta(x)=1/2+\delta\kappa(\delta x)$, where $\kappa(X)$ is the domain wall function,

$$\kappa(X)\to\pm\kappa_\infty\quad\text{as}\quad X\to\pm\infty.$$

We assume that $\kappa(X)$ is sufficiently smooth and approaches its asymptotic values sufficiently rapidly. The operator $H(s^\delta(x))=$

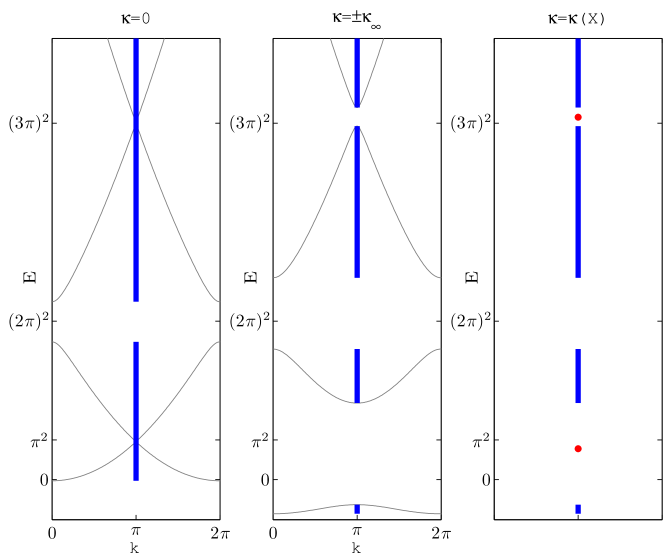

Fig. 2. Spectra (blue) for $H_{\delta=15} = -\partial_x^2 + 10\cos(2(2\pi x)) + 30\kappa(X)\cos(2\pi x)$ for different choices of $\kappa(X)$. (*Left*) $\kappa(X) \equiv 0$. Plotted are the first four Floquet–Bloch dispersion curves $k \mapsto E_b(k)$ (gray) and the Dirac points $(k_\star = \pi, E_\star)$ approximately at $E_\star = \pi^2, (3\pi)^2$. (*Center*) $\kappa(X) \equiv \pm\kappa_\infty$, a nonzero constant. Shown are open gaps about Dirac points of the unperturbed potential and smooth dispersion curves (gray). (*Right*) Midgap eigenvalues (red) are shown for periodic potential modulated by domain wall: $\kappa(X) = \tanh(X)$.

$-\partial_x^2 + \mathcal{Q}(x; s^\delta(x))$ is the Hamiltonian for a structure that adiabatically transitions between dimer periodic structures at $\pm\infty$ through a domain wall. For δ small,

$$\mathcal{Q}\big(x; s^\delta(x)\big) \approx V_e(x) + \delta\kappa(\delta x) W_o(x),$$

where $V_e(x)$ denotes an even-index cosine series, a structure with a Dirac point, and $W_o(x)$, an odd-index cosine series, is a noncompact perturbation induced by the domain wall (phase defect).

This motivates our study of the family of operators:

$$H_\delta = -\partial_x^2 + V_e(x) + \delta\kappa(\delta x) W_o(x).$$

The operator H_δ interpolates adiabatically, through a domain wall, between the operators

$$H\left(\frac{1}{2} - \kappa_\infty\delta\right) \approx H_{\delta,-} \equiv H\left(\frac{1}{2}\right) - \delta\kappa_\infty W_o \quad \text{at } x = -\infty,$$

$$H\left(\frac{1}{2} + \kappa_\infty\delta\right) \approx H_{\delta,+} \equiv H\left(\frac{1}{2}\right) + \delta\kappa_\infty W_o \quad \text{at } x = +\infty.$$

The noncompact perturbations $\delta\kappa(\delta x) W_o(x)$ and $\pm\delta\kappa_\infty W_o(x)$, which break the symmetries of $V_e(x)$, open a gap in the essential spectrum.

Proposition 2. *Let (k_*, E_*) denote a Dirac point of $-\partial_x^2 + V_e(x)$. Assume the condition $\vartheta_\sharp = \langle \Phi_1, W_o \Phi_2 \rangle_{L^2[0,1]} \neq 0$, which holds for generic W_o and V_e. Fix c less than but arbitrarily close to 1 and define the real interval*

$$\mathcal{I}_\delta \equiv \big(E_* - c\delta\kappa_\infty |\vartheta_\sharp|, \quad E_* + c\delta\kappa_\infty |\vartheta_\sharp|\big).$$

Then, there exists $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$,

$$\mathcal{I}_\delta \cap \sigma_{ess}(H_\delta), \quad \mathcal{I}_\delta \cap \sigma_{ess}(H_{\delta,-}), \quad \textit{and} \quad \mathcal{I}_\delta \cap \sigma_{ess}(H_{\delta,+})$$

are all empty sets.

Fig. 2, *Center* demonstrates the opening of gaps at the Dirac points in the essential spectrum for the perturbations $\pm\delta\kappa_\infty W_o(x)$.

Topologically Protected States

Our main result is the following:

Theorem 3. *Consider the eigenvalue problem for the Schrödinger operator H_δ. Let $(k_* = \pi, E_*)$ denote a Dirac point of $-\partial_x^2 + V_e(x)$ in the sense of Theorem 1. Furthermore, assume the condition*

$$\vartheta_\sharp = \langle \Phi_1, W_o \Phi_2 \rangle_{L^2([0,1])} \neq 0, \qquad [4]$$

which holds for generic V_e and W_o. *Assume that $\kappa(X) \to \pm\kappa_\infty$ sufficiently rapidly as $X \to \pm\infty$. Then, the following holds:*

1) *There exists $\delta_0 > 0$ and a branch of solutions to the eigenvalue problem $H_\delta \Psi^\delta = E^\delta \Psi^\delta$,*

$$\delta \mapsto \big(E^\delta, \Psi^\delta\big), \quad 0 < \delta < \delta_0,$$

bifurcating from energy E_* at $\delta = 0$, into the gap $\mathcal{I}_\delta$ (Proposition 2), with $|E^\delta - E_*| \lesssim \delta^2$, and with corresponding spatially localized (exponentially decaying) eigenstate, Ψ^δ.

2) *$\Psi^\delta(x)$ is approximated by a slowly varying and spatially decaying modulation of the degenerate Floquet–Bloch modes Φ_1 and Φ_2:*

$$\left\| \Psi^\delta(x) - \delta^{1/2}\left[\alpha_{*,1}(\delta x)\Phi_1(x) + \alpha_{*,2}(\delta x)\Phi_1(x)\right] \right\|_{H^1(\mathbb{R}_x)} \lesssim \delta.$$

3) *The amplitudes, $\alpha_*(X) = (\alpha_{*,1}(X), \alpha_{*,2}(X))$, are governed by the topologically protected zero eigenmode of the 2×2 matrix Dirac operator,*

$$\mathcal{D} \equiv i\lambda_\sharp \sigma_3 \partial_X + \vartheta_\sharp \kappa(X) \sigma_1, \qquad [5]$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *and* $\lambda_\sharp \vartheta_\sharp \neq 0$ (Eqs. **3** and **4**).

Theorem 3 is illustrated in Figs. 2–4. Fig. 2, *Right* displays the midgap eigenvalues, E^δ. Fig. 3 displays the unperturbed

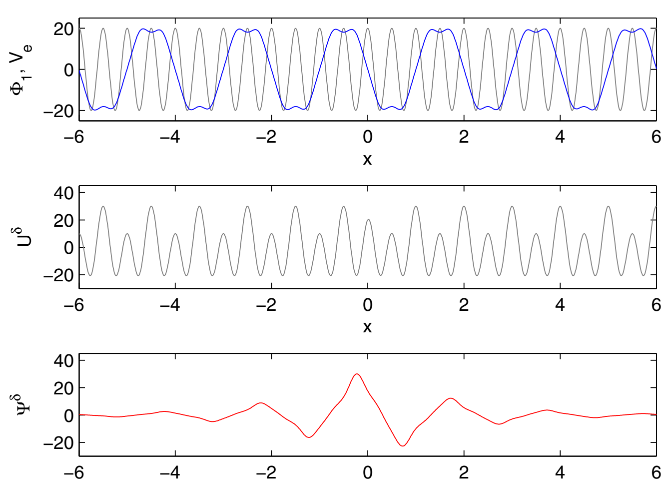

Fig. 3. Potentials and modes for $H^\delta = -\partial_x^2 + U_\delta(x)$, where $U_\delta(x) = 20\cos(2(2\pi x)) + 2\delta\tanh(\delta x)\cos(2(2\pi x))$. (*Top*) Floquet–Bloch eigenmode $\Phi_1(x) \in L^2_{\pi,e}$ (blue) corresponding to the degenerate eigenvalue $E_\star$ at the Dirac point of unperturbed Hamiltonian, $H_{\delta=0}$, superimposed on plot of $U_0(x)$. Here, $\Phi_2(x) = \Phi_1(-x) \in L^2_{\pi,o}$. (*Middle*) Domain-wall modulated periodic structure, $U_\delta(x)$, $\delta = 5$. (*Bottom*) Localized midgap eigenmode $\Psi^\delta(x)$ of H_δ.

Assume the following conditions:

I) *E_* is a simple $L^2_{k_*,\mathrm{e}}$ eigenvalue of H with one-dimensional eigenspace*

$$span\{\Phi_1(x)\} \subset L^2_{k_*,\mathrm{e}}.$$

II) *E_* is a simple $L^2_{k_*,\mathrm{o}}$ eigenvalue of H with one-dimensional eigenspace*

$$span\{\Phi_2(x) = \Phi_1(-x)\} \subset L^2_{k_*,\mathrm{o}}.$$

III) *Nondegeneracy condition*

$$0 \neq \lambda_\sharp \equiv 2i\langle \Phi_1, \partial_x \Phi_1 \rangle = -2\pi \left\{ 2 \sum_{m \in 2\mathbb{Z}} m |c_1(m)|^2 + 1 \right\}. \qquad \textbf{[6]}$$

Here, $\{c_1(m)\}_{m\in\mathbb{Z}}$ denote the $L^2_{k_*,\mathrm{e}}-$ Fourier coefficients of $\Phi_1(x)$. Then, (k_*, E_*) is a Dirac point in the sense of Theorem 1.

The proof follows that of theorem 4.1 of ref. 13 for the case of honeycomb lattice potentials in $\mathbb{R}^2$. To establish *Theorem 1*, that $-\partial_x^2 + V_\mathrm{e}$ has Dirac points for generic V_e, we consider the family of operators $H^{(\varepsilon)} = -\partial_x^2 + \varepsilon V_\mathrm{e}$. The conditions of *Theorem 4* can be verified for all $\varepsilon \in [0, \varepsilon_0)$, for some $\varepsilon_0(E_*) > 0$ by a perturbative/Lyapunov–Schmidt reduction argument about quasi-momentum energy pairs for $\varepsilon = 0$: $(k_*, E_{m,*}) = (\pi, (2m+1)^2\pi^2)$, $m = 0, 1, \ldots$ A continuation argument to ε of arbitrary size is implemented following the strategy of ref. 13. This shows the persistence of the conditions of *Theorem 4* and therefore the existence of a Dirac point, $(k_*, E_*^{(\varepsilon)})$, for all $\varepsilon \notin \tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}}$ denotes a countable and closed subset of $\mathbb{R}$.

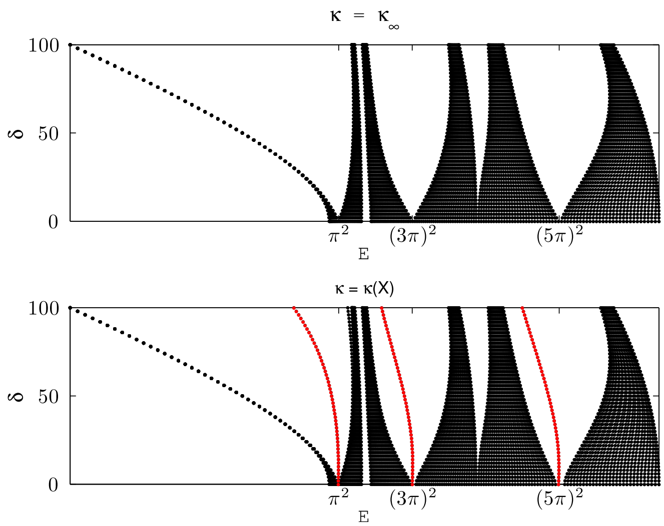

Fig. 4. Energy spectra (E) of H_δ for different choices of $\kappa(X)$ over a range of $\delta \geq 0$. (*Upper*) Spectrum of Hill's operator: $-\partial_x^2 + 10\cos(2(2\pi x)) + 2\delta\kappa_\infty \sum_{j=1,3,5}\cos(2\pi jx)$. For $\delta \neq 0$, spectral gaps of width $\mathcal{O}(\delta)$ open about Dirac points of H_0, approximately located at $\pi^2, (3\pi)^2, (5\pi)^2, \ldots$. (*Lower*) Spectrum of the domain-wall modulated periodic potential: $-\partial_x^2 + 10\cos(2(2\pi x)) + 2\delta\kappa(\delta x)\sum_{j=1,3,5}\cos(2\pi jx)$ with $\kappa(X) = \tanh(X)$. Solid (red) curves are the first three bifurcation curves of topologically protected states.

potential, $V_\mathrm{e}(x)$, and the mode $\Phi_1(x)$ (Fig. 3, *Top*), the domain-wall modulated potential $V_\mathrm{e}(x) + \delta\kappa(\delta x)W_\mathrm{o}(x)$ (Fig. 3, *Middle*), and the midgap mode $\Psi^\delta(x)$ (Fig. 3, *Bottom*). Fig. 4, *Lower* displays the bifurcation curves of topologically protected states from the first several Dirac points of $-\partial_x^2 + V_\mathrm{e}(x)$.

Remark 1:

1) The zero mode of a one-dimensional Dirac operator, $\mathcal{D}$, plays an important role in refs. 3, 21, and 22.

2) The bifurcation discussed in *Theorem 3* is associated with a noncompact perturbation of $H_0 = -\partial_x^2 + V_\mathrm{e}(x)$, namely a phase defect across the structure, which at once changes the essential spectrum and spawns a bound state. This is in contrast to bifurcations from the edge of continuous spectra, arising from localized perturbations (for example, refs. 26–31). A class of edge bifurcations due to a noncompact perturbation is studied in ref. 32.

3) Our model captures many aspects of the phenomenon of topologically protected edge states for 2D bulk structures such as the honeycomb structure of graphene (for example, refs. 3 and 13).

Proof of Theorem 1

To establish that (k_*, E_*) is a Dirac point, we verify the sufficient conditions of the following:

Theorem 4. *Consider $H = -\partial_x^2 + V_\mathrm{e}$, where V_e, given by Eq.* **2**, *is sufficiently smooth. Let $k_* = \pi$ and assume that E_* is a double eigenvalue, lying at the intersection of the b_*^{th} and $(b_*+1)^{\mathrm{st}}$ spectral bands:*

$$E_* = E_{b_*}(k_*) = E_{b_*+1}(k_*).$$

Proof of Theorem 3

A formal multiple-scale expansion anticipates the form of the solution at any finite order in δ. To establish the validity of this expansion, the corrector is decomposed into its "near-energy" and "far-energy" components, using the spectral decomposition of the unperturbed operator: $-\partial_x^2 + V_\mathrm{e}$. The near-energy regime corresponds to energies within the intersecting spectral bands, which are near E_* and quasi-momentum, k, satisfying $|k - k_*| \leq \delta^\tau$, where $0 < \tau < 1$. The far-energy components correspond to all other energies (within the intersecting bands and all other bands). The eigenvalue problem is equivalent to a coupled infinite system for the near- and far-energy spectral components. To solve this system we first solve for the far-energy components as a functional of the near-energy components. This leads to a (Lyapunov–Schmidt) reduction to a closed nonlocal system, which determines the near-energy components of the corrector and the correction to the degenerate eigenvalue E_*. Under rescaling, the latter equation may be written as a Dirac-type equation with Dirac operator $\mathcal{D}$ (Eq. **5**) band limited to rescaled momenta $|\xi| \leq \delta^{\tau-1}$. We then solve this system for all δ positive and sufficiently small.

ACKNOWLEDGMENTS. The authors thank P. Deift, J. B. Keller, J. Lu, A. Millis, M. Rechtsman, M. Segev, and C. W. Wong for stimulating discussions. This work was supported in part by National Science Foundation (NSF) Grants DMS-1265524 (to C.L.F.) and DMS-10-08855 and Columbia University Optics and Quantum Electronics Integrative Graduate Education and Research Traineeship (IGERT) NSF Grant DGE-1069420 (to M.I.W. and J.P.L.-T.).

1. Tamm IE (1932) A possible kind of electron binding on crystal surfaces. *Phys Z Sowjetunion* 1:733–735.
2. Shockley W (1939) On the surface states associated with a periodic potential. *Phys Rev* 56:317.
3. Raghu S, Haldane FDM (2008) Analogs of quantum-Hall-effect edge states in photonic crystals. *Phys Rev A* 78:033834.
4. Yu Z, Veronis G, Wang Z, Fan S (2008) One-way electromagnetic waveguide formed at the interface between a plasmonic metal under a static magnetic field and a photonic crystal. *Phys Rev Lett* 100(2):023902.
5. Wang Z, Chong YD, Joannopoulos JD, Soljačić M (2008) Reflection-free one-way edge modes in a gyromagnetic photonic crystal. *Phys Rev Lett* 100(1):013905.
6. Malkova N, Hromada I, Wang X, Bryant G, Chen Z (2009) Observation of optical Shockley-like surface states in photonic superlattices. *Opt Lett* 34(11):1633–1635.
7. Rechtsman MC, et al. (2013) Topological creation and destruction of edge states in photonic graphene. *Phys Rev Lett* 111:103901.
8. Rechtsman MC, et al. (2013) Photonic Floquet topological insulators. *Nature* 496(7444):196–200.

9. Kane CL, Mele EJ (2005) $\mathbb{Z}_2$ topological order and the quantum spin Hall effect. *Phys Rev Lett* 95(14):146802.
10. Hasan MZ, Kane CL (2010) Colloquium: Topological insulators. *Rev Mod Phys* 82:3045.
11. Castro Neto AH, Guinea F, Peres NMR, Novoselov KS, Geim AK (2009) The electronic properties of graphene. *Rev Mod Phys* 81:109–162.
12. Katsnelson M (2012) *Graphene: Carbon in Two Dimensions* (Cambridge Univ Press, Cambridge, UK).
13. Fefferman CL, Weinstein MI (2012) Honeycomb lattice potentials and Dirac points. *J Am Math Soc* 25:1169–1220.
14. Fefferman CL, Weinstein MI (2014) Wave packets in honeycomb lattice structures and two-dimensional Dirac equations. *Commun Math Phys* 326:251–286.
15. Wen X-G (1991) Gapless boundary excitations in the quantum Hall states and in the chiral spin states. *Phys Rev B Condens Matter* 43(13):11025–11036.
16. Thouless DJ, Kohmoto M, Nightingale MP, Den Nijs M (1982) Quantized hall conductance in a two-dimensional periodic potential. *Phys Rev Lett* 49:405.
17. Halperin BI (1982) Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. *Phys Rev B* 25:2185–2190.
18. Hatsugai Y (1993) Chern number and edge states in the integer quantum Hall effect. *Phys Rev Lett* 71(22):3697–3700.
19. Graf J-M, Porta M (2012) Bulk-edge correspondence for two-dimensional topological insulators. *Commun Math Phys* 324:851–895.
20. Avila JC, Schulz-Baldes H, Villegas-Blas C (2013) Topological invariants of edge states for periodic two-dimensional models. *Math Phys Anal Geom* 16:136–170.
21. Jackiw R, Rebbi C (1976) Solitons with Fermion number 1/2. *Phys Rev D Part Fields* 13:3398–3409.
22. Su WP, Schrieffer JR, Heeger AJ (1979) Solitons in polyacetylene. *Phys Rev Lett* 42: 1698–1701.
23. Fefferman CL, Lee-Thorp JP, Weinstein MI (2014) Topologically protected states in one-dimensional systems. arXiv: 1405.4569.
24. Moser J (1981) An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum. *Comment Math Helvetici* 56:198–224.
25. Magnus W, Winkler S (1979) *Hill's Equation*. (Dover, New York).
26. Simon B (1976) The bound state of weakly coupled Schrödinger operators in one and two dimensions. *Ann Phys* 97:279–288.
27. Deift PA, Hempel R (1986) On the existence of eigenvalues of the Schrödinger operator h-λw in a gap of σ(h). *Commun Math Phys* 103:461–490.
28. Figotin A, Klein A (1997) Localized classical waves created by defects. *J Stat Phys* 86(1–2): 165–177.
29. Borisov DI, Gadyl'shin RR (2008) On the spectrum of a periodic operator with a small localized perturbation. *Izv Math* 72:659–688.
30. Duchêne V, Vukićević I, Weinstein MI (2014) Scattering and localization properties of highly oscillatory potentials. *Commun Pure Appl Math* 67(1):83–128.
31. Duchêne V, Vukićević I, Weinstein MI Homogenized description of defect modes in periodic structures with localized defects. *Commun Math Sci*, in press.
32. Bronski J, Rapti Z (2011) Counting defect modes in periodic eigenvalue problems. *SIAM J Math Anal* 43:803–827.

MATHEMATICS