On the equivalence of Tu’s and G&P notions of manifolds, smooth maps, tangent spaces, differentials, etc.

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January 31, 2021

Here we will demonstrate the equivalence of the different notions of the above concepts as given in two standard textbooks: namely Loring Tu’s *An Introduction to Smooth Manifolds* as well as Alan Pollack and Victor Guillemin’s *Differential Topology*. Note we don’t necessarily have equality between these structures: the different definitions provide distinct structures. On the other hand, we do have equivalence: the distinct structures are isomorphic in the appropriate way. Let’s start with smooth manifolds. Note that G&P do not provide a notion of a topological manifold, so we must start with smooth manifolds and smooth maps. First, Tu’s definitions:

**Definition.** A *Tu smooth manifold* is a triple \((X, \mathcal{T}, \mathcal{A})\), where \(X\) is a set, \(\mathcal{T}\) is a locally Euclidean, Hausdorff, and second countable topology on \(X\), and \(\mathcal{A}\) is a maximal atlas. To define a maximal atlas, we must first define a chart. A *chart* is a pair \((U, \phi)\) where \(U\) is a nonempty open subset of \(X\) and \(\phi: U \to \mathbb{R}^n\) is a homeomorphism onto its image, where we require \(\phi(U)\) is an open subset of \(\mathbb{R}^n\). We say two charts \((U, \phi), (V, \psi)\) are *smoothly compatible* if \(\phi \circ \psi^{-1}: \psi(U \cap V) \to \phi(U \cap V)\) and \(\psi \circ \phi^{-1}: \phi(U \cap V) \to \psi(U \cap V)\) are \(C^\infty\) in the standard Euclidean sense (i.e. partial derivatives of all orders exist and are continuous). Note it makes sense to talk about these maps being \(C^\infty\) (also called smooth) as \(\phi(U \cap V), \phi(U \cap V)\) are easily verified to be open. An atlas is then collection of pairwise smoothly compatible charts \(\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}\) for some index set \(A\) s.t. \(\bigcup_{\alpha \in A} U_\alpha = M\). A maximal atlas is an atlas that is not properly contained in any other atlas.

Now that a Tu smooth manifold has been defined, we can define a Tu smooth map. Let \(M, N\) be Tu smooth manifolds. Then a continuous function \(f: M \to N\) is called *Tu smooth at* \(x\) if there is some chart \((U, \phi)\) in the atlas of \(M\) s.t. \(x \in U\) and some chart \((V, \psi)\) in the atlas of \(N\) s.t. \(f(x) \in V\) and \(\psi \circ f \circ \phi^{-1}: \phi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)\) is a smooth map in the standard Euclidean sense (which makes sense as \(\phi(U \cap f^{-1}(V))\) is an open subset of a Euclidean space). The function \(f\) is then called smooth if is smooth at all \(x \in M\). Then \(f\) is a *Tu diffeomorphism* if \(f\) is smooth bijection with a smooth inverse.

Note that if \(M\) is a Tu smooth manifold with atlas \(\mathcal{A}\) and \(U \subseteq M\) is an open set, there is a natural way to consider \(U\) as a Tu smooth manifold. Let \(U\) inherit the subspace topology and define an atlas on \(U\): \(\{(V \cap U, \phi|_{V \cap U}) | (V, \phi) \in \mathcal{A}\}\). Then if \(N\) is another Tu smooth
manifold, and \( f : U \to N \) is a continuous function. We can say that it is Tu smooth iff it is Tu smooth as map between \( U \) and \( N \) where \( U \) is given manifold structure in the above described way.

Also note that \( \mathbb{R}^n \) can be given manifold structure by taking \((\mathbb{R}^n, id_{\mathbb{R}^n})\) as an atlas. It is then easy to verify that a map \( f : \mathbb{R}^n \to \mathbb{R}^m \) is smooth as a map between manifold iff it is Euclidean smooth. Now that \( \mathbb{R}^n \) has manifold structure, we can state the following:

**Proposition.** If \( M \) is a Tu smooth manifold and \((U, \phi)\) is a chart, then \( \phi : U \to \mathbb{R}^k \) is a Tu diffeomorphism onto its image.

**Proof.** Note that \( \phi \) is by definition a homeomorphism, thus we only need to check \( \phi, \phi^{-1} \) are smooth. Let \( x \in U \). Take as a chart around \( x \), \((U, \phi)\). Take as a chart around \( \phi(x) \in \phi(U) \), \((\phi(U), id_{\phi(U)})\). Then \( id_{\phi(U)} \circ \phi \circ \phi^{-1} = id_{\phi(U)} \) is Euclidean smooth, so \( \phi \) is smooth. The proof goes similarly with \( \phi^{-1} \), using the same charts. Then \( \phi^{-1} \) is also smooth, so \( \phi \) is a Tu diffeomorphism onto its image. \( \square \)

It is an easy verification also that the composition of Tu smooth maps is Tu smooth. Another useful fact about Tu smooth maps is the following:

**Proposition.** If \( f : M \to N \) is a Tu smooth map between Tu smooth manifolds, \((U, \phi)\) a chart in the atlas on \( M \), \((V, \psi)\) a chart in the atlas on \( N \), then \( \psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V) \) is Euclidean smooth.

**Proof.** Suppose \( p \in U \cap f^{-1}(V) \). Then by \( f \) Tu smooth, there is a chart \((A, \alpha)\) with \( p \in A \) and \((B, \beta)\) with \( f(p) \in B \) s.t. \( \beta \circ f \circ \alpha^{-1} : \alpha(A \cap f^{-1}(B)) \to \beta(f(A) \cap B) \) is Euclidean smooth. Also \( \psi \circ \beta^{-1} : \beta(B \cap V) \to \psi(B \cap V) \) and \( \alpha \circ \phi^{-1} : \phi(A \cap U) \to \alpha(A \cap U) \) must be Euclidean smooth. Then the composition \( \psi \circ \beta^{-1} \circ \beta \circ f \circ \alpha^{-1} \circ \alpha \circ \phi^{-1} = \psi \circ f \circ \phi^{-1} \) is Euclidean smooth at \( \phi(p) \). Thus \( \psi \circ f \circ \phi^{-1} \) is Euclidean smooth at every point in the domain, hence it is Euclidean smooth. \( \square \)

Now for the G&P definitions:

**Definition.** To define a G&P smooth manifold, we must first define a G&P smooth function. A function \( f : X \subseteq \mathbb{R}^n \to Y \subseteq \mathbb{R}^m \) is called \( \text{G&P smooth} \) if for any \( x \in X \), there exists some open (in \( \mathbb{R}^n \)) neighborhood \( U \) of \( x \) and a function \( F : U \to \mathbb{R}^m \) s.t. \( F|_{U \cap X} = f|_{U \cap X} \) and \( F \) is smooth in the standard Euclidean sense. A function \( f : X \to Y \), where \( X, Y \) are subsets of some Euclidean spaces, is a \( \text{G&P diffeomorphism} \) if it is a smooth bijection with a smooth inverse.

A \( \text{G&P k-dimensional smooth manifold} \) is a set \( X \subseteq \mathbb{R}^N \) for some \( N \in \mathbb{N} \) s.t. for any \( x \in X \) there is open (in \( X \)) neighborhood \( V \) of \( x \) that is G&P diffeomorphic to an open subset of \( \mathbb{R}^k \).

Now we can begin to make comparisons. First it is worth noting that a Tu smooth manifold doesn’t necessarily have fixed dimension. It is a consequence of smooth invariance of domain that connected components of a Tu smooth manifold have a well defined dimension: i.e. all charts with neighborhoods of points in the component will have a codomain of \( \mathbb{R}^n \) for fixed \( n \in \mathbb{N} \). On the other hand, distinct components can have different dimensions: i.e. the disjoint union of \( \mathbb{R} \) with \( \mathbb{R}^2 \) will have two components, one that is 1 dimensional, one that
is 2-dimensional. As such we see Tu smooth manifolds are somewhat more general, but not in a particularly profound way. In order to facilitate our comparison, we will require that every connected component of our Tu smooth manifolds have the same dimension.

The next thing to note is that Tu smooth manifolds have sets which are not necessarily subsets of \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \). As a consequence of the Whitney embedding theorem (well actually this is a “hard” theorem, for the proof of which we may want to have already established the correspondence we are currently presenting. There is an “easier” theorem which gives the result we want, i.e. that every Tu smooth manifold is diffeomorphic to an embedded submanifold, but without the nice numerical bound given by Whitney), every Tu smooth manifold is Tu diffeomorphic to an embedded submanifold of \( \mathbb{R}^n \) for some \( n \). This effectively says every manifold is the same as a certain subset of some Euclidean space; the technical details will be explained shortly. In order to do so, we must make some further definitions:

**Definition.** Let \( M \) be a Tu smooth manifold. Let \( p \in M \). Consider the set: \( S := \{ f : U \to \mathbb{R} \mid p \in U, U \text{ open}, f \text{ is Tu smooth} \} \), note we are treating \( U \) as a Tu smooth manifold in the way described above. Define an equivalence relation \( \sim \) on \( S \) by \( f \sim g \) where \( f : U \to \mathbb{R}, g : V \to \mathbb{R} \in S \) iff there exists some open \( W \subseteq U \cap V \) s.t. \( f|_W = g|_W \). Then let \( C^\infty_p(M) \) be the collection of equivalence classes \( S/ \sim \). We call these equivalence classes the germs of smooth functions at \( p \). We give the germs of smooth functions ring structure by considering normal function addition and multiplication using representatives of the germs. It needs to be verified that this is well defined, but this is straightforward. Adding scalar multiplication by real numbers makes \( C^\infty_p(M) \) into a vector space with a multiplication operation on the vectors, i.e. an algebra. Then a point-derivation of \( C^\infty_p(M) \) is a linear map \( C^\infty_p(M) \to \mathbb{R} \) s.t. \( D(fg) = (Df)g(p) + f(p)Dg \). Let \( T_pM \) denote the collection of point derivations at \( p \). We call this the Tu tangent space of \( M \) at \( p \) and call its elements Tu tangent vectors. It is straightforward to verify that the Tu tangent space is in fact a vector space over \( \mathbb{R} \): linear combinations of point derivations are still point derivations. Note that a tangent vector is entirely determined by its action on germs of smooth functions. Let \( F : M \to N \) be a Tu smooth map between Tu smooth manifolds. Then define the Tu differential of \( F \) (at \( p \)) to be \( F_{*,p} : T_pM \to T_{F(p)}N \) where if \( X_p \in T_pM \), then \( F_{*,p}(X_p)([f]) \), where \( [f] \in C^\infty_{F(p)}(N) \) is the equivalence class with representative \( f \), is given as \( X_p([f \circ F]) \). It requires verification that this is a well-defined map from \( T_pM \to T_{F(p)}N \). Further, it can be shown that this map is linear and that a chain rule applies: if \( M, N, P \) are Tu smooth manifolds, \( F : M \to N \), and \( G : N \to P \), then \( (G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p} \).

Since we have defined the Tu tangent space and Tu differential, we can now define an Tu embedded submanifold.

**Definition.** Let \( f : M \to N \) be a Tu smooth map between Tu smooth manifolds. Let \( p \in M \). Say \( f \) is a Tu immersion at \( p \) if \( f_{*,p} \) is injective. Say \( f \) is a Tu immersion if it is a Tu immersion at \( p \) for every \( p \in M \). Say that \( f \) is a Tu embedding if \( f \) is a topological embedding (i.e. a homeomorphism onto its image) and a Tu immersion. \( N \subseteq M \) is said to be an embedded submanifold if the inclusion map is a Tu embedding.

The above discussion notes that every Tu smooth manifold is diffeomorphic to an embedded submanifold of Euclidean space. As such we need only consider the correspondence...
between Tu smooth manifolds that are embedded submanifolds of Euclidean space and G&P smooth manifolds. First we want a procedure which takes a G&P smooth manifold and gives a Tu smooth manifold, together with an inverse procedure going from Tu smooth manifolds to G&P smooth manifolds.

**Tu Smooth Manifolds from G&P Smooth Manifolds**

Let’s start with a G&P $k$-dimensional smooth manifold $M \subseteq \mathbb{R}^n$ for some $n$. For a Tu smooth manifold, we need a set, a topology, and a maximal atlas. Let’s take the same set and topology as the G&P smooth manifold has. We just need to construct a maximal atlas. It is an easy theorem that every atlas is contained a unique maximal atlas. As such, we just need to construct some atlas. By definition, for each $p$, there is some open (in $M$) neighborhood $U_p$ of $p$ and a map $\phi_p : U_p \to \mathbb{R}^k$ that is a G&P diffeomorphism onto its image, $\phi_p(U_p)$, which is an open subset of $\mathbb{R}^k$. Let $\mathcal{A} = \{(U_p, \phi_p)\}_{p \in M}$. I claim that $\mathcal{A}$ is an atlas. First it is clear that the $U_p$ cover $M$. Then since it is easy to verify G&P diffeomorphisms are homeomorphisms, we see that the $(U_p, \phi_p)$ are in fact charts. We just have to check they are smoothly compatible. Let $p, q \in M$. We want to show $\phi_p \circ \phi_q^{-1} : \phi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q)$ is smooth in the standard Euclidean sense (the other case follows analogously). First note that $\phi_p, \phi_q^{-1}$ are both G&P smooth, thus the desired result follows from two claims: (1) the composition of G&P smooth functions is G&P smooth and (2) if $f : U \subseteq \mathbb{R}^n \to X \subseteq \mathbb{R}^m$ is G&P smooth and $U$ is open, then $f$ is Euclidean smooth.

(1) Let $X, Y, Z$ be arbitrary subsets of Euclidean spaces of dimensions $\ell, m, n$ respectively. Let $f : X \to Y, g : Y \to Z$ be G&P smooth functions. Consider $g \circ f$. Let $x \in X$. Then there exists some $U \subseteq \mathbb{R}^\ell$ open such that $x \in U$ and a smooth function $F : U \to Y$ s.t. $F|_{U \cap X} = f|_{U \cap X}$. Similarly there is some $V \subseteq \mathbb{R}^m$ open such that $f(x) \in V$ and $G : V \to Z$ smooth s.t. $G|_{V \cap Y} = g|_{V \cap Y}$. Then $G \circ F : U \cap F^{-1}(V) \to Z$ is a smooth function (as the composition of Euclidean smooth functions is smooth). Also let $p \in U \cap X \cap F^{-1}(V)$. Then $F(p) = f(p) \in V \cap Y$ so $G(F(p)) = g(f(p))$. Thus $G \circ F|_{U \cap F^{-1}(V) \cap X} = g \circ f|_{U \cap F^{-1}(V) \cap X}$. Then since $F$ is smooth, it is continuous, so $F^{-1}(V)$ is open. Then $G \circ F : U \cap F^{-1}(V) \to Z$ is a Euclidean smooth function extending $g \circ f$ at $x$. Since $x$ was arbitrary, we have that $g \circ f$ is smooth, and the claim is established.

(2) This claim is straightforward to verify, as the restriction of the domain of a Euclidean smooth function to an open subset is still Euclidean smooth.

Then by the above discussion, we have shown that the two charts are smoothly compatible, hence $\mathcal{A}$ is an atlas. Thus our construction of a Tu smooth manifold is complete. Note that we have constructed a Tu smooth manifold in which every chart is into $\mathbb{R}^k$ where $k$ is fixed, i.e. this manifold is $k$ dimensional.

**G&P Smooth Manifolds from Tu Smooth Manifolds**

Now let $M$ be a Tu smooth manifold with atlas $\mathcal{A}$. As noted above, WLOG we can take $M$ to be an embedded submanifold of $\mathbb{R}^N$ for some $N$. Also as noted above, we must take it so that $M$ has a well defined dimension, or in other words that every connected component has the same dimension. Let $k \in \mathbb{N}$ be this dimension. We then claim that $M \subseteq \mathbb{R}^N$ is a G&P $k$-dimensional smooth manifold. Let $p \in M$. Then by assumption the inclusion map
\( i : M \rightarrow \mathbb{R}^N \) is, by assumption, Tu smooth, an immersion, and a topological embedding. Since \( i \) is an immersion, the immersion theorem (Theorem 11.5(i) in Tu) applies, giving charts \((V, \psi)\) centered at \( p \) (i.e. \( \psi(p) = 0 \)) and \((W, \lambda)\) centered at \( i(p) = p \) s.t. that for some neighborhood \( A \) of \( \psi(p) \), \( \lambda \circ i \circ \psi^{-1}(r^1, \ldots, r^k) = (r^1, \ldots, r^k, 0, \ldots, 0) \). We then claim that \( \psi \) is a G&P diffeomorphism from \( V \) to \( \psi(V) \). We already know \( \psi(V) \) is an open subset of \( \mathbb{R}^k \), so directly from the definition, establishing this is sufficient to establish that \( M \subseteq \mathbb{R}^N \) is a G&P \( k \)-dimensional smooth manifold. Consider \( i \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}^N \). Then as the composition of Tu smooth maps, this map is Tu smooth. Then as a Tu smooth map between Euclidean spaces, it is Euclidean smooth. Then \( \psi^{-1} : \psi(V) \rightarrow V \) is a Euclidean smooth map, hence it is G&P smooth.

Now we want to show \( \psi \) is G&P smooth. For this we need a Euclidean smooth extension of \( \psi \) to an open subset of \( \mathbb{R}^N \). Note that \( \psi^{-1}(A) \) is an open (in \( M \)) neighborhood of \( p \), since \( A \) is open and \( \psi^{-1} \) a homeomorphism. Since \( M \) has the subspace topology (since \( i \) is a topological embedding), there is some open set \( B \subseteq \mathbb{R}^N \) s.t. \( B \cap M = \psi^{-1}(A) \). Let \( U = B \cap W \). Then \( U \) is an open subset of \( \mathbb{R}^N \) that is a neighborhood of \( p \). Then consider \((\lambda^1, \ldots, \lambda^k) : U \rightarrow \psi(V)\), the projection of \( \lambda \) to the first \( k \) coordinates. First we show this is a smooth map. \( \lambda : U \rightarrow \mathbb{R}^N \) is a Tu smooth map between Euclidean spaces, hence is a Euclidean smooth map. Note also the projection map: \((r^1, \ldots, r^k, r^N) \mapsto (r^1, \ldots, r^k)\) is Euclidean smooth, thus \((\lambda^1, \ldots, \lambda^k)\) is the composition of Euclidean smooth maps, hence Euclidean smooth. So we have a smooth map defined on an open (in \( \mathbb{R}^N \)) neighborhood of \( p \); we just have to show that \((\lambda^1, \ldots, \lambda^k)|_{U \cap M} = \psi|_{U \cap M} \). Let \( x \in U \cap M \). Then \( x \in B \cap M \), hence \( x \in \psi^{-1}(A) \). Thus \( \psi(x) \in A \), and so \( \lambda \circ i \circ \psi^{-1}(\psi(x)) = \lambda \circ i \circ \psi^{-1}(\psi(x)^1, \ldots, \psi(x)^k) = (\psi(x)^1, \ldots, \psi(x)^k, 0, \ldots, 0) \). But also \( \lambda \circ i \circ \psi^{-1}(\psi(x)) = \lambda \circ i(x) = \lambda(x) \). Thus \( \lambda(x) = (\psi(x)^1, \ldots, \psi(x)^k, 0, \ldots, 0) \), therefore \((\lambda^1(x), \ldots, \lambda^k(x)) = (\psi(x)^1, \ldots, \psi(x)^k) = \psi(x) \) or in other words: \((\lambda^1, \ldots, \lambda^k) = \psi(x) \). Then since \( x \in U \cap M \) arbitrary, we have \((\lambda^1, \ldots, \lambda^k)|_{U \cap M} = \psi|_{U \cap M} \). Therefore \( \psi \) is G&P smooth, hence is a G&P diffeomorphism. Thus \( M \subseteq \mathbb{R}^N \) is a G&P manifold.

**There and Back Again: These Operations Are Inverse**

We have shown how to get a G&P smooth manifold from a Tu smooth manifold and vice versa. But for this to be a 1-1 correspondence, we have to show these operations are inverse. In order to prove this though we need a proposition:

**Proposition.** Suppose \( M \subseteq \mathbb{R}^n \) is a Tu smooth embedded submanifold, \( U \subseteq M \) an open set, and \( f : U \rightarrow \mathbb{R}^m \) a Tu smooth function. Then \( f : U \rightarrow \mathbb{R}^m \) is G&P smooth.

It is helpful to first prove two lemmas:

**Lemma.** If \( N, M, P \) are Tu smooth manifolds s.t. \( N \) is an embedded submanifold of \( M \) and \( M \) is an embedded submanifold of \( P \), then \( N \) is an embedded submanifold of \( P \).

**Proof.** Let \( i_N : N \rightarrow M \), \( i_M : M \rightarrow P \), \( i_P : N \rightarrow P \) be the respective inclusion maps. Then note that that \( i_P = i_M \circ i_N \). By assumption \( i_N, i_M \) are Tu smooth, immersions, and topological embeddings. It is easy to check the composition of topological embeddings is a topological embedding. Also the composition of Tu smooth functions is Tu smooth. Thus \( i_P \) is a smooth topological embedding. We just have to check its an immersion at every
Lemma. If \( U \subseteq M \) is an open subset of a Tu smooth manifold \( M \) given manifold as described above, then \( U \) is an embedded submanifold of \( M \).

Proof. It is easy to verify directly that the inclusion map \( i : U \to M \) is smooth and a topological embedding. So we just have to show that \( i_{*p} \) is injective for an arbitrary \( p \in U \). Since \( i_{*p} \) is a vector space homomorphism, it is sufficient to check the kernel is trivial. Let \( v \in T_p U \) and suppose \( i_{*p}(v) = 0 \). Let \([f] \in C^\infty_p(U)\). Then for some \( V \subseteq U \) open, we have \( f : V \to \mathbb{R} \) smooth. Since \([f] \in C^\infty_p(M)\), call that class \( C \). Then \( i_{*p}(v)(C) = v([f \circ i]) = 0 \). But \( f \circ i = f \), so \([f \circ i] = [f]\), and therefore \( v([f]) = 0 \). Thus \( v = 0 \), so the kernel is trivial and \( i \) is an immersion. Thus \( U \) is an embedded submanifold of \( M \).

Proof. First note by the proceeding two lemma, \( U \) itself is an embedded submanifold of \( \mathbb{R}^n \). Thus it is sufficient to prove the proposition for arbitrary embedded submanifolds of \( \mathbb{R}^n \) (without regard to open subsets).

Let \( M \subseteq \mathbb{R}^n \) be an Tu smooth embedded submanifold. Since \( i \) is an immersion, we can apply the immersion theorem as above, to get charts \((U, \phi)\) centered at \( p \) and \((V, \psi)\) centered at \( \phi(p) = 0 \) s.t. for some neighborhood \( W \) of \( \phi(p) = 0 \) we have \( \psi \circ i \circ \phi^{-1}(r^1, \ldots, r^k) = (r^1, \ldots, r^k, 0, \ldots, 0) \). Then \( \phi^{-1}(W) \) is an open (in \( M \)) neighborhood of \( p \). Since \( M \) has the subspace topology, there is some \( A \subseteq \mathbb{R}^n \) open s.t. \( A \cap M = \phi^{-1}(W) \). Let \( \pi : \mathbb{R}^n \to \mathbb{R}^k \) be given as \( \pi(r^1, \ldots, r^k, \ldots, r^n) = (r^1, \ldots, r^k) \). Then \( \pi \) is smooth. Then \( \psi^{-1}(\pi^{-1}(W)) \) is open and also \( \pi(\phi(p)) = 0 \in W \), so \( p \in \psi^{-1}(\pi^{-1}(W)) \). Let \( B = A \cap V \cap \psi^{-1}(\pi^{-1}(W)) \). Then \( B \) is an open neighborhood of \( p \). Define \( F : B \to \mathbb{R}^m \) by \( F = f \circ \phi^{-1} \circ \pi \circ \psi \). Let’s check this is well defined. If \( b \in B \), then \( \psi(b) \in \mathbb{R}^n \) so \( \pi(\psi(b)) \) is well defined. Also \( b \in \psi^{-1}(\pi^{-1}(W)) \), so \( \pi(\psi(b)) \in W \), so \( \phi^{-1} \circ \pi \circ \psi(b) \in M \) is well defined. Thus \( F \) is well defined. Also each map is Tu smooth, hence \( F \) is Tu smooth. Therefore in order to show \( f \) is G&P smooth, it is sufficient to show \( F|_{B \cap M} = f|_{B \cap M} \). Suppose \( x \in B \cap M \). Then \( x \in A \cap M \), so \( x \in \phi^{-1}(W) \) so \( \phi(x) \in W \). Then \( \psi \circ i \circ \phi^{-1}(\phi(x)) = \psi \circ i \circ \phi^{-1}(\phi(x)^1, \ldots, \phi(x)^k) = (\phi(x)^1, \ldots, \phi(x)^k, 0, \ldots, 0) \). But also \( \psi \circ \phi^{-1}(\phi(x)) = \psi(x) \). Thus \( F(x) = f \circ \phi^{-1} \circ \pi \circ \psi(x) = f \circ \phi^{-1} \circ \pi \circ \psi(x) = f \circ \phi^{-1}(\phi(x)^1, \ldots, \phi(x)^k, 0, \ldots, 0) \circ \phi^{-1}(\phi(x)^1, \ldots, \phi(x)^k) = f \circ \phi^{-1}(\phi(x)) = f(x) \). Therefore \( f \) is G&P smooth.

Now we proceed in showing the operations are inverse. Since we can only get G&P smooth manifolds from Tu smooth manifolds which are embedded submanifolds of Euclidean space, in order that the correspondence to be 1-1, we need the Tu smooth manifold gotten from a G&P smooth manifold to be an embedded submanifold of \( \mathbb{R}^n \). We check this now.

Consider the inclusion map \( i : M \to \mathbb{R}^n \). Since our Tu smooth manifold \( M \) inherited the topology from the G&P smooth manifold \( M \), which just has the subspace topology as a subset of \( \mathbb{R}^n \), we know our Tu smooth manifold just has the subspace topology, hence the inclusion map is a topological embedding. Thus we just have to show that \( i \) is Tu smooth and an immersion. Let \( p \in M \) and \((U_p, \phi_p)\) a chart as described above. Then note that there is a global chart on \( \mathbb{R}^n \), namely \((id_{\mathbb{R}^n}, \mathbb{R}^n)\). Then \( i \) is Tu smooth at \( p \) iff
id_{\mathbb{R}^n} \circ i \circ \phi^{-1} : \phi(U_p \cap i^{-1}(\mathbb{R}^n)) \to id_{\mathbb{R}^n}(i(U_p) \cap \mathbb{R}^n) \text{ is Euclidean smooth, or equivalently if } \phi^{-1} : \phi(U_p) \to \mathbb{R}^n \text{ is Euclidean smooth. But } \phi^{-1} : \phi(U_p) \to \mathbb{R}^n \text{ is G&P smooth and } \phi(U_p) \subseteq \mathbb{R}^k \text{ is open, so } \phi^{-1} : \phi(U_p) \to \mathbb{R}^n \text{ is Euclidean smooth. Thus } i \text{ is Tu smooth at } p \text{ for all } p \in M, \text{ so } i \text{ is Tu smooth.}

Thus all that’s left is to show \( i_{*, p} \) is injective for arbitrary \( p \in M \). Since \( i_{*, p} \) is a vector space homomorphism, it is sufficient to show the kernel is trivial. Let \( v \in T_p M \) and suppose \( i_{*, p}(v) = 0 \). Let \( [f] \in C^\infty_{\psi}(M) \). Then note that \( f : U \to \mathbb{R} \) is Tu smooth, hence G&P smooth by above. Therefore there exists some \( F : W \to \mathbb{R} \) smooth where \( W \subseteq \mathbb{R}^n \) is an open neighborhood of \( p \) and \( F|_{W \cap M} = f|_{W \cap M} \). Then \( [F] \in C^\infty_{\psi}(\mathbb{R}^n) = C^\infty_{\psi}(\mathbb{R}^n) \). Therefore \( i_{*, p}(v)([F]) = 0 = v([F \circ i]) = v([f]) \) since \( F \circ i : i^{-1}(W) = W \cap M \to \mathbb{R} \) agrees with \( f : U \to \mathbb{R} \) on the open neighborhood of \( p \), \( W \cap M \). Thus \( v = 0 \) as \( [f] \in C^\infty_{\psi}(M) \) has arbitrary. Therefore \( i_{*, p} \) has a trivial kernel and thus \( i \) is an immersion. Thus every Tu smooth manifold induced by a G&P smooth manifold is in fact an embedded submanifold of the Euclidean space it is a subset of.

Now we can easily show that the two operations are inverse in one direction. Let \( M \subseteq \mathbb{R}^n \) be G&P smooth manifold. Then the induced Tu smooth manifold \( M' \) is an embedded submanifold of \( \mathbb{R}^n \), as such it induces a G&P smooth manifold \( M'' \). Note though that two G&P smooth manifolds are the equal iff they are equal as sets, and that the operations taking G&P smooth manifolds to Tu smooth manifolds and vice versa do not affect the underlying set. Hence \( M'' = M \) as G&P smooth manifolds. Therefore going from a G&P smooth manifold to a Tu smooth manifold and back again acts as the identity.

Now we have to check the other direction. Let \( M \subseteq \mathbb{R}^n \) be a Tu smooth embedded submanifold with maximal atlas \( \mathcal{A} \). Then by above the set \( M \subseteq \mathbb{R}^n \) is a G&P smooth manifold. Then we can construct a smooth atlas on the G&P smooth manifold making it into a Tu smooth manifold. This atlas is given as \( \mathcal{B} = \{(U_p, \phi_p)\}_{p \in M} \) where \( \phi_p : U_p \to \mathbb{R}^k \) is a G&P diffeomorphism onto its image. Since every atlas is contained in a unique maximal atlas, it is sufficient to show the charts of \( \mathcal{B} \) are in \( \mathcal{A} \), which, since \( \mathcal{A} \) is maximal, is true iff the charts of \( \mathcal{B} \) are pairwise smoothly compatible with the charts in \( \mathcal{A} \). So consider an arbitrary pair of charts \((U, \phi) \in \mathcal{B} \) and \((V, \psi) \in \mathcal{A} \). Then we need to show that \( \phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \) and \( \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \) are both Euclidean smooth. Note that these maps are Euclidean smooth iff they are G&P smooth. The charts of \( \mathcal{B} \) are specifically chosen to be G&P diffeomorphisms, hence \( \phi, \phi^{-1} \) are G&P smooth. Then we note that \( \psi : V \to \mathbb{R}^k \) is a Tu smooth function from an open subset of an embedded submanifold to a Euclidean space, hence by the proposition above it is G&P smooth. Finally we consider \( \psi^{-1} : \psi(V) \to V \). It is certainly Tu smooth as coordinate maps are Tu diffeomorphisms, as was noted previously. Then since the inclusion map \( i : M \to \mathbb{R}^n \) is Tu smooth, we have \( i \circ \psi^{-1} : \psi(V) \to \mathbb{R}^n \) a Tu smooth map from an open subset of \( \mathbb{R}^k \) to \( \mathbb{R}^n \). Then taking the \((\psi(V), id) \) and \((\mathbb{R}^n, id) \) as charts, we have \( id \circ i \circ \psi^{-1} \circ id^{-1} = i \circ \psi^{-1} \) is Euclidean smooth. Thus \( \psi^{-1} \) is Euclidean smooth. Hence \( \psi^{-1} \) is G&P smooth. Then as compositions of G&P smooth maps, we have \( \phi \circ \psi^{-1} \) and \( \psi \circ \phi^{-1} \) are G&P smooth map, and hence Euclidean smooth maps. Thus by the above discussion, going from a Tu smooth manifold to a G&P smooth manifold and back again acts as the identity.

Putting it together shows the two processes really are inverses. Thus every G&P smooth manifold is naturally identified with a unique Tu smooth manifold with the same under-
lying set and topology such that the inclusion map into the ambient Euclidean space is a smooth immersion as well as a topological embedding. Further, every Tu smooth embedded submanifold of \( \mathbb{R}^n \) can be gotten in this way.

From now on, we can then refer without ambiguity to a subset \( M \subseteq \mathbb{R}^n \) as a smooth manifold iff it is a G&P smooth manifold. This raises a natural question as to whether smooth maps are preserved by this correspondence.

**Smooth Maps are Preserved by the Correspondence**

Note that WLOG we can take the ambient space for any finite collection of smooth manifolds to be \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \) by taking \( n \) to be the maximum of the dimensions of the ambient spaces of the respective smooth manifolds.

**Theorem.** Let \( M, N \subseteq \mathbb{R}^n \) be smooth manifolds and \( f : M \to N \) a function. Then \( f \) is Tu smooth iff \( f \) is G&P smooth.

**Proof.** Suppose \( f \) is Tu smooth. Let \( i : N \to \mathbb{R}^N \) be the inclusion map. We know \( i \) is Tu smooth. Thus \( i \circ f : M \to \mathbb{R}^N \) is Tu smooth. By the previous proposition \( i \circ f \) is then G&P smooth. Then since G&P smoothness is irrespective of codomain, \( f \) is G&P smooth.

Suppose \( f \) is G&P smooth. Let \( p \in M \). Let \( U \) be an open subset of \( \mathbb{R}^N \) s.t. \( p \in U \) and \( F : U \to \mathbb{R}^N \) be a smooth function s.t. \( F|_{U \cap M} = f|_{U \cap M} \). Then let \( (V, \phi) \) and \( (W, \psi) \) be charts about \( p, f(p) \) respectively. Then \( \psi \circ f \circ \phi^{-1} = \psi \circ F \circ \phi^{-1} \) on \( \phi(U \cap V \cap f^{-1}(W)) = \phi(V \cap f^{-1}(W) \cap U \cap M) \) an open subset of \( \mathbb{R}^m \) (where \( m := \dim M \)). Thus \( \psi \circ f \circ \phi^{-1} : \phi(U \cap V \cap f^{-1}(W)) \to \psi(f(V \cap U) \cap W) \) is Euclidean smooth. Then taking the charts \((\phi, U \cap V)\) and \((\psi, W)\), this proves \( f \) is Tu smooth at \( p \). Thus \( f \) is Tu smooth. \( \square \)

So now without ambiguity we can speak of smooth maps between smooth manifolds.

**G&P’s Notion of Tangent Space**

**Definition.** Let \( M \subseteq \mathbb{R}^n \) be a \( k \)-dimensional smooth manifold and let \( p \in M \). Let \( (U, \phi) \) be a chart centered at \( p \) (i.e. so that \( \phi(p) = 0 \)). Then \( \phi^{-1} : \phi(U) \subseteq \mathbb{R}^k \to U \subseteq \mathbb{R}^n \) is a Euclidean smooth function so has a Jacobian matrix at 0: \( d\phi^{-1}|_0 \), which is an \( n \times k \) matrix. Define the **G&P tangent space** \( T_pM := d\phi^{-1}|_0(\mathbb{R}^k) \) (where we are using the nonstandard \( S \) to differentiate from the Tu tangent space) to be the image of \( \mathbb{R}^k \) under the linear map \( d\phi|_0 \).

First we want to show that this definition of tangent space is indepent of our choice of chart. Suppose \((V, \psi)\) were another chart centered at \( p \). Then \( \phi^{-1} = \psi^{-1} \circ (\psi \circ \phi^{-1}) \) on \( \phi(U \cap V) \), which is open. Then applying the chain rule we have \( d\phi^{-1}|_0 = d\psi^{-1}|_{\psi(\phi^{-1}(0))} \circ d(\psi \circ \phi^{-1})|_0 = d\psi^{-1}|_0 \circ d(\psi \circ \phi^{-1})|_0 \) where \( d\alpha|_x \) is the Jacobian matrix of \( \alpha \) at \( x \). Then we note that this implies that the image of \( d\phi^{-1}|_0 \) must be contained in \( d\psi^{-1}|_0 \). Interchanging the roles of \( \phi \) and \( \psi \) above show that the reverse inclusion must also hold. Thus the two tangent spaces are in fact equal.

Note that as the image under linear map of a vector space, \( S_pM \) is a vector subspace of \( \mathbb{R}^n \). We now seek to provide a natural identification between the Tu tangent space \( T_pM \) and the G&P tangent space \( S_pM \). Before we can do that, we need to provide a natural
identification between the Tu tangent space of \( \mathbb{R}^n \) and \( \mathbb{R}^n \). Note that every element \( v \in \mathbb{R}^n \) gives a point derivation at \( p \) in \( T_p \mathbb{R}^n \) for any \( p \in \mathbb{R}^n \) by considering the directional derivative in the direction of \( v \). It can be verified that directional derivatives are in fact point derivations. Further it can be shown that the this correspondence actually provides a vector space isomorphism \( \mathbb{R}^n \rightarrow T_p \mathbb{R}^n \) (the details are all worked out in Tu chapter 2).

Thus we have a natural identification between elements of \( T_p \mathbb{R}^n \) and elements of \( \mathbb{R}^n \). For the time being call this map \( \gamma \). This motivates an identification between elements of \( T_p M \) and elements of \( \mathbb{R}^n \) where \( M \subseteq \mathbb{R}^n \), namely the map \( \lambda : T_p M \rightarrow \mathbb{R}^n \) given by \( \lambda := \gamma \circ i_* \), where \( i \) is the inclusion map. What we want then to show is that \( \lambda : T_p M \rightarrow S_p M \) is a well defined vector space isomorphism. First let’s consider \( \gamma^{-1}|_{S_p M} : S_p M \rightarrow i_*(T_p M) \), i.e. that \( \gamma^{-1}(S_p M) \subseteq i_*(T_p M) \). Let \( v \in S_p M \). Then \( \gamma^{-1}(v) \) is the point-derivation given by the directional derivative in the direction of \( v \). Define \( d \in T_p M \) as follows. Let \([f] \in C^\infty_p(M)\). Then \( f \in U \rightarrow \mathbb{R} \) is a smooth function with \( U \subseteq \mathbb{R} \) open. In particular it is G&P smooth, which means it admits a smooth extension \( F : W \rightarrow \mathbb{R} \) with \( W \subseteq \mathbb{R}^n \) open and \( p \in W \). Then \( d([f]) := \gamma^{-1}(v)([F]) \).

First we have to show \( d \) is well defined. Let \( g \in [f] \). Then there is some open \( A \subseteq M \) s.t. \( g : A \rightarrow \mathbb{R} \) is smooth and \( p \in A \). As above, we get an extension \( G : B \rightarrow \mathbb{R} \) s.t. \( B \subseteq \mathbb{R}^n \) is open and \( p \in B \). We want to show that \( \gamma^{-1}(v)([F]) = \gamma^{-1}(v)([G]) \). Or equivalently that \( F \) and \( G \) have the same directional derivative in the direction of \( v \). Note that there is some open \( V \subseteq M \) s.t. \( V \subseteq U, V \subseteq A \), and \( f|_V = g|_V \). Then on \( V \cap W \cap B \), an open subset of \( M \) containing \( p \), \( F \) and \( G \) agree. Thus on this set \( F - G = 0 \). Let \((C, \phi)\) be a chart centered at \( p \). Let \( D := C \cap V \cap W \cap B \). Then \((D, \phi)\) is also a chart centered at \( p \). Thus there is some \( w \in \mathbb{R}^k \) s.t. \( d\phi^{-1}|_0(w) = v \). Consider \((F - G) \circ \phi^{-1} : \phi(D) \rightarrow \mathbb{R} \). Then note that since on \( D \) \( F - G \) is identically 0, this function is identically 0. In particular its Jacobian matrix at 0 is 0. Putting this together with the chain rule yields 0 = \( d((F - G) \circ \phi^{-1})|_0 = d(F - G)|_p \circ d\phi^{-1}|_0 \). In particular then \( d(F - G)|_p(d\phi^{-1}|_0(w)) = d(F - G)|_p(v) = 0 \). Thus \( dF_p(v) = dG_p(v) \), i.e. \( F \) and \( G \) have the same directional derivative in the direction of \( v \). Thus, by the above discussion \( d \) is a well defined function \( C^\infty_p(M) \rightarrow \mathbb{R} \). For \( d \in T_p M \), we need that \( d \) is a point derivation. Well if \([f], [g] \in C^\infty_p(M)\), then \( d([f][g]) = d([[f]g]) = \gamma^{-1}(v)((FG)) \), where \( F, G \) are smooth extensions of \( f, g \) and we used that \( FG \) is a smooth extension of \( fg \) (which can be directly verified). Then since \( \gamma^{-1}(v) \) is a point derivation, we have \( \gamma^{-1}(v)([FG]) = \gamma^{-1}(v)([F][G]) = \gamma^{-1}(v)([F])G(p) + F(p)\gamma^{-1}(v)([G]) = d([f])g(p) + f(p)d([g]) \). This shows that \( d \) is in fact a point derivation, hence \( d \in T_p M \).

Now I claim that \( \gamma^{-1}(v) = i_*(d) \). Let \([F] \in C^\infty_p(\mathbb{R}^n)\). Then \( i_*(d)([F]) = d([F \circ i]) = \gamma^{-1}(v)([F]) \), where we used that \( F \circ i \) is a smooth function from an open neighborhood of \( p \) to \( \mathbb{R} \) and \( F \) is a smooth extension of this function. Thus the claim is established. In particular we have that \( \gamma^{-1}(v) \in i_*(T_p M) \). Thus \( \gamma^{-1}(S_p M) \subseteq i_*(T_p M) \). Also since \( \gamma \) is a vector space isomorphism, \( \gamma^{-1} : S_p M \rightarrow i_*(T_p M) \) is an injective linear map. Thus \( \gamma^{-1} \) is an isomorphism \( S_p M \rightarrow i_*(T_p M) \) provided that \( \dim S_p M = \dim i_*(T_p M) \). Since \( i_* \) is an injection, \( \dim i_*(T_p M) = \dim T_p M \), so we really just need to show \( \dim S_p M = \dim T_p M \). This result is true, and will be proven immediately following this discussion. Therefore \( \gamma^{-1} : S_p M \rightarrow i_*(T_p M) \) is a vector space isomorphism, hence \( \gamma : i_*(T_p M) \rightarrow S_p M \) is a vector space isomorphism. Then \( i_*(T_p M) \rightarrow i_*(T_p M) \) is also a vector space isomorphism, and so \( \lambda \), as the composition of isomorphisms, is an isomorphism. Thus we have a natural identification of \( T_p M \) with \( S_p M \). To complete the proof though, we need the following result.
Proposition. $T_p M = S_p M = \dim M$.

Lemma. Let $M, N$ be diffeomorphic smooth manifolds under $F : M \to N$. Let $p \in M$. Then the differential $F_* : T_p M \to T_{F(p)} N$ is a vector space isomorphism.

Proof. Just apply the chain rule for differential to $F \circ F^{-1} = id_N$ and $F^{-1} \circ F = id_M$ and note that $(id)_* = id$, i.e. that the differential of the identity map is the identity on the tangent space. Thus $F_*, F_*^{-1}$ are inverse linear maps, hence $F_*$ is an isomorphism.

Proof. First let’s consider the Tu case, $T_p M$. It’s first worth noting that for for $p \in M$ and $U$ a neighborhood of $p$, that it’s easy to verify the $C^\infty_p(M) = C^\infty(U)$. Thus $T_p M = T_p U$. Now let $(U, \phi)$ be a chart centered at $p$. So $\phi : U \to \mathbb{R}^k$ is a diffeomorphism, hence it’s differential is vector space isomorphism: $T_p U \to T_{\phi(U)} \mathbb{R}^k$. Then $\phi(U) \subseteq \mathbb{R}^k$ is open, so $T_0 \phi(U) = T_0 \mathbb{R}^k$, and we previously noted that $T_0 \mathbb{R}^k$ is $k$ dimensional. Hence $T_p M$ is $k$ dimensional, which is the dimension of $M$.

Now the G&P case. Let $p \in M$. Let $(U, \phi)$ be a chart centered at $p$. Let $\Phi : W \to \mathbb{R}^k$ be $\phi$’s smooth extension to $W \subseteq \mathbb{R}^n$ open. Let $V := W \cap U$. Then consider $\phi^{-1} : \phi(V) \to V$. First note $\phi(V) \subseteq \mathbb{R}^k$ is open as $\phi$ is a diffeomorphism hence an open map. Also $\Phi \circ \phi^{-1} = id_{\phi(V)}$. Then applying the chain rule for Jacobians, we have $d\Phi |_p \circ d\phi^{-1}|_0 = d\phi|_0$ by $id_{\mathbb{R}^k}$. Thus $d\phi^{-1}|_0$ must be injective. As such $S_p M = d\phi^{-1}|_0(\mathbb{R}^k)$ is a $k$ dimensional vector subspace of $\mathbb{R}^n$, where $k = \dim M$.

G&P’s Notion of the Differential

So we have a nice identification between $T_p M$ and $S_p M$, the two different notions of a tangent space. To show why this identification is nice, we have to first introduce G&P’s notion of the differential. The differential takes morphisms of smooth manifolds (smooth maps) to morphisms of tangent spaces (linear maps). We want to show under this identification of the tangent spaces, the two notions of the differential are identified. First let’s define the G&P differential.

Definition. Let $M, N \subseteq \mathbb{R}^n$ be smooth manifolds of dimension $m, n$ respectively and $f : M \to N$ a smooth map. Let $p \in M$. Let $(U, \phi)$ be a chart centered at $p$, and $(V, \psi)$ be a chart centered at $f(p)$. Let $h := \psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)$. Then $f = \psi^{-1} \circ h \circ \phi$ in a neighborhood of $\phi(p) = 0$. Define the G&P differential $df_p := d\psi^{-1}|_0 \circ dh|_0 \circ (d\phi^{-1}|_0)^{-1}$, where $\psi$ and $h$ are Euclidean smooth maps defined on neighborhoods of 0, hence $T_p M = \dim M$ that the linear map $d\phi^{-1}|_0 : \mathbb{R}^m \to \mathbb{R}^n$ is an isomorphism onto its image $S_p M$. Thus $(d\phi^{-1}|_0)^{-1} : S_p M \to \mathbb{R}^m$ is just the inverse of this isomorphism.

Let’s show that our definition of $df|_p$ is independent of our choice of charts. WLOG we can take the charts to have the same open sets (just take intersections to shrink as necessary). So let $(U, \alpha), (V, \beta)$ be charts centered at $p$, $f(p)$ respectively. Define $g := \beta \circ f \circ \alpha^{-1}$. Then $g = (\beta \circ \psi^{-1}) \circ h \circ (\phi \circ \alpha^{-1})$ on some neighborhood of 0. Then note that each bracketed part of that composition is Euclidean smooth, so applying the chain rule, we get: $dg|_0 = d(\beta \circ \psi^{-1})|_0 \circ dh|_0 \circ d(\phi \circ \alpha^{-1})|_0$. Then we want to show that $d\beta^{-1}|_0 \circ dg|_0 \circ (d\alpha^{-1}|_0)^{-1} = d\psi^{-1}|_0 \circ dh|_0 \circ (d\phi^{-1}|_0)^{-1}$, for which it is sufficient to establish $d\beta^{-1}|_0 \circ d(\beta \circ \psi^{-1})|_0 = d\psi^{-1}|_0$
and that \(d(\phi \circ \alpha^{-1})|_0 \circ (d\alpha^{-1}|_0)^{-1} = (d\phi^{-1}|_0)^{-1}\). The first relation is easy to see as \(\beta^{-1}, \beta \circ \psi^{-1}\) are both Euclidean smooth functions, so we can just use the chain rule. For the second relation, note that \(\phi \circ \alpha^{-1}\) is a diffeomorphism \(\mathbb{R}^m \to \mathbb{R}^m\), thus \(d(\phi \circ \alpha^{-1})|_0\) is a vector space isomorphism \(\mathbb{R}^m \to \mathbb{R}^m\). Also note that \((d\alpha^{-1}|_0)^{-1} : S_pM \to \mathbb{R}^m\) and \((d\phi^{-1}|_0)^{-1} : S_pM \to \mathbb{R}^m\) are vector space isomorphisms, so we can instead consider the inverse of the relation: \(d\alpha^{-1}|_0 \circ (d(\phi \circ \alpha^{-1})|_0)^{-1} = d\phi^{-1}|_0\). Then apply the inverse function theorem to \(\phi \circ \alpha^{-1}\) at 0, which is a diffeomorphism, to get \((d(\phi \circ \alpha^{-1})|_0)^{-1} = d(\alpha \circ \phi^{-1})|_0\). Finally, we get \(d\alpha^{-1}|_0 \circ d(\alpha \circ \phi^{-1})|_0 = d(\alpha^{-1} \circ \alpha \circ \phi^{-1})|_0 = d\phi^{-1}|_0\) by the chain rule. Thus the two relations are established, and so the differential is well defined independent of chart choice.

**Proposition** (Chain Rule for G&P Differential). Let \(M, N, P \subseteq \mathbb{R}^n\) be smooth manifolds and \(f : M \to N, g : N \to P\) be smooth maps. Let \(p \in M\). Then the smooth map \(g \circ f\) has G&P differential at \(p\): \(d(g \circ f)|_p = dg|_{f(p)} \circ df|_p\).

**Proof.** Let \((U, \phi), (V, \psi), (W, \gamma)\) be charts centered at \(p, f(p), g(f(p))\) respectively. Then \(d(g \circ f)|_p = d\gamma^{-1}|_0 \circ d(\gamma \circ g \circ f \circ \phi^{-1})|_0 \circ (d\phi^{-1}|_0)^{-1}\). Then \(dg|_{f(p)} \circ df|_p = d\gamma^{-1}|_0 \circ d(\gamma \circ g \circ \psi^{-1})|_0 \circ (d\psi^{-1}|_0)^{-1} \circ d\psi^{-1}|_0 \circ d(\psi \circ f \circ \phi^{-1})|_0 \circ (d\phi^{-1}|_0)^{-1}\). Thus it is sufficient to establish \(d(\gamma \circ g \circ \phi^{-1})|_0 \circ d(\psi \circ f \circ \phi^{-1})|_0 = d(\gamma \circ g \circ f \circ \phi^{-1})|_0\), but this follow directly from applying the chain rule to \((\gamma \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) = \gamma \circ g \circ f \circ \phi^{-1}\).

\[\square\]

**The Two Differentials are Identified Under the Previous Identifications**

Thus we see the G&P differential plays the same role of taking smooth functions to G&P tangent space homomorphisms as the \(T_u\) differential does for \(T_u\) tangent spaces. To show that these notions of the differential are actual identified under the previous identifications, we have the following theorem.

**Theorem.** Let \(M, N \subseteq \mathbb{R}^N\) be smooth manifolds of dimension \(m, n\) respectively, \(p \in M\), and \(f : M \to N\) a smooth function. Let \(\Lambda : T_pM \to S_pM\) and \(\Gamma : T_{f(p)}N \to S_{f(p)}N\) be the natural isomorphisms identifying the two tangent spaces. Then the following diagram commutes.

\[
\begin{array}{ccc}
T_pM & \xrightarrow{f_{*,p}} & T_{f(p)}N \\
\downarrow \Lambda & & \downarrow \Gamma \\
S_pM & \xrightarrow{df|_p} & S_{f(p)}N
\end{array}
\]

**Proof.** Recall there is a canonical identification \(T_p\mathbb{R}^N \to \mathbb{R}^N\). Denote this identification \(\gamma\). Then \(\Lambda = \gamma \circ i_{*,p}\) and \(\Gamma = \gamma \circ i_{*,f(p)}\). Let \((U, \phi), (V, \psi)\) be charts centered at \(p, f(p)\) respectively. We want to show \(df|_p = \Gamma \circ f_{*,p} \circ \Lambda^{-1}\). Note that both sides are linear maps, so it is sufficient to show they agree on some basis of \(S_pM\). Since \(d\phi^{-1}|_0 : \mathbb{R}^m \to S_pM\) is an isomorphism, it takes a basis to a basis. Denote the standard basis on \(\mathbb{R}^m\) by \(\{e_i\}_1^m\). Then \(\{d\phi^{-1}|_0 e_i\}_1^m\) is a basis for \(S_pM\).
First let's compute $df_p \circ d\phi^{-1}|_p e_i$. Note that $df_p\circ d\psi^{-1}|_p d(\psi \circ f \circ \phi^{-1})|_0 (d\phi^{-1}|_0)\circ f \circ \phi^{-1}|_0$, so $df_p\circ d\phi^{-1}|_0 e_i = d\psi^{-1}|_0 d(\psi \circ f \circ \phi^{-1})|_0 (d\phi^{-1}|_0)\circ f \circ \phi^{-1}|_0$. Then we note that $d\psi^{-1}|_0 d(\psi \circ f \circ \phi^{-1})|_0 = d(f \circ \phi^{-1})|_0$ by the chain rule. Thus $df_p\circ d\phi^{-1}|_0 e_i = d(f \circ \phi^{-1})|_0 e_i$.

For notational convenience, denote $v := d\phi^{-1}|_0 e_i$. Then we want to show that $\Gamma \circ f_{*,p} \circ \Lambda^{-1}(v) = d(f \circ \phi^{-1})|_0 e_i$. Well $\Gamma \circ f_{*,p} \circ \Lambda^{-1}(v) = \gamma \circ i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v)$, so we have the above equality iff $i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v)$ is the directional derivative induced by $d(f \circ \phi^{-1})|_0 e_i$, which we will denote by $\gamma$. We know $i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v), \chi \in T_0^i \mathbb{R}^n$, which pulling back the standard basis on $\mathbb{R}^n$ under $\gamma$, has a basis given by $\{\frac{\partial}{\partial x^j}\}_{j=1}^n$. Then we know that $\gamma$ in this basis is given as $\sum_{j=1}^n (d(f \circ \phi^{-1})|_0 e_i)^j \frac{\partial}{\partial x^j}$, since it is just a directional derivative. Then write $i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v) = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}$. Then $a^j = i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v)(r^j)$, where $r^j$ is just the standard coordinate function $(x^1,...,x^j,...,x^n) \mapsto x^j$. This we just need to show that $i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v)(r^j) = (d(f \circ \phi^{-1})|_0 e_i)^j$.

$(i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v))(r^j) = (i_{*,f(p)} \circ f_{*,p} \circ \gamma^{-1}(v))(r^j)$. Denote $D_v := \gamma^{-1}(v)$, which is just the directional derivative in the direction of $v$. Then denote $u := i_{*,f(p)}(D_v)$. Then $(i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v))(r^j) = (i_{*,f(p)} \circ f_{*,p}(u))(r^j) = ((i \circ f)_{*,p}(u))(r^j) = u(r^j \circ i \circ f) = u(f^j)$, where $f^j$ is just the $j$th coordinate of the $f: M \rightarrow \mathbb{R}^n$ where we construe $f$ as a smooth function with codomain $\mathbb{R}^n$. Since $f$ is smooth, it admits a smooth extension to a neighborhood of $p$ in $\mathbb{R}^n$, $F: W \rightarrow \mathbb{R}^n$. Then $i_{*,f(p)}(F^j) = u(F^j \circ i) = u(f^j)$ since $F^j \circ i = f^j$ in a $M$ neighborhood of $p$. But also $i_{*,f(p)}(F^j) = i_{*,f(p)}(D_v)(F^j) = D_v(F^j)$, which is just the directional derivative of $F^j$ in the direction of $v$, which then equal to $dF^j|_p u = (dF^j|_p v)^j$. Then expanding out $v$, we get $u(f^j) = (dF|_p v)^j = (dF|_p d\phi^{-1}|_0 e_i)^j = (d(F \circ \phi^{-1})|_0 e_i)^j = (d(f \circ \phi^{-1})|_0 e_i)^j$, where second equality follows from the chain rule and the final equality follows since $F \circ \phi^{-1} = f \circ \phi^{-1}$ is a neighborhood of 0 since $F$ is a smooth extension of $f$ and $\phi^{-1}$ has a codomain of $M$. Thus we have shown $(i_{*,f(p)} \circ f_{*,p} \circ \Lambda^{-1}(v))(r^j) = u(f^j) = (d(f \circ \phi^{-1})|_0 e_i)^j$, and so $\Gamma \circ f_{*,p} \circ \Lambda^{-1}(d\phi^{-1}|_0 e_i) = d(f \circ \phi^{-1})|_0 e_i = df_p\circ d\phi^{-1}|_0 e_i$. Thus $\Gamma \circ f_{*,p} \circ \Lambda^{-1}$ and $df_p$ agree on a basis of $S_p M$, so they are in fact equal. Thus the above diagram commutes.

\[\square\]

**Concluding Remarks**

We have demonstrated an explicit one to one correspondence between Tu smooth manifolds which are embedded submanifold submanifolds of $\mathbb{R}^n$ and G&P smooth manifolds. This correspondence preserves the underlying set, so any map $f: M \rightarrow N$ between smooth manifolds can be interpreted as either a map between Tu smooth manifolds or between G&P smooth manifolds. We showed that $f$ is Tu smooth as a map between Tu smooth manifolds iff it is G&P smooth as a map between G&P smooth manifolds. An important part of smooth manifold theory is the tangent space construction. G&P and Tu give distinct constructions of the tangent space of a smooth manifolds at point $p$. We showed that the G&P tangent space and the Tu tangent space of a smooth manifold $M \subseteq \mathbb{R}^n$ at a point $p \in M$ are isomorphic is a natural way. Further there is the notion of a differential, which takes a smooth map between manifolds to a vector space homomorphism between tangent spaces. G&P and Tu give different definitions of the differential of a smooth map. We showed that under the previous identification of smooth manifolds and tangent spaces, that the Tu differential is identified with the G&P differential. This correspondence then gives a way to take theorems about Tu smooth manifolds and prove them for G&P smooth manifolds and
vice versa.