Euler Equations for Gas Dynamics

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References: Finite Volume Methods for Hyperbolic Problems

Randall Leveque

Consider a gas in one dimension whose mass density is denoted \( \rho(x,t) \).

Since the particles in the gas are far enough from each other that they can be compressed or expanded, the gas is a compressible fluid.

We expect the pressure to increase where the density is increased, and vice-versa.

If the fluid velocity is \( u(x,t) \), it follows from conservation of mass that for \( x_1 < x_2 \),

\[
\frac{d}{dt} \int_{x_1}^{x_2} \rho(x,t) \, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial x} \, dx.
\]

where the flux \( f(x) \) is given by \( f(x) = \rho(x)u(x,t) \).

Hence,

\[
\int_{x_1}^{x_2} \frac{2\rho - \frac{d}{dx}(\rho u)}{\rho} \, dx = 0.
\]

As \( x_1 \) and \( x_2 \) are arbitrary, this means

\[
\frac{\partial}{\partial t} \int_{x_1}^{x_2} \rho u \, dx = 0 \quad (1)
\]

This is the equation of continuity that expresses local conservation of mass.
Momentum, with density \( \rho(x,t)u(x,t) \), is also conserved. The momentum flux out the boundaries of an interval is a combination of macroscopic and microscopic fluxes

\[
f(x,t) = \rho u \cdot u + \rho = \rho(x,t)u(x,t)^2 + \rho(x,t)
\]

\[\Rightarrow \rho \text{ is microscopic momentum flux}
\]

As in the equation of continuity,

\[
(\rho u)_t + (\rho u^2 + \rho)_x = 0 \quad (2)
\]

We have introduced a new variable — \( \rho \), the pressure.

Two ways to close the system: speed of sound, simple way: if there are no shock waves \( (u << c) \), then, the entropy of the gas is constant, and

\[
\rho = \rho(\rho) = k \rho^{\gamma} \quad (3) \quad (\gamma = 1.4) \quad \text{for air}
\]

Equations (1), (2), \& (3) form a system of two coupled, nonlinear conservation laws

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + P(\rho))_x &= 0
\end{align*}
\]

General way: in the case that we are concerned with a general situation, where there
might be shocks, we need to introduce another conserved quantity, the energy \( E(x, t) \).

Energy flux is given by

\[
 f(x, t) = E_u + p u \text{microscopic} = \frac{E}{\gamma - 1} \sqrt{\gamma \rho} \text{macroscopic}
\]

Thus, \( E_t + (E u + p u) x = 0 \), and we get the system

\[
\begin{pmatrix}
\rho u \\
\rho u^2 + p \\
\frac{E}{\gamma - 1}\end{pmatrix}_t + \begin{pmatrix}
\rho u \\
\rho u^2 + p \\
\frac{E}{\gamma - 1}\end{pmatrix}_x = 0
\]

For an ideal polytropic gas, \( E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \), and in primitive variables \((u, p, \rho)\) the system becomes

\[
\begin{pmatrix}
\rho \\
\rho u \\
\rho u^2 + p \end{pmatrix}_t + \begin{pmatrix}
u & \rho & 0 \\
0 & u & 1/\gamma \\
0 & \gamma p & u \end{pmatrix} \begin{pmatrix}
\rho \\
\rho u \\
\rho u^2 + p \end{pmatrix}_x = 0
\]

Matrix has eigenvalues and eigenvectors given by

\[
\begin{align*}
\lambda_1 &= u - c \\
\lambda_2 &= u + \sqrt{\gamma p} \\
\lambda_3 &= u + c
\end{align*}
\]

where \( c = \sqrt{\gamma p} \).

Show simulation of shock bubble in CLAWPACK software.

Initial data - bubble containing low density gas at constant pressure. A shock wave approaches from the left.