Computing the flow of an incompressible fluid can be classified in 2 categories depending on choice of governing equations:

1. Velocity-Pressure formulation
   - Solve the Navier-Stokes equation and continuity equation subject to appropriate boundary conditions, initial conditions and possibly supplemental constraints
   - primary variables
   - have to combine equations to solve for the pressure, or do some kind of projection to assure incompressibility
   - sometimes two interwoven grids are used
   - can be extended to 3D

2. Vorticity-Velocity (a.k.a. Vorticity-Stream) formulation
   - based on vorticity transport then solving for the velocity
   - no need to solve for the pressure
   - simpler in conception + implementation
   - if velocity is expressed in terms of a streamfunction, the continuity equation is automatically satisfied
   - no need to derive appropriate BC for the pressure
   - need to derive boundary conditions for the vorticity and streamfunction
   - cumbersome to extend to 3D
Velocity - Pressure Equations

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} &= -\mathbf{u} \cdot \nabla \mathbf{u} - \frac{i}{\rho} \mathbf{u} \cdot \nabla \mathbf{p} \quad \text{(1)} \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]

Note that by taking a divergence of the first eqn and using \( \nabla \cdot \mathbf{u} = 0 \), we can get for the pressure

\[
\nabla^2 p = \rho \nabla \cdot (-\mathbf{u} \cdot \nabla \mathbf{u})
\]

or we could use some projection to solve for it.

Velocity - Stream Equations (2D)

\[
\begin{align*}
\frac{\partial \mathbf{\omega}}{\partial t} &= -\mathbf{u} \cdot \nabla \mathbf{\omega} + \nu \nabla^2 \mathbf{\omega} \\
\nabla^2 \psi &= -\mathbf{\omega} \\
\mathbf{u} &= \nabla^+ \psi = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)
\end{align*}
\]

Note that in 2D, \( \mathbf{u} = (u_x, u_y, 0) \) \( \mathbf{\omega} = (0, 0, \omega_3) \) so the velocity transport PDE is just for \( \omega_3 \):

\[
\frac{\partial \omega_3}{\partial t} = -u_x \frac{\partial \omega_3}{\partial x} - u_y \frac{\partial \omega_3}{\partial y} + \nu \left( \frac{\partial^2 \omega_3}{\partial x^2} + \frac{\partial^2 \omega_3}{\partial y^2} \right)
\]
Explicit scheme

\[
\frac{q_i^{n+1} - q_i^n}{\Delta t} = \gamma \frac{q_{i-1}^n - 2q_i^n + q_{i+1}^n}{\Delta y^2}
\]

for \( i = 2 \ldots N \)

\[
\rightarrow \quad q_i^{n+1} = \alpha q_i^n + (1 - 2\alpha) q_i^{n+1} + \alpha q_{i+1}^n
\]

\[
\rightarrow \quad C \cdot q^{n+1} = d
\]

where \( C \) is tridiagonal with diagonals \( \alpha \), \( i - 2\alpha \) and \( \alpha \),

while \( d = \begin{bmatrix}
\Delta y^2 q_2 + U_1 \\
\Delta y^2 q_3 \\
\vdots \\
\Delta y^2 q_{N-1} \\
\Delta y^2 q_N + U_2
\end{bmatrix} \)

Also, simultaneously solve the equation \( \frac{\partial^2 y}{\partial y^2} = -q \)

once we have \( \frac{\partial q}{\partial y} \) from the above

\[
\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta y^2} = -q_i
\]

which gives a tridiagonal system \( A \cdot u^{n+1} = (-\Delta y^2) \cdot u^{n+1} \)

with diagonals of \( A \) \( 1, -2, 1 \)

( Don’t forget \( u_1^{n+1} = U_1 \) and \( u_2^{n+1} = U_2 \) )

Note: Simpler implementation for this formulation.
Unidirectional Flow: Vorticity-Velocity

\[ \frac{\partial \omega}{\partial t} = - \frac{\partial (\omega w)}{\partial y} \]

\[ \omega = - \frac{\partial u}{\partial y} \]

\[ \rightarrow u(y) = U_1 - \int_0^y \omega(y') \, dy' \] ①

Note that we need two boundary conditions for \( \omega \), since the vorticity transport is 2nd order equation. They need to be such that

\[ \int_0^h \omega(y') \, dy' = U_1(+) - U_2(+) \] ②

Also, we must either specify the flow rate, or pressure gradient. Evaluating the momentum equation

\[ \frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \]

at the lower and upper walls and rearrange

\[ \left( \frac{\partial \omega}{\partial y} \right) y=0 = \frac{1}{\nu} \frac{\partial u_1}{\partial t} - \frac{2}{\mu} \frac{\partial p}{\partial x} \]

\[ \left( \frac{\partial \omega}{\partial y} \right) y=h = \frac{1}{\nu} \frac{\partial u_2}{\partial t} - \frac{2}{\mu} \frac{\partial p}{\partial x} \]

so Neumann conditions at the walls.

Next, we must ensure the vorticity BC satisfies ④

\[ \frac{d}{dt} \int_0^h \omega(y') \, dy' = \frac{d}{dt} (U_1 - U_2) \]

Alternate set of equations:

\[ \frac{\partial \omega}{\partial t} \right|_{y=0} = \frac{\partial \omega}{\partial y} \right|_{y=0} = -q \]

where \( q = \frac{\partial u}{\partial y} \)

By

\[ \frac{\partial w}{\partial t} = - \frac{\partial \omega}{\partial y} \rightarrow \frac{\partial q}{\partial t} = \nu \frac{\partial^2 q}{\partial y^2} \]

and

\[ q \right|_{y=0} = -\frac{1}{\nu} \frac{\partial u_1}{\partial t} - \frac{1}{\mu} \frac{\partial p}{\partial x} \] and

\[ q \right|_{y=h} = -\frac{1}{\nu} \frac{\partial u_2}{\partial t} - \frac{1}{\mu} \frac{\partial p}{\partial x} \]
Flow in a Cavity

\[ \Delta \rightarrow U(t) \]

**Vertically-Stream Formulation**

Let \( \psi \) be such that

\[ U = \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x} \]

No penetration condition requires the normal velocity at each wall to be zero,

\[ \psi = 0 \quad \text{over all walls} \]

(tangential derivative of \( \psi \) is zero).

No slip BC requires the tangential velocity at the bottom or side walls to be zero, and equal to \( U(t) \) on the lid.

\[ \frac{\partial \psi}{\partial y} = 0 \quad \text{at the bottom} \]
\[ \frac{\partial \psi}{\partial x} = 0 \quad \text{at the sides} \]
\[ \frac{\partial \psi}{\partial y} = U(t) \quad \text{at the top} \]

Recall equations:

\[ \omega = -\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right), \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

And

\[ \frac{\partial \omega}{\partial t} = -U \frac{\partial \omega}{\partial x} - V \frac{\partial \omega}{\partial y} + \nu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \]

Other BC's need to be reworked:

\[ V = 0, \quad \frac{\partial V}{\partial x} = 0 \quad \text{at bottom wall}, \quad \Rightarrow \omega = -\frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} \]

And similarly,

\[ \omega = -\frac{\partial^2 \psi}{\partial x^2} \quad \text{on the top and bottom} \]
\[ \omega = -\frac{\partial^2 \psi}{\partial x^2} \quad \text{at the sides} \]
Unidirectional flow: Velocity-Presure formulation

Compute the flow in a channel where the pressure gradient is specified.

\[ \frac{\partial u}{\partial t} = \frac{1}{S} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \]

\[ \begin{align*}
    u(y=0) &= U_1(t) \\
    u(y=h) &= U_2(t)
\end{align*} \]

For \( i = 2, N \)

\[ \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{S} \left( \frac{\partial p}{\partial x} \right)_i^n + \nu \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta y^2} \]

Two parallel walls at \( y=0, y=h \)
moving with velocities \( U_1(t) \) and \( U_2(t) \) respectively.

\[ \frac{dp}{dx} \] specified

- No vertical velocity
- No \( x \)-dependence for \( u \)

- Discretize with \( N+1 \) points
- Use implicit time-stepping

\[ u_i^n = u_i(i\Delta y, n\Delta t) \]

\[ u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} \]

\[ \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{S} \left( \frac{\partial p}{\partial x} \right)_i^n \]

we can rearrange, with \( \alpha = \frac{\nu \Delta t}{\Delta y^2} \)

\[ \alpha u_i^{n+1} + (1+2\alpha) u_i^n - \alpha u_{i-1}^{n+1} = u_i^n - \left( \frac{\Delta t}{S} \right) \left( \frac{\partial p}{\partial x} \right)_i^n \]

or

\[ A u_i^{n+1} = u_i^n + b \]

where \( A \) is tridiagonal with diagonals \(-\alpha, 2\alpha+1, -\alpha\).

- System can be solved efficiently with tridiagonal solvers
- Method is unconditionally stable.
**Def.** Irrotational flow \( \nabla \times \vec{u} = 0 \)

**Def.** Vorticity \( \vec{\omega} = \nabla \times \vec{u} \)

\[ \vec{\omega} = (0, 0, \omega) \]

\[ \omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \]

**Momentum Eqn.:**
\[ \frac{\partial \vec{u}}{\partial t} + (\vec{\omega} \times \vec{u}) \times \vec{u} = -\nabla \left( \frac{P}{\rho} + \frac{1}{2} u^2 + \chi \right) \]

For an ideal fluid, \( (\vec{u} \cdot \nabla) H = 0 \) where \( H = \frac{P}{\rho} + \frac{1}{2} u^2 + \chi \)

In conservative form, \( \chi = \nabla \cdot \nabla \vec{v} \)

\[ \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} = -\nabla H \]

Take a unit \( \vec{u} = 0 \) \( \frac{\partial \vec{u}}{\partial t} + \nabla (\vec{\omega} \times \vec{u}) = 0 \)

Simplify and use the fact that \( \nabla \cdot \vec{u} = 0 \) for incompressible flows.

\[ \frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u} \]

Or \( \frac{\partial \vec{\omega}}{\partial t} = (\vec{\omega} \cdot \nabla) \vec{u} \) for Euler flows.

**2D:** Pressure is eliminated.

\[ \vec{u} = (u_1, u_2, 0) \]

\[ \vec{\omega} = (0, 0, \omega) \]

\[ (\vec{\omega} \cdot \nabla) \vec{u} = \omega \frac{\partial \vec{u}}{\partial z} = 0 \]

\[ \frac{\partial \vec{\omega}}{\partial t} = 0 \]

In 2D ideal fluid subject to a conservative force, the vorticity \( \vec{\omega} \) of each individual fluid particle is conserved.
By Stokes Theorem, \[ \oint_C \vec{u} \cdot d\vec{s} = \iint_S (\nabla \times \vec{u}) \cdot d\vec{S} \]

\[ M = \oint_C (y\,dx + u_2\,dy) = \iint_S \left( \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \, dx \, dy \]

\[ \to M = 0 \quad \text{for any} \quad C \quad \text{not encircling a wing} \]

**Velocity potential** \( \Phi \) exists only if \( \nabla \times \vec{u} = 0 \) \( \Rightarrow \vec{u} = \nabla \Phi \)

\[ M = \oint_C \vec{u} \cdot d\vec{s} = \iint_S \nabla \Phi \cdot d\vec{S} = \left[ \Phi \right]_C \]

**Streamfunction** useful for representing incompressible 2D flow.

\[ u_1 = \frac{\partial \psi}{\partial y}, \quad u_2 = \frac{-\partial \psi}{\partial x} \]

\[ \Rightarrow \text{automatically satisfies incompressibility} \quad \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \]

Another property: \( (\vec{u} \cdot \nabla) \psi = u_1 \frac{\partial \psi}{\partial x} + u_2 \frac{\partial \psi}{\partial y} = 0 \)

**Complex potential** 2D, incompressible, irrotational.

\[ u_1 = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad u_2 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \]

Cauchy-Riemann Eqs. with \( W = \phi + i\psi \) of \( z = x + iy \)

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \]

Properties of \( W \)

\[ \frac{dW}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = u_1 - iu_2 \]

Flow speed at any point \( q = (u^2 + v^2)^{1/2} = \left| \frac{dW}{dz} \right| \)
Vorticity Equation for Euler Flow

\[ \frac{\partial \omega}{\partial t} = (\omega \cdot \nabla) \mathbf{u} \]

Example (Vortex stretching) Assume \( \omega = \omega \mathbf{k} \)

\[ \frac{\partial \omega}{\partial t} = \omega \frac{\partial \mathbf{u}}{\partial z} \Rightarrow \text{2-component is} \quad \frac{\partial \omega}{\partial t} = \omega \frac{\partial u_3}{\partial z} \]

So vorticity of a particular fluid element increases with time if \( \frac{\partial u_3}{\partial z} > 0 \), that is, if instantaneous vertical velocity increases with \( z \).

Example: Axisymmetric flow \( \mathbf{u} = u_R \mathbf{e}_R + u_\theta \mathbf{e}_\theta \)

\[ \frac{\partial \omega}{\partial t} = 0 \quad \Rightarrow \quad \omega = \frac{\partial u_R}{\partial z} - \frac{\partial u_\theta}{\partial R} \]

\[ \Rightarrow \text{we can show} \quad \frac{D}{Dt} \left( \frac{\omega}{R} \right) = 0 \]

Vorticity of any fluid element changes in proportion to \( R \) in time. (Here vortex tubes are aligned with \( z \)-axis)

Note: (Vorticity Transport)

In 2D, \( \frac{\partial \omega}{\partial t} = 0 \)

\[ \omega (X(x,t),t) = \omega_0 (x) \]

That is, vorticity is transported by the fluid; vorticity of each individual fluid element is conserved.

(This is a Lagrangian flow result.)
Vertically Eqn for N-S flow

In a viscous fluid (N-S), there’s also diffusion of vorticity.

Taking a curl of the momentum eqn

\[ \frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = (\omega \cdot \nabla) \mathbf{u} + \nu \nabla^2 \omega \]

or \[ \frac{\partial \omega}{\partial t} = (\omega \cdot \nabla) \mathbf{u} + \nu \nabla^2 \omega \]

One exact soln: Burger’s Vortex

\[ u_r = \frac{-1}{\alpha^2} \frac{x r}{2}, \quad u_\theta = \alpha x, \quad u_z = \frac{M}{2\pi r} \left(1 - e^{-\alpha r^2/4\nu}\right) \]

(Cylindrical coordinates)

\[ \alpha = \text{const} > 0 \]

\[ M = \text{circulation const} \]

\[ \omega = \frac{\alpha M}{4\pi \nu} e^{-\alpha r^2/4\nu} \frac{\partial}{\partial z} \]

Vertically is concentrated in a vortex core of radius \( O\left(\sqrt{\frac{1}{\nu}}\right) \)

- Core is smaller for small viscosity fluids
- Without diffusion (\( \nu = 0 \)), the secondary flow makes the vortex stronger as time \( t \)

(That is known as vortex stretching.)
\[ \begin{align*}
\mathbf{N-S} : & \quad \begin{cases}
\frac{D\mathbf{u}}{Dt} = -\nabla p + \nu \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} = 0
\end{cases} \\
\text{Taking a curl,} & \quad \frac{D\mathbf{\omega}}{Dt} = \mathbf{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{\omega}
\end{align*} \]

Because \( \nabla \cdot \mathbf{u} = 0 \) \( \Rightarrow \) unique stream function \( \Psi \) so that

\[ \mathbf{u} = (-\Psi_y, \Psi_x) = \nabla^\perp \Psi \]

Then note \( \nabla \times \mathbf{u} = \nabla^2 \Psi \) so \( \mathbf{\omega} = \nabla^2 \Psi \).

Thus we can write \( \mathbf{N-S} \) as a closed system with these variables,

\[ \begin{cases}
\frac{D\mathbf{\omega}}{Dt} = \mathbf{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{\omega} \\
\mathbf{u} = \nabla^\perp \Psi \\
\nabla^2 \Psi = \mathbf{\omega}
\end{cases} \]

Notes:
- closed system
- no pressure
- incompressibility is included.