Hedetniemi's Conjecture

and

Shannon Capacity

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Hedetniemi

Hed – et – <u>knee</u>– eh – me

“ The name Hedetniemi comes from Finland, although the spelling of the name was created by a clerk at Ellis Island, because that is what the name, when spoken, sounded like to him. The suffix -niemi in Finn means peninsula. The prefix Hedet-, I was told by a Finnish friend, translates to wet or swampy. Many names in Finland in the old days were descriptive of the location of a farm where the person lived and/or worked. ”

Hedetniemi’s Conjecture is a well-known conjecture about graph coloring.

Recently Yaroslav Shitov disproved it.

Replacing graph coloring by other concepts gives other “Hedetniemi problems”.

Gabor Simonyi poses the “Hedetniemi problem for Shannon capacity” and makes a connection to the asymptotic spectrum of graphs.

Graphs

The basic algebraic operations on numbers…

$$a + b$$

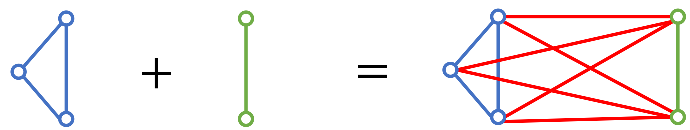
$$a * b$$

$$a \leq b$$

$$\min\{a, b\}$$

…extend to graphs.

We can add graphs

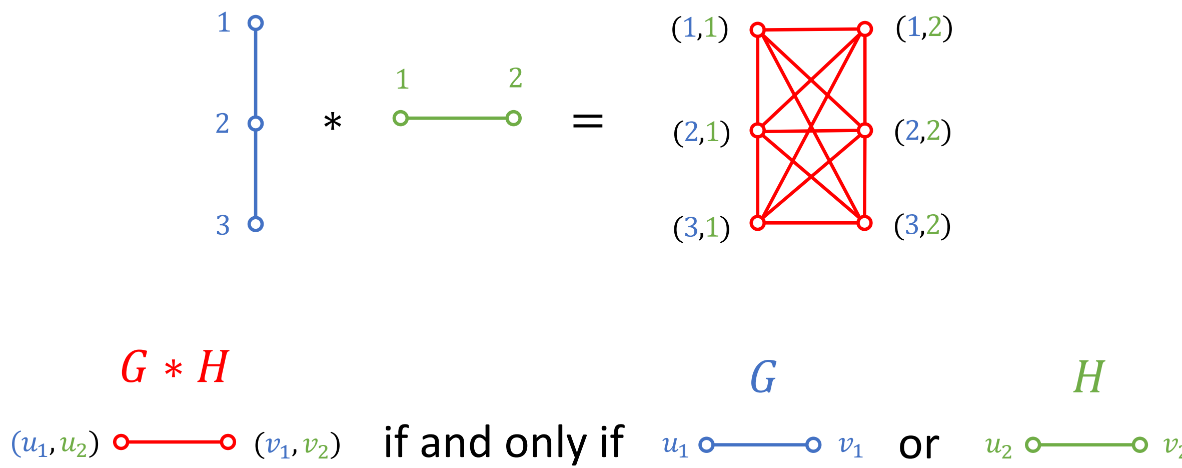


0	1	1
1	0	1
1	1	0

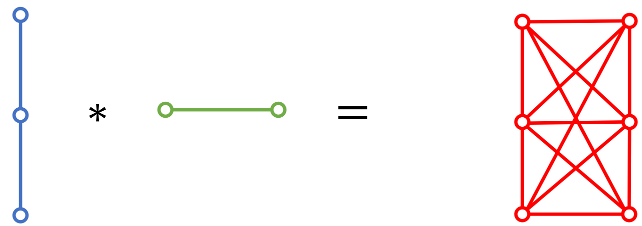
0	1
1	0

0	1	1	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

We can multiply graphs



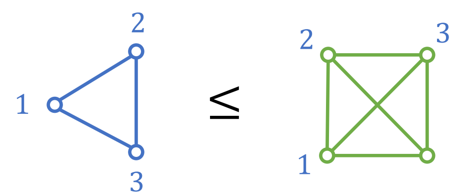
We can multiply graphs

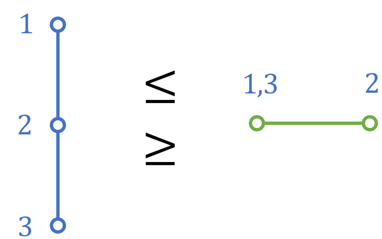


$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \otimes \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} + \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \otimes \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} = \begin{matrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{matrix}$$

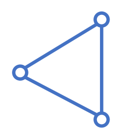
We can compare graphs

$G \leq H$ if and only if homomorphism $G \rightarrow H$



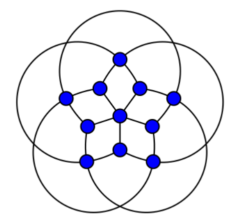


Actually, some graphs are incomparable:



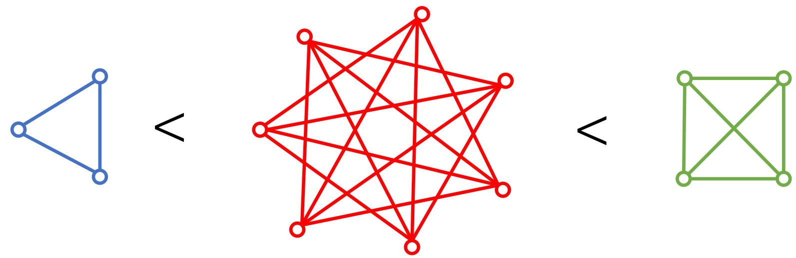
$\not\leq$

$\not\geq$



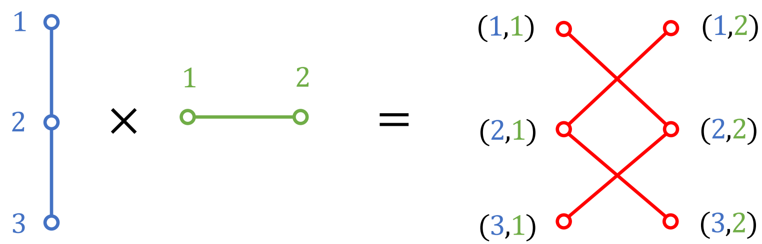
Grötzsch graph

In other ways, graphs behave like real numbers again:

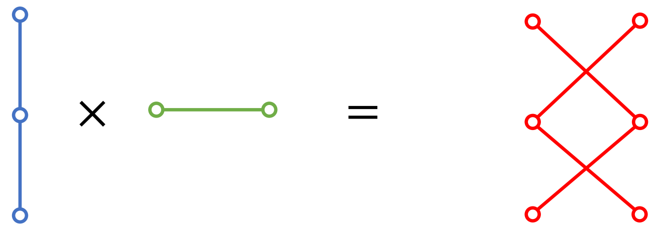


K_3 $K_{7/2}$ K_4

We can take the minimum of two graphs

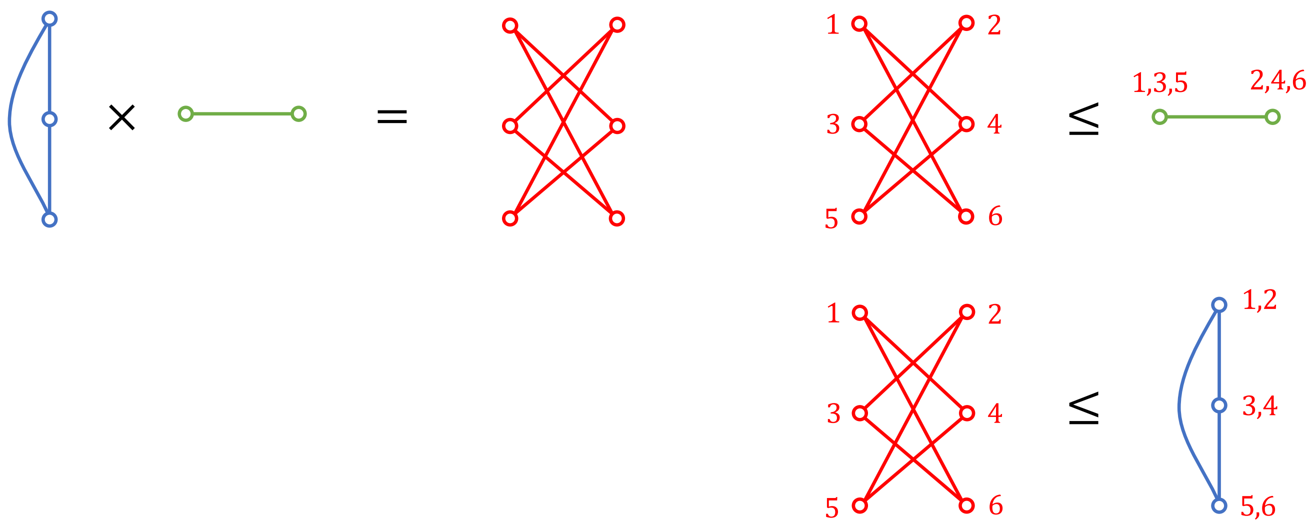


We can take the <u>minimum</u> of two graphs



$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \otimes \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} = \begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix}$$

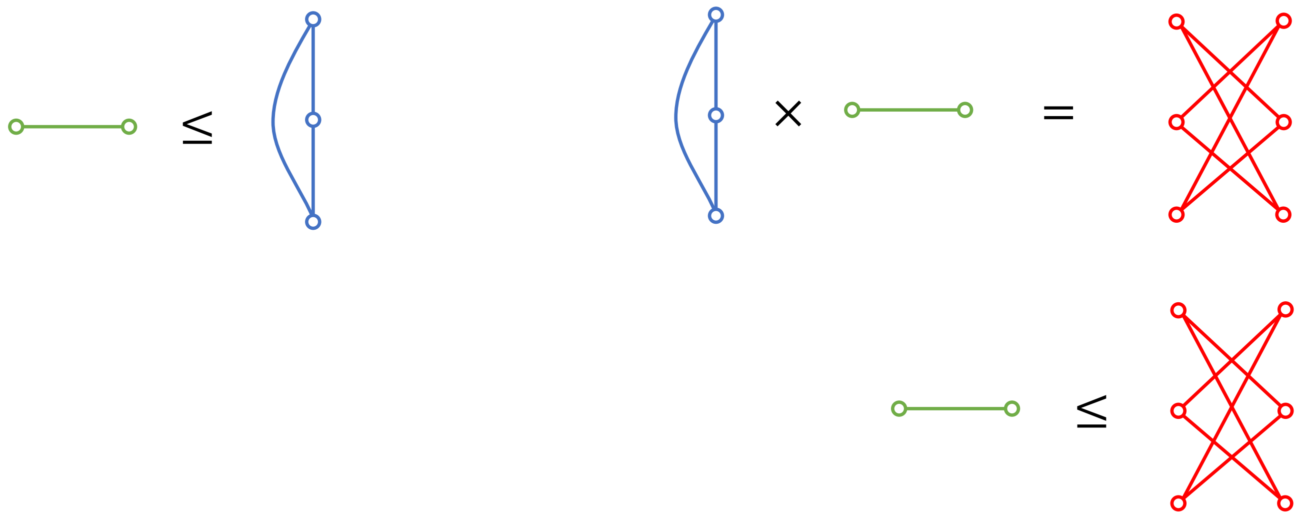
Surely the minimum is smaller than the operands?



Lemma

$G \times H \leq G$ and $G \times H \leq H$

Surely the smaller graph is at most the minimum?



Lemma

$G \leq H \Rightarrow G \leq G \times H$

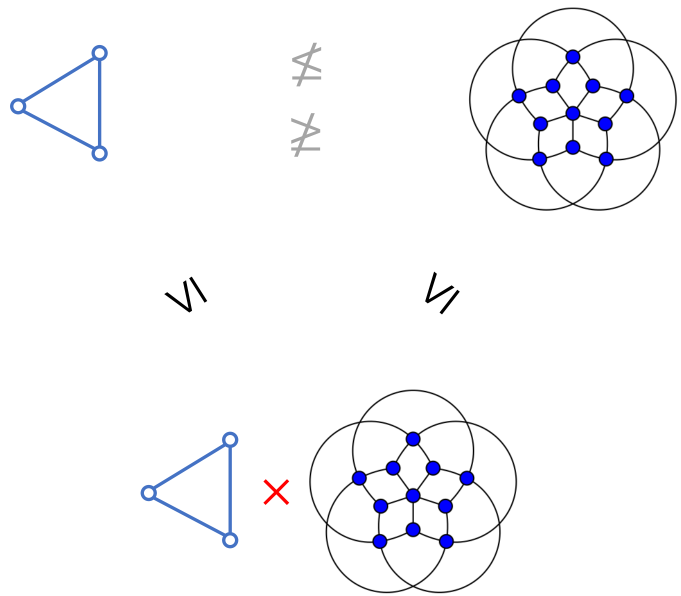
(and $G \times H \leq G$ always holds)

Lemma

$G \times H \leq G$ and $G \times H \leq H$

$G \leq H \Rightarrow G \leq G \times H$

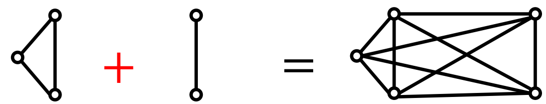
The point is that minimum is also defined for *incomparable* graphs

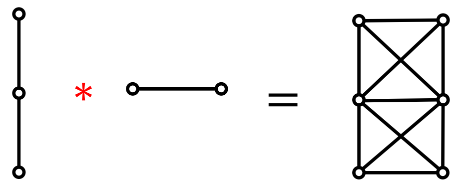


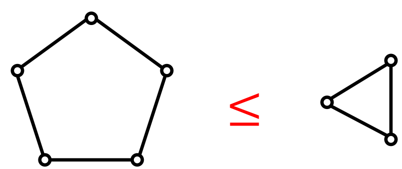
Lemma

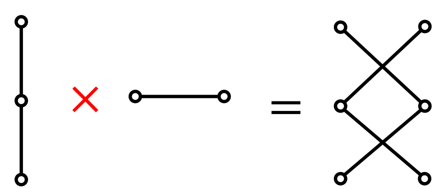
If $K \leq G$ and $K \leq H$ then $K \leq G \times H$.

The basic algebraic operations on numbers extend to graphs

addition  join

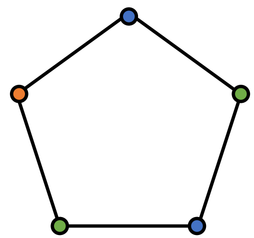
multiplication  or-product

comparison  homomorphism

minimum  categorical product

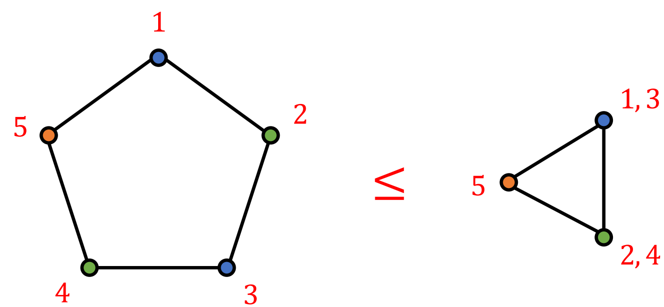
Hedetniemi's Conjecture

Chromatic number $\chi(G)$ is the minimum number of colors in proper coloring of G



has chromatic number 3

Chromatic number $\chi(G)$ is monotone



Lemma: If $G \leq H$ then $\chi(G) \leq \chi(H)$.

Hedetniemi's conjecture (1966) states that:

"Chromatic number of minimum equals minimum of chromatic numbers."

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

Recall:

$G \times H \leq H$ and $G \times H \leq G$.

If $A \leq B$ then $\chi(A) \leq \chi(B)$.

So the *upper bound* is always true:

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$$

The conjecture is true for comparable graphs:

If $G \leq H$ then $G \times H \;\substack{\leq \\ \geq}\; G$

so $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$

The general conjecture is false! (Shitov 2019)

There are G and H such that $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$.

To prove: There are G and H such that $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$.

Exponential graph: $K_n^{\,G}$

Vertices: all maps $\phi : V(G) \to [n]$

Edges: $\phi \sim \psi \iff \forall\, x \sim y \;\; \phi(x) \neq \psi(y)$

$\chi(G \times K_n^G) \leq n$

Proof: coloring $\psi(v, \phi) = \phi(v)$

New goal: Find a graph G such that $\chi(G) > n$ and $\chi(K_n^G) > n$.

Theorem: If H satisfies a bunch of properties, then

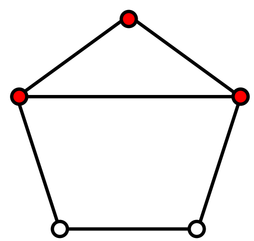
$\chi(H[K_q]) > n$

$\chi(K_n^{\,H[K_q]}) > n$

($H[K_q]$ is the q-blow-up of H)

Shannon capacity

Clique number $\omega(G)$ is the size of the largest clique in G



has clique number 3

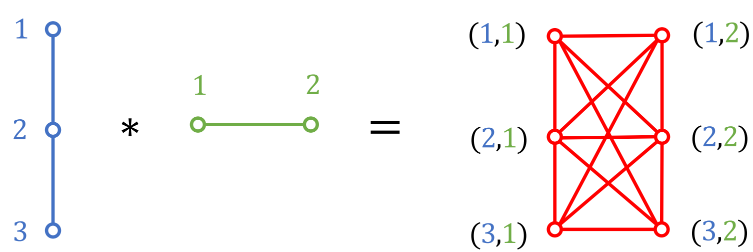
The clique number is monotone and actually satisfies the Hedetniemi equality!

$$\omega(G \times H) = \min\{\omega(G), \omega(H)\}$$

Shannon capacity $\Theta(G)$ is the rate of growth of the clique number under $*$-powers

$$\Theta(G) = \lim_{n\to\infty} \omega(G^{*n})^{1/n}$$

Recall:



$$\Theta(G) = \lim_{n\to\infty} \omega(G^{*n})^{1/n}$$

Question: Does the Shannon capacity satisfy the Hedetniemi equality? (Simonyi)

$$\Theta(G \times H) = \min\{\Theta(G), \Theta(H)\}\,?$$

Again, the upper bound is always true:

$$\Theta(G \times H) \leq \min\{\Theta(G), \Theta(H)\}$$

The asymptotic spectrum of graphs models graphs as real numbers

Model of graphs as real numbers

$$\varphi : \{\text{graphs}\} \to \mathbb{R}_{\geq 0}$$

that maintains consistency:

$$\varphi(G * H) = \varphi(G) * \varphi(H)$$

$$\varphi(G + H) = \varphi(G) + \varphi(H)$$

$$\varphi(K_1) = 1$$

$$G \leq H \Rightarrow \varphi(G) \leq \varphi(H)$$

Examples: Lovász theta number ϑ, fractional chromatic number

$$\Theta(G) = \lim_{n\to\infty} \omega(G^{*n})^{1/n}$$

Shannon capacity equals the minimum over consistent models (Strassen)

For every G

$$\Theta(G) = \min\{\varphi(G) : \varphi \text{ consistent}\}.$$

Connecting Hedetniemi for Shannon capacity to asymptotic spectrum (Simonyi)

For every G and H, if

$$\Theta(G\times H) < \min\{\Theta(G),\Theta(H)\}$$

then there is a consistent model φ such that

$$\varphi(G\times H) < \min\{\varphi(G),\varphi(H)\}$$

Proof:

1. $\varphi(G\times H) = \Theta(G\times H)$
2. Assume: $\varphi(G\times H) = \min\{\varphi(G),\varphi(H)\}$
3. $\min\{\Theta(G),\Theta(H)\} \geq \Theta(G\times H) = \varphi(G\times H) = \min\{\varphi(G),\varphi(H)\} \geq \min\{\Theta(G),\Theta(H)\}$
4. Conclude: $\min\{\Theta(G),\Theta(H)\} = \Theta(G\times H)$

Does the asymptotic spectrum of graphs change if we require the Hedetniemi equality?

Model of graphs as real numbers

$$\varphi : \{\text{graphs}\} \to \mathbb{R}_{\geq 0}$$

that maintains stronger consistency:

$$\varphi(G * H) = \varphi(G) * \varphi(H)$$

$$\varphi(G + H) = \varphi(G) + \varphi(H)$$

$$\varphi(K_1) = 1$$

$$G \leq H \Rightarrow \varphi(G) \leq \varphi(H)$$

$$\varphi(G \times H) = \min\{\varphi(G), \varphi(H)\}$$

Examples: Lovász theta number ϑ, fractional chromatic number

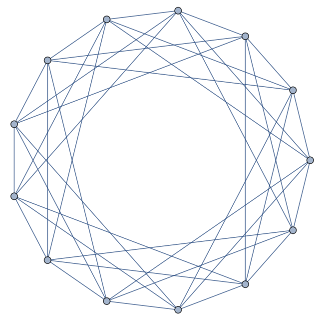
Candidate counter example needs to avoid the general bounds…

$$\max\{\Theta(G'), \Theta(H') : \begin{array}{l} G' \subseteq G,\ H' \subseteq H, \\ G' \leq H,\ H' \leq G \end{array}\} \leq \Theta(G \times H) \leq \min\{\Theta(G), \Theta(H)\}$$

…being equal.

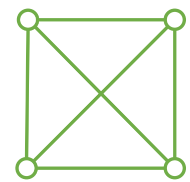
Candidate counter example to Hedetniemi equality for Shannon capacity

(Simonyi)



P_{13}

$\times$



K_4

$$\Theta(P_{13} \times K_4) \overset{?}{<} \min\{\Theta(P_{13}), \Theta(K_4)\} = \sqrt{13} = 3.6$$

Question

Do all elements in the asymptotic spectrum of graphs satisfy the Hedetniemi equality?

Question

What problems in mathematics and computer science naturally involve $+, \ *, \ \leq, \ \min$?