Practical Advances in Complex Root Clustering

Collaborative and ongoing works

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Example

System: Let $\sigma \geq 3$ and $f(z) = 0$ be:

$$\begin{align*}
(z_1 - 2^{-\sigma})(z_1 + 2^{-\sigma}) &= 0 \\
(z_2 + 2^\sigma z_1^2)(z_2 - 1)z_2 &= 0
\end{align*}$$

Solutions: $f(z) = 0$ has 6 solutions, all real:

$$\begin{align*}
a_1 &= (2^{-\sigma}, 0) \\
a_2 &= (2^{-\sigma}, 1) \\
a_3 &= (-2^{-\sigma}, 1) \\
a_4 &= (-2^{-\sigma}, 0) \\
a_5 &= (-2^{-\sigma}, -2^{-\sigma}) \\
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Natural clusters:

- $(\Delta^1, 4)$
- $(\Delta^2, 2)$

Notations: $m(a, f)$: multiplicity of $a$ as a sol. of $f$
Example

System: Let $\sigma \geq 3$ and $f(z) = 0$ be:

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\begin{align*}
(z_1 - 2^{-\sigma})^2(z_1 + 2^{-\sigma}) &= 0 \\
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Solutions: $f(z) = 0$ has 6 solutions, all real:

- $a^1 = (2^{-\sigma}, 0) \quad \leftarrow m(a^1, f) = 2$
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Local solution Clustering Problem (LCP)

Input: a polynomial map \( f : \mathbb{C}^n \to \mathbb{C}^n \) (assume \( f(z) = 0 \) is 0-dim), a polybox \( B \subset \mathbb{C}^n \), the Region of Interest (RoI), \( \epsilon > 0 \)

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Notations: \( f = (f_1, \ldots, f_n) \), \( B = (B_1, \ldots, B_n) \) where the \( B_i \)'s are square complex boxes
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- \( Z(B, f) \subseteq \bigcup_{j=1}^\ell Z(\Delta^j, f) \subseteq Z((1+\delta)B, f) \) for a small \( \delta \)

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Definition: a pair \( (\Delta, m) \) is called natural cluster (relative to \( f \)) when it satisfies:

\[
m = \#(\Delta, f) = \#(3\Delta, f) \geq 1
\]

if \( r(\Delta) \leq \epsilon \), it is a natural \( \epsilon \)-cluster
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$(\Delta^3, 3)$, $(\Delta^4, 6)$ are not natural clusters
Why root clustering instead of root isolation?

Root isolation:
- input polynomials with \( \mathbb{Z} \) or \( \mathbb{Q} \) coefficients, or
- input polynomials squarefree

Root clustering:
- input polynomials with any \( \mathbb{C} \) coefficients
- robust to multiple roots
0 - Univariate case:

Complexity analysis of root clustering for a complex polynomial.

Near optimal: bit complexity $\tilde{O}(d^2(\sigma + d))$
for the benchmark problem

Efficient implementation *Ccluster* described in

Implementation of a near-optimal complex root clustering algorithm.

Notations: $d, \sigma$: degree, bit-size of $f$
Menu

0 - Univariate case:

1 - Multivariate triangular case

[IPY19] Rémi Imbach, Marc Pouget, and Chee Yap.
Clustering complex zeros of triangular systems of polynomials.

\[
\begin{align*}
  f_1(z_1) &= 0 \\
  f_2(z_1, z_2) &= 0 \\
  &\vdots \\
  f_n(z_1, z_2, \ldots, z_n) &= 0
\end{align*}
\]

, \deg_{z_i}(f_i) \geq 1

with: finite number of sols
Symbolic-Numeric solving of systems of polynomials:

\[
\begin{align*}
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rewriting step

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Symbolic-Numeric solving of systems of polynomials:

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seq. times in s on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz
asked precision: 53 bits

\[ S_4 \begin{cases} 
 z_1^4 - 57 * z_1^2 * z_2 - 86 * z_1 * z_2^2 - 160 * z_2^3 + 95 * z_2^2 * z_3 + 35 * z_1^2 - 106 * z_3 = 0 \\
 z_2^4 - 64 * z_2^3 - 190 * z_1 * z_2 + 186 * z_1 * z_3 - 119 * z_2 * z_3 + 188 * z_3 + 93 = 0 \\
 z_3^4 + 116 * z_1 * z_2^2 - 168 * z_1 * z_2 * z_3 + 135 * z_1 * z_3^2 + 29 * z_3^3 - 8 * z_1 * z_3 + 119 * z_2 * z_3 = 0 
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Symbolic-Numeric solving of systems of polynomials:

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\]
Menu

0 - Univariate case:

1 - Multivariate triangular case

2 - Back to univariate case

- polynomials with real coefficients
- new counting test

New practical advances in polynomial root clustering.
In *MACIS 19*, 2019.
Menu

0 - Univariate case:

Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

**Oracle for $\alpha$:** function $O_\alpha : \mathbb{Z} \to \square \mathbb{C}$

such that $\alpha \in O_\alpha (L)$ and $w(O_\alpha (L)) \leq 2^{-L}$

**Notations:** $\square \mathbb{C}$: set of complex interval
Oracle numbers and polynomials

Let \( \alpha \in \mathbb{C} \).

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Let \( f \in \mathbb{C}[z_1, \ldots, z_n] \)

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\[ \text{s.t. } f \in O_f(L) \text{ and } w(O_f(L)) \leq 2^{-L} \]

\( \simeq \text{ oracles for the coeffs of } f \)

**Notations:**
- \( \Box \mathbb{C} \): set of complex interval
- \( \Box \mathbb{C}[z_1, \ldots, z_n] \): polynomials with coefficients in \( \Box \mathbb{C} \)
Outline of [BSS+16]

Counting test: \( T^* : (\Delta, \mathcal{O}_f) \mapsto m \in \{-1, 0, \ldots, d\} \)
\[ T^*(\Delta, \mathcal{O}_f) \geq 0 \Rightarrow \#(\Delta, f) = m \]

Discarding test: \( T^0 : (\Delta, \mathcal{O}_f) \mapsto m \in \{-1, 0\} \)
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Subdivision approach:

Notations: \( \#(S, f) : \text{sum of multiplicities of roots of } f \text{ in } S \)
\( d: \text{degree of } f \)
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The Pellet’s test

**Pellet’s Theorem:** Let $\Delta$ be a complex disc centered in $c$ and radius $r$. Let $f \in \mathbb{C}[z]$, $d = \deg(f)$ and $f_\Delta = f(c + rz)$.

If $\exists \ 0 \leq m \leq d$ s.t.

$$|(f_\Delta)_m| > \sum_{i \neq k} |(f_\Delta)_i|$$ (1)

then $f$ has exactly $m$ roots in $\Delta$.

Notations: $(f)_m$: coeff. of the monomial of degree $m$ of $f$
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then $f$ has exactly $m$ roots in $\Delta$.

If $f$ has no root in this annulus $\rightarrow$ 
$\exists m$ s.t. eq. (1) holds.

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**Pellet’s Theorem:** Let $\Delta$ be a complex disc centered in $c$ and radius $r$. Let $f \in \mathbb{C}[z]$, $d = \text{deg}(f)$ and $f_\Delta = f(c + rz)$.

If $\exists 0 \leq m \leq d$ s.t.

$$ |(f_\Delta)_m| > \sum_{i \neq k} |(f_\Delta)_i| $$

then $f$ has exactly $m$ roots in $\Delta$.

With Dandelin-Gräffe’s iterations:
If $f$ has no root in this annulus $\rightarrow$
$\exists m$ s.t. eq. (1) holds.

**Notations:** $(f)_m$: coeff. of the monomial of degree $m$ of $f$
The Pellet’s test

Pellet’s Theorem: Let $\Delta$ be a complex disc centered in $c$ and radius $r$. Let $f \in \mathbb{C}[z]$, $d = \deg(f)$ and $f_{\Delta} = f(c + rz)$.

If $\exists \ 0 \leq m \leq d \ s.t.
\begin{equation}
|(f_{\Delta})_m| > \sum_{i \neq k} |(f_{\Delta})_i|
\end{equation}

then $f$ has exactly $m$ roots in $\Delta$.

PelletTest($\Delta, f$) //Output in \{-1, 0, 1, \ldots, d\}

1. compute $f_{\Delta}$
2. for $m$ from 0 to $d$ do
3. \hspace{1em} if $|(f_{\Delta})_m| > \sum_{i \neq k} |(f_{\Delta})_i|$
4. \hspace{1em} return $m$ //m roots (with mult.) in $\Delta$
5. return $-1$ //Roots near the boundary of $\Delta$
The soft Pellet’s test: for interval polynomials

Pellet’s Theorem: Let $\Delta$ be a complex disc centered in $c$ and radius $r$. Let $f \in \mathbb{C}[z]$, $d = \deg(f)$ and $f_\Delta = f(c + rz)$.

If $\exists \ 0 \leq m \leq d$ s.t.
\[
|f_\Delta|^m > \sum_{i \neq k} |f_\Delta|^i
\]  

then $f$ has exactly $m$ roots in $\Delta$. 

\[
\text{(2)}
\]
The soft Pellet’s test: for interval polynomials

Pellet’s Theorem: Let $\Delta$ be a complex disc centered in $c$ and radius $r$. Let $f \in \mathbb{C}[z]$, $d = \text{deg}(f)$ and $f_\Delta = f(c + rz)$.

If $\exists 0 \leq m \leq d$ s.t.

$$|(f_\Delta)_m| > \sum_{i \neq k} |(f_\Delta)_i|$$  \hspace{1cm} (2)

then $f$ has exactly $m$ roots in $\Delta$.

SoftCompare($\Box a$, $\Box b$)  // $\Box a$, $\Box b$ are real intervals

**Input:** $\Box a$, $\Box b$ real intervals

**Output:** a number in $\{-2, -1, 1\}$ s.t.:

1. $1 \Rightarrow \Box a > \Box b$
2. $-1 \Rightarrow \Box a < \Box b$ or $\Box a$, $\Box b$ are too close
3. $-2 \Rightarrow \Box a \cap \Box b \neq \emptyset$
The soft Pellet’s test: for interval polynomials

**SoftPelletTest**(\(\Delta, \square f\))  
// Output in \(\{-2, -1, 0, 1, \ldots, d\}\)

1. compute \(\square f_\Delta\)
2. for \(m\) from 0 to \(\text{deg}\) do
3. \(R \leftarrow \text{SoftCompare}(|(\square f_\Delta)_m|, \sum_{i \neq k} |(\square f_\Delta)_i|)\)
4. if \(R \geq 0\) then return \(m\)  // any \(f \in \square f\) has \(m\) roots  
   // (with mult.) in \(\Delta\)
5. if \(R = -2\) then return \(-2\)  // \(\square f\) is too wide
6. return \(-1\)  // Roots near the boundary of \(\Delta\)

**SoftCompare**(\(\square a, \square b\))  // \(\square a, \square b\) are real intervals

**Input:** \(\square a, \square b\) real intervals

**Output:** a number in \(\{-2, -1, 1\}\) s.t.:  
1 \(\Rightarrow \square a > \square b\)

\(-1 \Rightarrow \square a < \square b\) or \(\square a, \square b\) are too close

\(-2 \Rightarrow \square a \cap \square b \neq \emptyset\)
The soft Pellet’s test: for oracle polynomials

\begin{align*}
\text{SoftPelletTest}(\Delta, \mathcal{F}) & \quad \text{//Output in \{-2, -1, 0, 1, \ldots, d\}} \\
1. & \text{compute } \mathcal{F}\Delta \\
2. & \text{for } m \text{ from } 0 \text{ to } \deg \text{ do} \\
3. & \quad R \leftarrow \text{SoftCompare}(|(\mathcal{F}\Delta)_m|, \sum_{i \neq k} |(\mathcal{F}\Delta)_i|) \\
4. & \quad \text{if } R \geq 0 \text{ then return } m \quad \text{//any } f \in \mathcal{F} \text{ has } m \text{ roots} \\
5. & \quad \text{if } R = -2 \text{ then return } -2 \quad \text{// } \mathcal{F} \text{ is too wide} \\
6. & \quad \text{return } -1 \quad \text{//Roots near the boundary of } \Delta \\
\end{align*}

Loop on precision:
The soft Pellet’s test: for oracle polynomials

**SoftPelletTest(Δ, □f)**

1. compute □fΔ
2. for m from 0 to deg do
3. \( R \leftarrow \text{SoftCompare}(|(□f_\Delta)_m|, \sum_{i \neq k} |(□f_\Delta)_i|) \)
4. if \( R \geq 0 \) then return \( m \)
   \( \quad \text{//any } f \in □f \text{ has } m \text{ roots} \)
   \( \quad \text{// (with mult.) in } \Delta \)
5. if \( R = -2 \) then return \(-2\)
   \( \quad \text{//□f is too wide} \)
6. return \(-1\)
   \( \quad \text{//Roots near the boundary of } \Delta \)

Loop on precision:

**T*(Δ, O_f)**

1. \( L \leftarrow 53, □f \leftarrow O_f(L), m \leftarrow \text{SoftPelletTest}(\Delta, □f) \)
2. while \( m = -2 \) do
3. \( L \leftarrow 2L, □f \leftarrow O_f(L), m \leftarrow \text{SoftPelletTest}(\Delta, □f) \)
4. return \( m \)
Univariate root clustering algorithms

ClusterOracle: solves the LCP in 1D ([BSS+16])

\( T^* \) embedded in a subdivision framework
accepts oracle polynomials in input

Complexity analysis of root clustering for a complex polynomial.
Univariate root clustering algorithms

ClusterOracle: solves the LCP in 1D ([BSS+16])
- $T^*$ embedded in a subdivision framework
- accepts oracle polynomials in input

ClusterInterval: solves the LCP in 1D
- Input: interval polynomial
- Output: a flag in $\{\text{success, fail}\}$, a list of natural clusters
  - SoftPelletTest embedded in a subdivision framework
  - returns fail when SoftPelletTest returns -2

Complexity analysis of root clustering for a complex polynomial.

Menu

0 - Univariate case:

1 - Multivariate triangular case


Rational, bivariate

\[
\begin{align*}
\left\{ \begin{array}{l}
f_1(z_1) = 0 \\
f_2(z_1, z_2) = 0
\end{array} \right. , \text{deg}_z(f_i) \geq 1, f_i \in \mathbb{Q}[z_1, z_2]
\]

Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

**Oracle for $\alpha$:** function $O_\alpha : \mathbb{Z} \rightarrow \square \mathbb{C}$

s.t. $\alpha \in O_\alpha(L)$ and $w(O_\alpha(L)) \leq 2^{-L}$

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$

**Oracle for $f$:** function $O_f : \mathbb{Z} \rightarrow \square \mathbb{C}[z_1, \ldots, z_n]$

s.t. $f \in O_f(L)$ and $w(O_f(L)) \leq 2^{-L}$

$\simeq$ oracles for the coeffs of $f$

Let $f_2 \in \mathbb{Q}[z_1, z_2]$ and $\alpha_1 \in \mathbb{C}$

Partial specialization of $f_2$: $f_2(\alpha_1) \in \mathbb{C}[z_2]$

**Notations:**

- $\square \mathbb{C}$: set of complex interval
- $\square \mathbb{C}[z_1, \ldots, z_n]$: polynomials with coefficients in $\square \mathbb{C}$
Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

**Oracle for $\alpha$:** function $O_{\alpha} : \mathbb{Z} \rightarrow \mathcal{L} \mathbb{C}$

such that $\alpha \in O_{\alpha}(L)$ and $w(O_{\alpha}(L)) \leq 2^{-L}$

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$

**Oracle for $f$:** function $O_{f} : \mathbb{Z} \rightarrow \mathcal{L} \mathbb{C}[z_1, \ldots, z_n]$

such that $f \in O_{f}(L)$ and $w(O_{f}(L)) \leq 2^{-L}$

$\simeq$ oracles for the coeffs of $f$

Let $f_2 \in \mathbb{Q}[z_1, z_2]$ and $\bar{\alpha}_1 \in \mathcal{L} \mathbb{C}$

**Partial specialization of $f_2$:** $f_2(\bar{\alpha}_1) \in \mathcal{L} \mathbb{C}[z_2]$

**Notations:**

- $\mathcal{L} \mathbb{C}$: set of complex interval
- $\mathcal{L} \mathbb{C}[z_1, \ldots, z_n]$: polynomials with coefficients in $\mathcal{L} \mathbb{C}$
Number of solutions in a polydisc

Let $\Delta = (\Delta_1, \Delta_2)$ and $m = (m_1, m_2)$.

**Proposition 1:** Suppose

(i) $f_1$ has $m_1$ roots in $\Delta_1$ with multiplicity

(ii) $\forall \alpha_1 \in Z(\Delta_1, f_1)$, $f_2(\alpha_1)$ has $m_2$ roots in $\Delta_2$ with multiplicity

Then $f(z) = 0$ has $m_2 \times m_1$ solutions in $\Delta$ with multiplicity.
Number of solutions in a polydisc

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Then $f(z) = 0$ has $m_2 \times m_1$ solutions in $\Delta$ with multiplicity.

**Proof**: direct consequence of

**Theorem [ZFX11]**: Let $\alpha \in Z(\mathbb{C}^2, f), \alpha = (\alpha_1, \alpha_2)$. Then

$$m(\alpha, f) = m(\alpha_2, f_2(\alpha_1)) \times m(\alpha_1, f_1)$$

[ZFX11] Zhihai Zhang, Tian Fang, and Bican Xia.

Real solution isolation with multiplicity of zero-dimensional triangular systems.

Example

System: Let $\sigma \geq 3$ and $f(z) = 0$ be:

$$\begin{cases}
(z_1 - 2^{-\sigma})^2(z_1 + 2^{-\sigma}) = 0 \\
(z_2 + 2^{\sigma} z_1^2)^2(z_2 - 1)z_2 = 0
\end{cases}$$

Solutions: $f(z) = 0$ has 6 solutions, all real:

\[a^1 = (2^{-\sigma}, 0) \quad \leftarrow \quad m(a^1, f) = \frac{2}{1 \times 2}\]
\[a^2 = (2^{-\sigma}, 1) \quad \leftarrow \quad m(a^2, f) = \frac{2}{1 \times 2}\]
\[a^3 = (-2^{-\sigma}, 1) \quad \leftarrow \quad m(a^3, f) = \frac{1}{1 \times 1}\]
\[a^4 = (-2^{-\sigma}, 0) \quad \leftarrow \quad m(a^4, f) = \frac{1}{1 \times 1}\]
\[a^5 = (-2^{-\sigma}, -2^{-\sigma}) \quad \leftarrow \quad m(a^5, f) = \frac{2}{2 \times 1}\]
\[a^6 = (2^{-\sigma}, -2^{-\sigma}) \quad \leftarrow \quad m(a^6, f) = \frac{4}{2 \times 2}\]

Natural clusters:

$(\Delta^1, 9)$
$(\Delta^2, 3)$

Notations: $m(a, f)$: multiplicity of $a$ as a sol. of $f$
Example

System: Let $\sigma \geq 3$ and $f(z) = 0$ be:

$$\begin{cases} (z_1 - 2^{-\sigma})^2(z_1 + 2^{-\sigma}) = 0 \\ (z_2 + 2^\sigma z_1^2)^2(z_2 - 1)z_2 = 0 \end{cases}$$

Solutions: $f(z) = 0$ has 6 solutions, all real:

- $a_1 = (2^{-\sigma}, 0) \leftarrow m(a_1, f) = 2 = 1 \times 2$
- $a_2 = (2^{-\sigma}, 1) \leftarrow m(a_2, f) = 2 = 1 \times 2$
- $a_3 = (-2^{-\sigma}, 1) \leftarrow m(a_3, f) = 1 = 1 \times 1$
- $a_4 = (-2^{-\sigma}, 0) \leftarrow m(a_4, f) = 1 = 1 \times 1$
- $a_5 = (-2^{-\sigma}, -2^{-\sigma}) \leftarrow m(a_5, f) = 2 = 2 \times 1$
- $a_6 = (2^{-\sigma}, -2^{-\sigma}) \leftarrow m(a_6, f) = 4 = 2 \times 2$

Natural clusters:

- $(\Delta^1, 9) \leftarrow 9 = 3 \times 3$
- $(\Delta^2, 3) \leftarrow 3 = 1 \times 3$

Notations: $m(a, f)$: multiplicity of $a$ as a sol. of $f$
Number of solutions in a polydisc

Let $\Delta = (\Delta_1, \Delta_2)$ and $m = (m_1, m_2)$.

**Proposition 1:** Suppose

(i) $f_1$ has $m_1$ roots in $\Delta_1$ with multiplicity

(ii) $\forall \alpha_1 \in Z(\Delta_1, f_1), f_2(\alpha_1)$ has $m_2$ roots in $\Delta_2$ with multiplicity

Then $f(z) = 0$ has $m_2 \times m_1$ solutions in $\Delta$ with multiplicity.

**Definition:** A pair $(\Delta, m)$ is a natural tower (relative to $f$) if

(i) $(\Delta_1, m_1)$ is a natural cluster relative to $f_1$

(ii) $\forall \alpha_1 \in \Delta_1, (\Delta_2, m_2)$ is a natural cluster relative to $f_2(\alpha_1)$
Number of solutions in a polydisc

Let $\Delta = (\Delta_1, \Delta_2)$ and $m = (m_1, m_2)$.

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**Definition:** A pair $(\Delta, m)$ is a natural $\epsilon$-tower (relative to $f$) if

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(ii) $\forall \alpha_1 \in \Delta_1$, $(\Delta_2, m_2)$ is a natural cluster relative to $f_2(\alpha_1)$

**Corollary 2:** If $(\Delta, m)$ is a natural tower,

$f(z) = 0$ has $m_2 \times m_1$ solutions in $\Delta$ with multiplicity.
Number of solutions in a polydisc

Let $\Delta = (\Delta_1, \Delta_2)$ and $m = (m_1, m_2)$.

**Proposition 1:** Suppose

(i) $f_1$ has $m_1$ roots in $\Delta_1$ with multiplicity

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Example

System: Let $\sigma \geq 3$ and $f(z) = 0$ be:

$$\begin{align*}
(z_1 - 2^{-\sigma})^2(z_1 + 2^{-\sigma}) &= 0 \\
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\end{align*}$$

Solutions: $f(z) = 0$ has 6 solutions, all real:

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Natural clusters:

- $(\Delta_1, 9) \quad \leftarrow \quad 9 = 3 \times 3$
- $(\Delta_2, 3) \quad \leftarrow \quad 3 = 1 \times 3$

Natural towers:

- $(\Delta_1, (3, 3))$
- $(\Delta_2, (1, 3))$
Pellet’s test and natural towers

Definition: A pair \((\Delta, m)\) is a natural tower (relative to \(f\)) if

(i) \((\Delta_1, m_1)\) is a natural cluster relative to \(f_1\)

(ii) \(\forall \alpha_1 \in \Delta_1, (\Delta_2, m_2)\) is a natural cluster relative to \(f_2(\alpha_1)\)

\(f(z) = 0\) has \(m_2 \times m_1\) solutions in \(\Delta\) with multiplicity.
Pellet’s test and natural towers

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\(f(z) = 0\) has \(m_2 \times m_1\) solutions in \(\Delta\) with multiplicity.

Proposition 3: Suppose

(i) \(\text{SoftPelletTest}(\Delta_1, f_1)\) returns \(m_1 \geq 1\)
(ii) \(\text{SoftPelletTest}(\Delta_2, f_2(\Delta_1))\) returns \(m_2 \geq 1\)

Then \((\Delta, m)\) is a natural tower relative to \(f\).
Pellet’s test and natural towers

**Definition:** A pair $(\Delta, m)$ is a natural tower (relative to $f$) if

(i) $(\Delta_1, m_1)$ is a natural cluster relative to $f_1$

(ii) $\forall \alpha_1 \in \Delta_1$, $(\Delta_2, m_2)$ is a natural cluster relative to $f_2(\alpha_1)$

$f(z) = 0$ has $m_2 \times m_1$ solutions in $\Delta$ with multiplicity.

**Proposition 3:** Suppose

(i) $\text{SoftPelletTest}(\Delta_1, f_1)$ returns $m_1 \geq 1$

(ii) $\text{SoftPelletTest}(\Delta_2, f_2(\mathcal{G} \Delta_1))$ returns $m_2 \geq 1$

Then $(\Delta, m)$ is a natural tower relative to $f$. 
Pellet's test and natural towers

Definition: A pair $(\Delta, m)$ is a natural tower (relative to $f$) if

(i) $(\Delta_1, m_1)$ is a natural cluster relative to $f_1$

(ii) $\forall \alpha_1 \in \Delta_1, (\Delta_2, m_2)$ is a natural cluster relative to $f_2(\alpha_1)$

$f(z) = 0$ has $m_2 \times m_1$ solutions in $\Delta$ with multiplicity.

Proposition 3: Suppose

(i) SoftPelletTest$(\Delta_1, f_1)$ returns $m_1 \geq 1$

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Then $(\Delta, m)$ is a natural tower relative to $f$.
Pellet’s test and natural towers

Definition: A pair $(\Delta, m)$ is a natural tower (relative to $f$) if

(i) $(\Delta_1, m_1)$ is a natural cluster relative to $f_1$

(ii) $\forall \alpha_1 \in \Delta_1$, $(\Delta_2, m_2)$ is a natural cluster relative to $f_2(\alpha_1)$

$f(z) = 0$ has $m_2 \times m_1$ solutions in $\Delta$ with multiplicity.

Proposition 3: Suppose

(i) $\text{SoftPelletTest}(\Delta_1, f_1)$ returns $m_1 \geq 1$

(ii) $\text{SoftPelletTest}(\Delta_2, f_2(\Box \Delta_1))$ returns $m_2 \geq 1$

Then $(\Delta, m)$ is a natural tower relative to $f$. 
Main data structure

A tower is a triple $T = \langle \ell, B, L \rangle$ where

- $\ell$ is an integer in $\{0, 1, 2\}$ called level
- $B = (B_1, B_2)$ is a polybox called domain
- $L = (L_1, L_2)$ is a vector in $(\mathbb{Z})^2$ called precision
Main data structure

A tower is a triple $\mathcal{T} = \langle \ell, B, L \rangle$ where

- $\ell$ is an integer in $\{0, 1, 2\}$ called **level**
- $B = (B_1, B_2)$ is a polybox called **domain**
- $L = (L_1, L_2)$ is a vector in $(\mathbb{Z})^2$ called **precision**

We will guarantee that if $\ell = 1$, $\exists m_1$ so that:

1. $\text{SoftPelletTest}(\Delta(B_1), f_1)$ returns $m_1$ and $r(\Delta(B_1)) < 2^{-L_1}$
Main data structure

A **tower** is a triple $\mathcal{T} = \langle \ell, B, L \rangle$ where

- $\ell$ is an integer in $\{0, 1, 2\}$ called **level**
- $B = (B_1, B_2)$ is a polybox called **domain**
- $L = (L_1, L_2)$ is a vector in $\mathbb{Z}^2$ called **precision**

We will guarantee that if $\ell = 2$, $\exists (m_1, m_2)$ so that:

(i) $\text{SoftPelletTest}(\Delta(B_1), f_1)$ returns $m_1$ and $r(\Delta(B_1)) < 2^{-L_1}$

(ii) $\text{SoftPelletTest}(\Delta(B_2), f_2(\Delta(B_1)))$ returns $m_2$ and $r(\Delta(B_2)) < 2^{-L_2}$

From proposition 3: $(\Delta(B), m)$ is a natural tower (relative to $f$) and $f(z) = 0$ has $m_2 \times m_1$ sols in $\Delta(B)$ with mult.
Lift of a tower from level 0 to level 1

Cluster1( \( f, \mathcal{T} \) ) //for \( f \) with exact coefficients

**Input:** \( f = (f_1, f_2) \), \( \mathcal{T} = \langle \ell, B, L \rangle \) a tower at any level

**Output:** a list of towers at level 1

1. calls ClusterOracle ([BSS+16]) for \( f_1, B_1, 2^{-L_1} \)
Lift of a tower from level 1 to level 2

\[ \Delta(B_1) \rightarrow \ldots \]

\[ \Delta(B_2) \]

**Cluster2( f, \mathcal{T} )**  // for \( f \) with exact coefficients

**Input:** \( f = (f_1, f_2), \mathcal{T} = \langle \ell, B, L \rangle \) a tower at level 1

**Output:** a flag in \{success, fail\} and a list of towers at level 2

1. calls ClusterInterval for \( f_2(\Delta(B_1)), B_2, 2^{-L_2} \)
   - **fail** if SoftPelletTest returns -2 (i.e. not enough prec. on \( \Delta(B_1) \))
Main algorithm

<table>
<thead>
<tr>
<th>ClusterTri($f, B, L$)</th>
<th>//for $f$ with exact coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> a triangular system $f(z) = 0$, a polybox $B$, $L &gt; 0$</td>
<td><strong>Output:</strong> a set of natural $2^{-L}$-towers solving the LCP</td>
</tr>
<tr>
<td>1. $Q$.push($\langle 0, B, (L, L) \rangle$)</td>
<td></td>
</tr>
<tr>
<td>2. <strong>while</strong> $Q$ contains towers of level $&lt; 2$ do</td>
<td></td>
</tr>
<tr>
<td>3. $T = \langle \ell, B, (L_1, L_2) \rangle \leftarrow Q$.pop() with $\ell &lt; 2$</td>
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<tr>
<td>4.</td>
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<td>5.</td>
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<td>10.</td>
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<tr>
<td>11.</td>
<td></td>
</tr>
<tr>
<td>12. return $Q$</td>
<td></td>
</tr>
</tbody>
</table>
Main algorithm

ClusterTri($f$, $B$, $L$)  // for $f$ with exact coefficients

Input: a triangular system $f(z) = 0$, a polybox $B$, $L > 0$
Output: a set of natural $2^{-L}$-towers solving the LCP

1. $Q.push(\langle 0, B, (L, L) \rangle)$
2. while $Q$ contains towers of level $< 2$ do
3. \hspace{1em} $T = \langle \ell, B, (L_1, L_2) \rangle \leftarrow Q.pop()$ with $\ell < 2$
4. \hspace{1em} if $\ell = 0$ then
5. \hspace{2em} $Q.push(Cluster1( f, T ))$
6. \hspace{1em} else
7. \hspace{1em} else
8. \hspace{1em} else
9. \hspace{1em} else
10. \hspace{1em} else
11. \hspace{1em} else
12. return $Q$
Main algorithm

ClusterTri($f$, $B$, $L$) //for $f$ with exact coefficients

Input: a triangular system $f(z) = 0$, a polybox $B$, $L > 0$
Output: a set of natural $2^{-L}$-towers solving the LCP

1. $Q.push(\langle 0, B, (L, L) \rangle)$
2. while $Q$ contains towers of level $< 2$ do
3. \hspace{1em} $T = \langle \ell, B, (L_1, L_2) \rangle \leftarrow Q.pop()$ with $\ell < 2$
4. \hspace{1em} if $\ell = 0$ then
5. \hspace{2em} $Q.push(Cluster1(\ f, T ))$
6. \hspace{1em} else
7. \hspace{2em} flag, $S \leftarrow Cluster2(\ f, T )$
8. \hspace{1em} if flag = success then
9. \hspace{2em} \hspace{1em} $Q.push(S)$
10. \hspace{1em} else \hspace{1em} // not enough precision on $B_1$
11. \hspace{2em} $Q.push(\langle 0, B, (2L_1, L_2) \rangle)$
12. return $Q$
Our implementation

**Ccluster**: library in C based on

- FLINT\(^1\): arithmetic for the geometric algorithm
- Arb\(^2\): arbitrary precision floating arithmetic with error bounds

Available at https://github.com/rimbach/Ccluster

**Ccluster.jl**: package for Julia\(^3\) based on \(\mathbb{N}e^mO^4\)

- interface for Ccluster
- Tcluster: implementation of ClusterTri

Available at https://github.com/rimbach/Ccluster.jl

---

\(^1\)https://github.com/wbhart/flint2
\(^2\)http://arblib.org/
\(^3\)https://julialang.org/
\(^4\)http://nemocas.org/
### Benchmark: systems

**Type of a triangular system:**

\[ f(z) = 0 \] has type \((d_1, \ldots, d_n)\) if \(f_i\) has degree \(d_i\) in \(z_i\), \(\forall 1 \leq i \leq n\).

**Table:** for each type, average on 5 random dense systems

Seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz

<table>
<thead>
<tr>
<th>Type</th>
<th>#Clus</th>
<th>#Sols</th>
<th>$t(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Systems with only simple solutions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9,9,9)</td>
<td></td>
<td></td>
<td>0.24</td>
</tr>
<tr>
<td>(6,6,6,6)</td>
<td></td>
<td></td>
<td>0.10</td>
</tr>
<tr>
<td>(9,9,9,9)</td>
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<td>0.37</td>
</tr>
<tr>
<td>(6,6,6,6)</td>
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<tr>
<td>(9,9,9,9,9)</td>
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<td>0.03</td>
</tr>
<tr>
<td>(2,2,2,2,2,2,2,2,2,2,2)</td>
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<td>0.05</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Systems with multiple solutions</th>
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<tbody>
<tr>
<td>(9,9)</td>
</tr>
<tr>
<td>(6,6,6)</td>
</tr>
<tr>
<td>(9,9,9)</td>
</tr>
<tr>
<td>(6,6,6,6)</td>
</tr>
<tr>
<td>(9,9,9,9)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type</th>
<th>#Clus</th>
<th>#Sols</th>
<th>$t(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,2,2,2,2,2,2,2,2,2,2,2)</td>
<td></td>
<td></td>
<td>0.03</td>
</tr>
</tbody>
</table>

**Tcluster local:**

\[ B = ([-1, 1] + i[-1, 1])^2, \epsilon = 2^{-53} \]

**Tcluster global:**

chosen with upper bound for roots

**HomCont.jl:**

HomotopyContinuation.jl

**triang solve:**

Singular solver for triangular systems
Benchmark: local vs global comparison

Type of a triangular system:
\( f(z) = 0 \) has type \((d_1, \ldots, d_n)\) if \( f_i \) has degree \( d_i \) in \( z_i \), \( \forall 1 \leq i \leq n \)

Table: for each type, average on 5 random dense systems
seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz

<table>
<thead>
<tr>
<th>type</th>
<th>Tcluster local</th>
<th>Tcluster global</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(#Clus, #Sols)</td>
<td>t(s)</td>
</tr>
<tr>
<td>(9,9,9)</td>
<td>(149 : 149)</td>
<td>0.24</td>
</tr>
<tr>
<td>(6,6,6,6)</td>
<td>(63.4 : 63.4)</td>
<td>0.10</td>
</tr>
<tr>
<td>(9,9,9,9)</td>
<td>(559 : 559)</td>
<td>1.06</td>
</tr>
<tr>
<td>(6,6,6,6,6)</td>
<td>(155 : 155)</td>
<td>0.37</td>
</tr>
<tr>
<td>(9,9,9,9,9,9)</td>
<td>(1739 : 1739)</td>
<td>4.83</td>
</tr>
<tr>
<td>(2,2,2,2,2,2,2,2,2)</td>
<td>(0 : 0)</td>
<td>0.13</td>
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Systems with multiple solutions

<table>
<thead>
<tr>
<th></th>
<th>Tcluster local</th>
<th>Tcluster global</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>(#Clus, #Sols)</td>
<td>t(s)</td>
</tr>
<tr>
<td>(9,9)</td>
<td>(23.8 : 13.6)</td>
<td>0.03</td>
</tr>
<tr>
<td>(6,6,6)</td>
<td>(35.2 : 8.80)</td>
<td>0.05</td>
</tr>
<tr>
<td>(9,9,9)</td>
<td>(113 : 37.6)</td>
<td>0.22</td>
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<tr>
<td>(6,6,6,6)</td>
<td>(81.6 : 10.2)</td>
<td>0.21</td>
</tr>
</tbody>
</table>

\( \text{Tcluster local} : B = ([−1, 1] + \nu[−1, 1])^2, \epsilon = 2^{-53} \)

\( \text{Tcluster global} : B \) chosen with upper bound for roots
Benchmark: extern comparison

Type of a triangular system:
\( f(z) = 0 \) has type \((d_1, \ldots, d_n)\) if \( f_i \) has degree \( d_i \) in \( z_i \), \( \forall 1 \leq i \leq n \)

Table: for each type, average on 5 random dense systems
seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz

<table>
<thead>
<tr>
<th>Type</th>
<th>Tcluster local</th>
<th>Tcluster global</th>
<th>HomCont.jl</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(#Clus, #Sols)</td>
<td>(#Clus, #Sols)</td>
<td>#Sols</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>t (s)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>t (s)</td>
</tr>
<tr>
<td>Systems with only simple solutions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9,9)</td>
<td>(149 : 149)</td>
<td>(729 : 729)</td>
<td>1.21</td>
</tr>
<tr>
<td></td>
<td>0.24</td>
<td>729</td>
<td>4.21</td>
</tr>
<tr>
<td>(6,6,6)</td>
<td>(63.4 : 63.4)</td>
<td>(1296 : 1296)</td>
<td>1.73</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>1296</td>
<td>4.70</td>
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<tr>
<td>(9,9,9)</td>
<td>(559 : 559)</td>
<td>(6561 : 6561)</td>
<td>12.9</td>
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<td>1.06</td>
<td>6561</td>
<td>14.0</td>
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<td>(6,6,6,6)</td>
<td>(155 : 155)</td>
<td>(7776 : 7776)</td>
<td>11.1</td>
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<td>0.37</td>
<td>7776</td>
<td>11.5</td>
</tr>
<tr>
<td>(9,9,9,9,9)</td>
<td>(1739 : 1739)</td>
<td>(59049 : 59049)</td>
<td>113</td>
</tr>
<tr>
<td></td>
<td>4.83</td>
<td>59049</td>
<td>116</td>
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<tr>
<td>(2,2,2,2,2,2,2,2)</td>
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<td>(1024 : 1024)</td>
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<tr>
<td></td>
<td>0.13</td>
<td>1024</td>
<td>4.84</td>
</tr>
<tr>
<td>Systems with multiple solutions</td>
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<td></td>
</tr>
<tr>
<td>(9,9)</td>
<td>(23.8 : 13.6)</td>
<td>(81 : 45)</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>33.6</td>
<td>3.27</td>
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<tr>
<td>(6,6,6)</td>
<td>(35.2 : 8.80)</td>
<td>(216 : 54)</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>53.2</td>
<td>2.75</td>
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<tr>
<td>(9,9,9)</td>
<td>(113 : 37.6)</td>
<td>(729 : 225)</td>
<td>1.06</td>
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<tr>
<td></td>
<td>0.22</td>
<td>159</td>
<td>28.4</td>
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<tr>
<td>(6,6,6,6)</td>
<td>(81.6 : 10.2)</td>
<td>(1296 : 162)</td>
<td>1.28</td>
</tr>
<tr>
<td></td>
<td>0.21</td>
<td>134</td>
<td>8.06</td>
</tr>
</tbody>
</table>

**Tcluster local**: \( B = ([-1, 1] + \iota[-1, 1])^2, \, \epsilon = 2^{-53} \)

**Tcluster global**: \( B \) chosen with upper bound for roots

**HomCont.jl**: HomotopyContinuation.jl
Benchmark:

**Type of a triangular system:**

\( f(z) = 0 \) has type \( (d_1, \ldots, d_n) \) if \( f_i \) has degree \( d_i \) in \( z_i \), \( \forall 1 \leq i \leq n \)

**Table:** for each type, average on 5 random dense systems seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz

| Type | Tcluster local | | Tcluster global | | HomCont.jl | |
|------|----------------|---|----------------|---|-------------|
|      | (#Clus, #Sols) | t (s) | (#Clus, #Sols) | t (s) | #Sols | t (s) |
| Systems with only simple solutions | | | | | |
| (9,9,9) | (149 : 149) | 0.24 | (729 : 729) | 1.21 | 729 | 4.21 |
| (6,6,6,6) | (63.4 : 63.4) | 0.10 | (1296 : 1296) | 1.73 | 1296 | 4.70 |
| (9,9,9,9) | (559 : 559) | 1.06 | (6561 : 6561) | 12.9 | 6561 | 14.0 |
| (6,6,6,6,6) | (155 : 155) | 0.37 | (7776 : 7776) | 11.1 | 7776 | 11.5 |
| (9,9,9,9,9) | (1739 : 1739) | 4.83 | (59049 : 59049) | 113 | 59049 | 116 |
| (2,2,2,2,2,2,2,2,2,2) | (0 : 0) | 0.13 | (1024 : 1024) | 2.42 | 1024 | 4.84 |
| Systems with multiple solutions | | | | | |
| (9,9) | (23.8 : 13.6) | 0.03 | (81 : 45) | 0.15 | 33.6 | 3.27 |
| (6,6,6) | (35.2 : 8.80) | 0.05 | (216 : 54) | 0.24 | 53.2 | 2.75 |
| (9,9,9) | (113 : 37.6) | 0.22 | (729 : 225) | 1.06 | 159 | 28.4 |
| (6,6,6,6) | (81.6 : 10.2) | 0.21 | (1296 : 162) | 1.28 | 134 | 8.06 |

Tcluster **local** : \( B = (-1, 1] + \epsilon [-1, 1])^2 \), \( \epsilon = 2^{-53} \)

Tcluster **global** : \( B \) chosen with upper bound for roots

HomCont.jl: HomotopyContinuation.jl
Benchmark:

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seq. times on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz

<table>
<thead>
<tr>
<th>type</th>
<th>Tcluster local</th>
<th>Tcluster global</th>
<th>HomCont.jl</th>
<th>triang_solve</th>
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<td>(#Clus, #Sols)</td>
<td>(#Clus, #Sols)</td>
<td>#Sols</td>
<td>t (s)</td>
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<tr>
<td></td>
<td>t (s)</td>
<td>#Sols</td>
<td>t (s)</td>
<td>#Sols</td>
</tr>
<tr>
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<td></td>
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<td>0.13</td>
<td>(1024 : 1024)</td>
<td>2.42</td>
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Tcluster local : \( B = (-1, 1] + \iota[-1, 1]^2, \epsilon = 2^{-53} \)
Tcluster global: \( B \) chosen with upper bound for roots
HomCont.jl: HomotopyContinuation.jl
triang_solve: Singular solver for triangular systems
Menu

0 - Univariate case:

1 - Multivariate triangular case

2 - Back to univariate case

• polynomials with real coefficients
• new counting test

New practical advances in polynomial root clustering.
In MACIS 19, 2019.
Pols with real coefficients

Example:
\[ \text{Mign}_d(z) = z^d - 2(2^{14}z - 1)^2 \]

\( d \) even \( \Rightarrow \) 4 real roots

\( d = 64 \)

Subdivision tree:
Pols with real coefficients (II)

Example:
\[ \text{Bern}_d(z) = \sum_{k=0}^{d} \binom{d}{k} b_{d-k} z^k \]

\(b_i\)'s: Bernoulli numbers

d even \(\Rightarrow\) \(d/4\) real roots

\(d = 64\)

Subdivision tree:

2492 \(T^0\)-tests

1476 \(T^0\)-tests (ratio \(\approx 0.6\))
Results (I)

**Ccluster**: version of [IPY18]

- $t_1$: time; $s_1$: number of $T^0$-tests

**CclusterR**: Ccluster for polynomials in $\mathbb{R}[z]$

- $t_2$: time; $s_2$: number of $T^0$-tests

<table>
<thead>
<tr>
<th></th>
<th>Ccluster (#Clus, #Sols)</th>
<th>s$_1$</th>
<th>t$_1$</th>
<th>s$_2$</th>
<th>t$_1$/t$_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bern$_{128}$</td>
<td>(128, 128)</td>
<td>4732</td>
<td>6.30</td>
<td>2712</td>
<td>1.72</td>
</tr>
<tr>
<td>Bern$_{191}$</td>
<td>(191, 191)</td>
<td>7220</td>
<td>20.2</td>
<td>4152</td>
<td>1.74</td>
</tr>
<tr>
<td>Bern$_{256}$</td>
<td>(256, 256)</td>
<td>9980</td>
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<td>5698</td>
<td>1.67</td>
</tr>
<tr>
<td>Bern$_{383}$</td>
<td>(383, 383)</td>
<td>14504</td>
<td>120</td>
<td>8198</td>
<td>1.82</td>
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<tr>
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<td>(127, 128)</td>
<td>4508</td>
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<td>1.92</td>
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<tr>
<td>Mign$_{191}$</td>
<td>(190, 191)</td>
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<td>Mign$_{256}$</td>
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<td>Mign$_{383}$</td>
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<td>6410</td>
<td>1.98</td>
</tr>
</tbody>
</table>

Sequential times in s. on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux
Menu

0 - Univariate case:

1 - Multivariate triangular case

2 - Back to univariate case

- polynomials with real coefficients
- new counting test

New practical advances in polynomial root clustering.
In MACIS 19, 2019.
Approximating Power Sums

Let $\Delta = \Delta(0, 1), f$ has deg. $d$, dist. roots $\alpha_1, \ldots, \alpha_{d\Delta}$ in $\Delta$ with mults $m_1, \ldots, m_{d\Delta}$

**Power Sums:** let $h \in \mathbb{Z}$

$$s_h = m_1 \times \alpha_1^h + \ldots + m_{d\Delta} \times \alpha_{d\Delta}^h$$
Approximating Power Sums

Let $\Delta = \Delta(0, 1)$, $f$ has deg. $d$, dist. roots $\alpha_1, \ldots, \alpha_d$ in $\Delta$ with mults $m_1, \ldots, m_d$

**Power Sums:** let $h \in \mathbb{Z}$

$$s_h = m_1 \times \alpha_1^h + \ldots + m_d \times \alpha_d^h$$

**Theorem [S82, P18]:**

If no root in $\{z \in \mathbb{C} | \frac{1}{\rho} < |z| < \rho\}$

Use evaluations of $f$ and $f'$ at $q$ points to approximate $s_h$ within error $\simeq d\rho^{-q}$

---


Old and new nearly optimal polynomial root-finders.


[Sch82] Arnold Schönhage.

The fundamental theorem of algebra in terms of computational complexity.

*Manuscript. Univ. of Tübingen, Germany, 1982.*
Approximating 0-th Power Sum

Let $\Delta = \Delta(0, 1)$, $f$ has deg. $d$, dist. roots $\alpha_1, \ldots, \alpha_{d\Delta}$ in $\Delta$ with mults $m_1, \ldots, m_{d\Delta}$

**Power Sums:** let $h \in \mathbb{Z}$

$$s_0 = m_1 \times \alpha_1^0 + \ldots + m_{d\Delta} \times \alpha_{d\Delta}^0 = \#(\Delta, f)$$

**Theorem [S82, P18]:**
if no root in $\{z \in \mathbb{C} | \frac{1}{\rho} < |z| < \rho\}$
use evaluations of $f$ and $f'$ at $q$ points to approximate $s_h$ within error $\simeq d \rho^{-q}$

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0-th Power Sum:

$s_0 = \#(\Delta, f)$

Approximation formula: let $q \in \mathbb{N}_*$, $\omega = e^{\frac{2\pi i}{q}}$

$$s_0^* = \frac{1}{q} \sum_{g=0}^{q-1} \omega^g \frac{f'(\omega^g)}{f(\omega^g)}$$
Approximating 0-th Power Sum

Let $\Delta = \Delta(0, 1)$, $f$ has deg. $d$, dist. roots $\alpha_1, \ldots, \alpha_{d\Delta}$ in $\Delta$ with mults $m_1, \ldots, m_{d\Delta}$

0-th Power Sum:

$$s_0 = \#(\Delta, f)$$

Approximation formula: let $q \in \mathbb{N}_*$, $\omega = e^{\frac{2\pi i}{q}}$

$$s_0^* = \frac{1}{q} \sum_{g=0}^{q-1} \omega^g \frac{f'(\omega^g)}{f(\omega^g)}$$

Corollary of [S82, P18]: if no root in $\{z \in \mathbb{C} | \frac{1}{\rho} < |z| < \rho \}$, $\theta = 1/\rho$, then

(i) $|s_0^* - s_0| \leq \frac{d\theta^q}{1 - \theta^q}$.

(ii) Fix $\delta > 0$. If $q = \lceil \log_\theta \left( \frac{\delta}{d+\delta} \right) \rceil$ then $|s_0^* - s_0| \leq \delta$. 
Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

**Oracle for $\alpha$:** function $O_\alpha : \mathbb{Z} \to \mathbb{C}$

$$s.t. \quad \alpha \in O_\alpha(L) \quad \text{and} \quad w(O_\alpha(L)) \leq 2^{-L}$$

Let $f \in \mathbb{C}[z]$

**Evaluation oracle for $f$:** function $I_f : \mathbb{Z} \times (\mathbb{Z} \to \mathbb{C}) \to \mathbb{C}$

$$s.t. \quad f(\alpha) \in I_f(L, O_\alpha) \quad \text{and} \quad w(I_f(L, O_\alpha)) \leq 2^{-L}$$

Notations: $\mathbb{C}$: set of complex interval
$\mathbb{Z} \to \mathbb{C}$: set of oracle numbers
The $P^*$-test

\[ P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho) \quad //\text{Output in } \{0, 1, \ldots, d\} \]

**Input:** \( \mathcal{I}_f, \mathcal{I}_{f'} \) evaluation oracles for \( f \) and \( f' \), \( \Delta \) a disc \( \rho \)-isolated

**Output:** \#(\( \Delta, f \))

1. \( \delta \leftarrow 1/4, \theta \leftarrow 1/\rho \)
2. \( q \leftarrow \left\lceil \log_{\theta} \left( \frac{\delta}{d+\delta} \right) \right\rceil \)
3. 
4. 
5.
The $P^*$-test

$$P^* (I_f, I_{f'}, \Delta, \rho)$$  //Output in $\{0, 1, \ldots, d\}$

Input: $I_f, I_{f'}$ evaluation oracles for $f$ and $f'$, $\Delta$ a disc $\rho$-isolated

Output: $\#(\Delta, f)$

1. $\delta \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta (\delta/(d+\delta)) \rceil$
3. compute $\square s_0^*$ with $q, I_f, I_{f'}$ so that $w(\square s_0^*) < 1/2$
4. 
5. 

Example: $f$ has degree 500, $\rho = 2$

Example: evaluate $f$ and $f'$ at $q = 11$ points

Example: then get $\#(\Delta, f)$ in $O(q)$ arithmetic operations

Efficiency: directly related to evaluation but requires $\rho$ to be known and $\rho > 1$. 
The $P^*$-test

\[ P^*(I_f, I_{f'}, \Delta, \rho) \quad //\text{Output in } \{0, 1, \ldots, d\} \]

**Input:** \( I_f, I_{f'} \) evaluation oracles for \( f \) and \( f' \), \( \Delta \) a disc \( \rho \)-isolated

**Output:** \( \#(\Delta, f) \)

1. \( \delta \leftarrow 1/4, \theta \leftarrow 1/\rho \)
2. \( q \leftarrow \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil \)
3. compute \( \Box s_0^* \) with \( q, I_f, I_{f'} \) so that \( w(\Box s_0^*) < 1/2 \)
4. \( \Box s_0 \leftarrow \Box s_0^* + [-1/4, 1/4] + i[-1/4, 1/4] \quad // w(\Box s_0) < 1 \)
The $P^*$-test

\[ P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho) \] //Output in \{0, 1, \ldots , d\}

**Input:** $\mathcal{I}_f, \mathcal{I}_{f'}$ evaluation oracles for $f$ and $f'$, $\Delta$ a disc $\rho$-isolated

**Output:** $\#(\Delta, f)$

1. $\delta \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta (\frac{\delta}{d+\delta}) \rceil$
3. compute $\mathcal{R}s_0^*$ with $q, \mathcal{I}_f, \mathcal{I}_{f'}$ so that $w(\mathcal{R}s_0^*) < 1/2$
4. $\mathcal{R}s_0 \leftarrow \mathcal{R}s_0^* + [-1/4, 1/4] + \iota[-1/4, 1/4]$ // $w(\mathcal{R}s_0) < 1$
5. return the unique integer in $\mathcal{R}s_0$

**Example:** $f$ has degree 500, $\rho = 2$

- evaluate $f$ and $f'$ at $q = 11$ points
- then get $\#(\Delta, f)$ in $O(q)$ arithmetic operations
The \( P^* \)-test

\[
P^*(I_f, I_{f'}, \Delta, \rho) \\
\text{Input: } I_f, I_{f'} \text{ evaluation oracles for } f \text{ and } f', \Delta \text{ a disc } \rho\text{-isolated} \\
\text{Output: } #(\Delta, f)
\]

1. \( \delta \leftarrow 1/4, \theta \leftarrow 1/\rho \)
2. \( q \leftarrow \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil \)
3. compute \( \square s^*_0 \) with \( q, I_f, I_{f'} \) so that \( w(\square s^*_0) < 1/2 \)
4. \( \square s_0 \leftarrow \square s^*_0 + [-1/4, 1/4] + i[-1/4, 1/4] \) \( \text{// } w(\square s_0) < 1 \)
5. return the unique integer in \( \square s_0 \)

**Example:** \( f \) has degree 500, \( \rho = 2 \)

- evaluate \( f \) and \( f' \) at \( q = 11 \) points
- then get \( #(\Delta, f) \) in \( O(q) \) arithmetic operations

**Efficiency:** directly related to evaluation
The \( P^\ast \)-test

<table>
<thead>
<tr>
<th>( P^\ast )-tests</th>
<th>Discarding tests</th>
<th>( T^\ast )-tests</th>
<th>( P^\ast )-tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>nb</td>
<td>( t_0 )</td>
<td>( t_0 / t ) (%)</td>
<td>( t'_0 )</td>
</tr>
<tr>
<td>( Bern_{128} )</td>
<td>4732</td>
<td>5.50</td>
<td>1.38</td>
</tr>
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<td>( Bern_{256} )</td>
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<td>27.8</td>
<td>0.60</td>
</tr>
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</table>

\( P^\ast \)-tests: \( P^\ast(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, 2) \)

nb: nb of discarding tests performed

\( t \): time in Ccluster

\( t_0 \): time in discarding \( T^\ast \)-tests

\( t'_0 \): time in \( P^\ast \)-tests

**Example:** \( f \) has degree 500, \( \rho = 2 \)

- evaluate \( f \) and \( f' \) at \( q = 11 \) points
- then get \( \#(\Delta, f) \) in \( O(q) \) arithmetic operations

**Efficiency:** directly related to evaluation
The $P^*$-test

\[ P^*(I_f, I_{f'}, \Delta, \rho) \]

//Output in \{0, 1, \ldots, d\}

**Input:** \( I_f, I_{f'} \) evaluation oracles for \( f \) and \( f' \), \( \Delta \) a disc \( \rho \)-isolated

**Output:** \#(\( \Delta, f \))

1. \( \delta \leftarrow 1/4, \theta \leftarrow 1/\rho \)
2. \( q \leftarrow \lceil \log_\theta \left( \frac{\delta}{d+\delta} \right) \rceil \)
3. compute \( \mathcal{I}s^*_0 \) with \( q, I_f, I_{f'} \) so that \( w(\mathcal{I}s^*_0) < 1/2 \)
4. \( \mathcal{I}s_0 \leftarrow \mathcal{I}s^*_0 + [-1/4, 1/4] + \mathcal{I}[-1/4, 1/4] \) \quad \text{// } w(\mathcal{I}s_0) < 1
5. `return` the unique integer in \( \mathcal{I}s_0 \)

**Example:** \( f \) has degree 500, \( \rho = 2 \)

- evaluate \( f \) and \( f' \) at \( q = 11 \) points
- then get \#(\( \Delta, f \)) in \( O(q) \) arithmetic operations

**Efficiency:** directly related to evaluation

**But:** requires \( \rho \) to be known and \( > 1 \).
The $P^*$-test

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<th>$P^*$-tests</th>
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<td>Mign$_{256}$</td>
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<td>27.8</td>
</tr>
</tbody>
</table>

$P^*$-tests: $P^*(I_f, I_{f'}, \Delta, 2)$

nb: nb of discarding tests performed
$n_{-1}$: nb of times $\left\lfloor s_0 \right\rfloor$ does not contains integer
$n_{err}$: nb of times result is not correct

Example: $f$ has degree 500, $\rho = 2$

- evaluate $f$ and $f'$ at $q = 11$ points
- then get $\#(\Delta, f)$ in $O(q)$ arithmetic operations

Efficiency: directly related to evaluation

But: requires $\rho$ to be known and $> 1$. 
Using the $P^*$-test as a filter

The $C^0$-test:

$$C^0(\Delta) := \begin{cases} 
-1 & \text{if } P^*(I_f, I_{f'}, \Delta, 2) \neq 0, \\
-1 & \text{if } P^*(I_f, I_{f'}, \Delta, 2) = 0 \text{ and } T^*(\Delta, O_f) \neq 0, \\
0 & \text{if } P^*(I_f, I_{f'}, \Delta, 2) = 0 \text{ and } T^*(\Delta, O_f) = 0.
\end{cases}$$
Results (I)

**Ccluster:** version of [IPY18]
- $t_1$: time; $s_1$: number of $T^0$-tests

**CclusterR:** Ccluster for polynomials in $\mathbb{R}[z]$
- $t_2$: time; $s_2$: number of $T^0$-tests

**CclusterP:** CclusterR with $P^*$-test as a filter
- $t_3$: time; $s_3$: number of $T^0$-tests

<table>
<thead>
<tr>
<th></th>
<th>Ccluster (#Clus, #Sols)</th>
<th>CclusterR $s_2$ $t_1/t_2$</th>
<th>CclusterP $s_3$ $t_3$ $t_2/t_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bern$_{128}$</td>
<td>(128, 128) 4732 6.30</td>
<td>2712 1.72</td>
<td>1983 3.30 1.10</td>
</tr>
<tr>
<td>Bern$_{191}$</td>
<td>(191, 191) 7220 20.2</td>
<td>4152 1.74</td>
<td>3073 10.7 1.08</td>
</tr>
<tr>
<td>Bern$_{256}$</td>
<td>(256, 256) 9980 41.8</td>
<td>5698 1.67</td>
<td>4067 21.9 1.14</td>
</tr>
<tr>
<td>Bern$_{383}$</td>
<td>(383, 383) 14504 120</td>
<td>8198 1.82</td>
<td>5813 53.5 1.23</td>
</tr>
<tr>
<td>Mign$_{128}$</td>
<td>(127, 128) 4508 5.00</td>
<td>2292 1.92</td>
<td>1668 1.81 1.43</td>
</tr>
<tr>
<td>Mign$_{191}$</td>
<td>(190, 191) 6260 15.5</td>
<td>3180 2.01</td>
<td>2431 4.34 1.77</td>
</tr>
<tr>
<td>Mign$_{256}$</td>
<td>(255, 256) 8452 31.8</td>
<td>4304 2.04</td>
<td>3223 10.7 1.44</td>
</tr>
<tr>
<td>Mign$_{383}$</td>
<td>(382, 383) 12564 79.7</td>
<td>6410 1.98</td>
<td>4883 26.8 1.49</td>
</tr>
</tbody>
</table>

sequential times in s. on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux
Pols with real coefficients

Example:

\[\text{Mign}_d(z) = z^d - 2(2^{14}z - 1)^2\]

\(d\) even \(\Rightarrow\) 4 real roots

only 4 non-zero coeffs

\[d = 64\]

Subdivision tree:

2044 \(T^0\)-tests

1072 \(T^0\)-tests (ratio \(\approx 0.52\))
Procedural polynomials

**Procedure:** Mand\(_k\)(z)

**Input:** \(k \in \mathbb{N}^*, z \in \mathbb{C}\)

**Output:** \(r \in \mathbb{C}\)

1. if \(k = 1\) then
2. return \(z\)
3. else
4. return \(z\)Mand\(_{k-1}\)(z\(^2\) + 1

\[k = 6 \text{ (deg = 63)}\]
**Procedural polynomials**

**Procedure:** $\text{Mand}_k(z)$

**Input:** $k \in \mathbb{N}^*, z \in \mathbb{C}$

**Output:** $r \in \mathbb{C}$

1. if $k = 1$ then
2. return $z$
3. else
4. return $z\text{Mand}_{k-1}(z)^2 + 1$

![Graph for $k = 6$ (deg = 63)](image)

**Procedure:** $\text{Runn}_k(z)$

**Input:** $k \in \mathbb{N}, z \in \mathbb{C}$

**Output:** $r \in \mathbb{C}$

1. if $k = 0$ then
2. return 1
3. else if $k = 1$ then
4. return $z$
5. else
6. return $\text{Runn}_{k-1}(z)^2 + z\text{Runn}_{k-2}(z)^4$

![Graph for $k = 8$ (deg = 170)](image)
## Results (II)

**Ccluster**: version of [IPY18]

\[ t_1 \]: time

**CclusterR**: Ccluster for polynomials in \( \mathbb{R}[z] \)

\[ t_2 \]: time

**CclusterP**: CclusterR with \( P^* \)-test as a filter

\[ t_3 \]: time

<table>
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<tr>
<th></th>
<th>Ccluster (#Clus, #Sols)</th>
<th>( t_1 )</th>
<th>CclusterR ( t_1/t_2 )</th>
<th>CclusterP</th>
<th>( t_2/t_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mand6</td>
<td>(63, 63)</td>
<td>0.99</td>
<td>1.69</td>
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</tr>
<tr>
<td>Mand7</td>
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<td>Runn9</td>
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Results (II)

**Ccluster**: version of [IPY18]

\[ t_1 : \text{time} \]

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\[ t_2 : \text{time} \]

**CclusterP**: CclusterR with \( P^* \)-test as a filter

\[ t_3 : \text{time} \]

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<th>t/t2</th>
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<th>t/t3</th>
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**CclusterP**: CclusterR with \( P^* \)-test as a filter
- \( t_3 \): time

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<th>t₁</th>
<th>CclusterR t₁/t₂</th>
<th>CclusterP t₃</th>
<th>t₂/t₃</th>
<th>MPSolve t₄</th>
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</thead>
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Thank you for your attention!