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Leading a continuation method by geometry for solving geometric constraints

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ABSTRACT

Geometric constraint problems arise in domains such as CAD, Robotics, Molecular Chemistry, whenever one expects 2D or 3D configurations of some geometric primitives fulfilling some geometric constraints. Most well-constrained 3D problems are resistant to geometric knowledge based systems. They are often solved by purely numerical methods that are efficient but provide only one solution. Finding all the solutions can be achieved by using, among others, generic homotopy methods, that become costly when the number of constraints grows. This paper focuses on using geometric knowledges to specialize a so-called coefficient parameter continuation to 3D geometric constraint systems. Even if the proposed method does not ensure obtaining all the solutions, it provides several real ones. Geometric knowledges are used to justify it and lead the search of new solutions.

1. Introduction

Solving geometric constraints is addressed in many fields such as CAD, molecular modeling, robotics. It consists of finding positions of geometric entities such as points, lines, circles in 2D and also planes and spheres in 3D. These entities must satisfy some constraints related to distances, angles, tangencies, incidences, distance ratio and so on. Moreover, most softwares of geometric modeling and simulation incorporate constraint solvers to allow the user to define objects by constraints. Constraint systems are usually considered to be well-constrained in the sense that there exists a finite number of solutions up to rigid body motions. Indeed rotations and translations have no effect on the satisfaction of the considered constraints.

This issue has been widely studied in the case of 2D design in CAD. Several approaches have been described in literature, see [1,2] for recent surveys. Basically, algebraic approaches translate the problem into equations and then use algebraic techniques. Geometric approaches yield solutions by means of classical theorems or constructions of geometry. Solvers often perform first a decomposition of the problem into sub-problems easier to solve. In addition, solving the whole problem is faster. The decomposition is based on the geometric nature of the statements. We can notice that some geometric methods as well as decomposition processes rely on basic geometric constructions, allowing efficient implementations through graph algorithms.

Solving 3D geometric constraint systems is much more difficult particularly if all the solutions of the systems, assumed well-constrained, are wanted. Indeed, geometric methods or decomposition processes used in 2D cannot be easily extended in 3D. This is illustrated for instance in rigidity theory where a combinatorial characterization of rigidity is known for 2D realizations of graphs but not for 3D ones. Considering decomposition, most problems related to 3D polyhedrons are not decomposed by usual combinatorial methods based on graphs. This is why industrial software products are usually based on numerical methods such as Newton–Raphson or Homotopy. Unlike the Newton–Raphson method, methods that use homotopy could propose several solutions. This is a reason why they have been particularly studied for solving geometric constraints:

(i) in [3], a classical homotopy approach is used which is able to find all the solutions. But yielding all the solutions is very inefficient.
(ii) in [4], a specialized version of homotopy is used where the sketch given by the designer is a starting system. In this case, only one solution could be provided because the goal is to find the solution closest to the sketch.

In this paper, we design a homotopy based method specialized in 3D geometric constraints solving. In this field several features must be considered. First, the number of solutions can be exponential with the number of constraints. It is therefore not necessary to produce all solutions at once but only some of them. Indeed, some methods yield one solution that may not be the one expected by the user. Next, the user provides a sketch that should be used to guide the search for similar solutions. Finally, the solutions must be provided within a reasonable time for the user that is

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Unknowns:
point $P_0, ..., P_5$
Parameters:
length $h_0, ..., h_8$
Constraints:
$distance(P_0, P_1) = h_0$
...
$distance(P_2, P_4) = h_8$
$coplanar(P_0, P_1, P_2, P_3)$
$coplanar(P_0, P_3, P_4, P_5)$
$coplanar(P_1, P_2, P_4, P_5)$

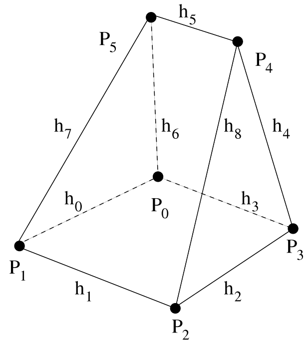

Fig. 1. A symbolic statement (left part) and a dimensioned sketch (right part) of the pentahedron problem. Edges are distances constraints of parameters h_i.

to say less than 15 s. Our approach takes into account all these features and quickly generates solutions to many problems of 3D constraints. The sketch and a plan for geometric construction, obtained by reparametrization, are used to control the search for solutions.

The rest of the text is structured as follows. Section 2 introduces notations about geometrical constraint systems. Basics on homotopy methods are given in Section 3 before outlining our method. Section 4 discusses specific homotopy paths and how they help in converging towards solutions. Section 5 addresses the issue of degenerated cases for instance involving coplanar faces. Section 6 introduces construction plans and their contribution in paths tracking. Sections 7 and 8 give some results and prospects.

2. Geometric constraint solving problems and construction plans

2.1. GCSP

In this paper, a Geometric Constraint Solving Problem (GCSP) is denoted by $G = (C, X, A)$ or also $C[X, A]$. It consists of a set C of constraints on a set X of geometric objects, depending on parameters A. A GCSP is defined over a geometric universe which specifies the dimension of the space where constructions are made, the types of geometric objects (points, circles, planes, *etc.*) and types of constraints (distances between two points, coplanarity of four points, *etc.*).

The 3D GCSP shown in Fig. 1 represents a well-constrained pentahedron with points $P_0, \ldots, P_5$. Among constraints some are *dimensioned* constraints such as distances, others are *Boolean* constraints such as coplanarity constraints. A numerical value for $x \in X \cup A$ is a *valuation* of x and is denoted $\sigma(x)$. For instance, in 3D, $\sigma(P_0) = (0, 0, 0)$ is a valuation of P_0. The notion of valuation is extended to any subset Y of $X \cup A$; a valuation $\sigma(Y)$ is a set of valuations for elements of Y.

2.2. Solutions

For a valuation of parameters $\sigma(A)$, a valuation $\sigma(X)$ is a *solution* (or a *figure*) of $G = C[X, A]$ if all the constraints c_i of G are satisfied.

A 3D dimensioned sketch of the pentahedron problem depicted in Fig. 1. This sketch is itself a valuation $\sigma_{sk}(X)$, and if it satisfies coplanarity constraints, it is a solution of G for some values $\sigma_{sk}(A)$ that can be read on the sketch.

Notice that if $\sigma(X)$ is a solution of $G = C[X, A]$, all solutions obtained by rigid body motions (translations and rotations) of $\sigma(X)$ are solutions of G since constraints are invariant under rigid body motions. The set of all solutions of G can be partitioned into equivalence classes such that two figures are in the same class if and only if they correspond each others by a rigid body motion. Only one solution of each class is relevant and some coordinates (6 in 3D) are fixed: we say that solutions are sought up to rigid body motions. In the pentahedron problem, one can fix the three coordinates of P_0, two coordinates of P_1 and one of P_2.

2.3. Construction plans

A construction plan $CP[O, A_p]$ is a sequence of terms $o_i = f(o_1, \ldots, o_{i-1}, A_p)$ with O a set of geometrical objects and A_p a set of parameters. It expresses a geometric construction where object o_i is built from previously built objects. Functions f usually compute loci intersections such as circle–line, circle–circle, sphere–sphere, and so on. Construction plans are commonly used in *geometric solver* and were introduced in [5] for instance to express symbolic solutions.

For a given GCSP, a geometric solver tries to produce a construction plan. Such a plan can then be evaluated numerically when numerical values are given for the parameters. Note that, some functions provide several values for results. For example, the intersection of two circles can contain two points. Therefore, numerical evaluation of a construction plan gives rise to a tree of solutions. Each branch corresponds to a solution. The benefit of having a construction plan is the ability to produce several or all solutions.

Unfortunately, most of the 3D examples from CAD cannot be fully treated by such geometric solvers. A method presented in [6] proposes to transform a GCSP by removing some numerical constraints and replacing them with others in order to make possible the creation of a construction plan. It is therefore a method of reparametrization of GCSP. In a second phase, numerical values for the added distances and angles are sought in order to satisfy the removed ones. This approach is effective when the number of added constraints is one or two, but is very expensive otherwise (see however [7]).

Our method is used for cases in which the geometric solvers fail or are not efficient enough. However, we will use a construction plan to guide our homotopy method. This plan will be obtained by reparametrization but it will not be to find values for the added parameters. It will be used to test some properties of solutions and to discover paths leading to new solutions.

2.4. Reading solutions

Homotopy methods proceed by deforming initial solutions. In our case, an initial solution is formed by the sketch on which the values for variables in $A \cup X$ are read. Values of unknowns are easily obtained from the sketch but the values of parameters must be calculated.

For each constraint $c(x_1, \ldots, x_n, a)$ where x_i is an unknown and a is a parameter, a function $a = f(x_1, \ldots, x_n)$ is assumed that computes the value of parameter a. For instance, for a constraint of distance between two points, function f simply calculates the Euclidean distance between two points.

In our method, getting parameter values is useful in various ways. Indeed, at key moments a construction plan is consulted. This construction plan comes from a reparametrization and corresponds to parameter values that slightly differ from the initial system. These values have then to be read from the current deformed figure.

3. Homotopy solving

Our method adapts a usual continuation method whose principles are summarized in the following.

3.1. Continuation methods

The goal of a classical homotopy resolution is to find the roots of a function $F : \mathbb{K}^m \to \mathbb{K}^m$. Roughly speaking the method starts

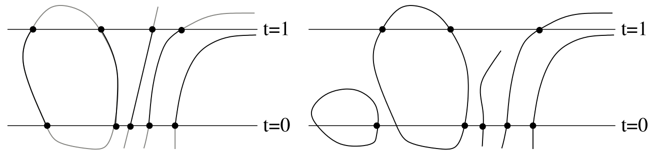

Fig. 2. Left: homotopy paths possibly encountered in a classical homotopy method (restrictions to hypervolume $\mathbb{K}^m \times [0, 1]$ are in black lines). Right: homotopy paths possibly encountered in the method described in this paper. Black points on lines $t = 0$ are initial solutions, those on lines $t = 1$ are found solutions.

from a function E having at least as many roots as F and whose roots are known. Then, paths from roots of E are tracked to roots of F. A function $H : \mathbb{K}^m \times [0, 1] \to \mathbb{K}^m$ is built such that $\forall x \in \mathbb{K}^m$, H satisfies $H(x, 0) = E(x)$ and $H(x, 1) = F(x)$. The interpolation variable of H is usually denoted by t.

If H is smooth and 0 is a regular value of H, $H^{-1}(0)$ consists of smooths manifolds of $\mathbb{K}^m \times [0, 1]$ of dimension 1, with boundaries equal to their intersections with $\mathcal{P}_0 \cup \mathcal{P}_1$ where $\mathcal{P}_0 = \mathbb{K}^m \times \{0\}$ and $\mathcal{P}_1 = \mathbb{K}^m \times \{1\}$. The left part of Fig. 2 shows five paths in $\mathbb{K}^m \times [0, 1]$, $\mathcal{P}_0$ and $\mathcal{P}_1$ are depicted by lines $t = 0$ and $t = 1$. For each x such that $F(x) = 0$, there exists a connected component $\mathcal{S}$ of $H^{-1}(0)$ with $(x, 1) \in \mathcal{S}$. In Fig. 2, solutions x are depicted by black circles on line $t = 1$.

Suppose now that some solutions of $H(x, 0) = 0$ are known (black circles on line $t = 0$ on Fig. 2), each one lies on a connected component that can be followed by a so-called path-tracking method until it crosses $\mathcal{P}_1$.

Several cases can occur depending on topology of components. Recall that a 1-dimensional connected smooth manifold is diffeomorphic either to a circle, or to an interval of real numbers. With $t \in [0, 1]$, a component of $H^{-1}(0)$ intersecting $\mathcal{P}_0$ cannot be diffeomorphic to a circle, since a circle does not have any boundary and does not intersects $\mathcal{P}_0$. It is thus diffeomorphic to an interval of real numbers that is either closed, and intersects $\mathcal{P}_1$, or half-closed. The rightmost paths of the left part of Fig. 2 do not cross $\mathcal{P}_1$ and have infinite length. Elements of such connected components are called solution at infinity.

In literature (see [8–10] for example), for polynomial systems $F = 0$, H is constructed by interpolating a simple algebraic system $E = 0$ with known solutions, called *start system*, with system $F = 0$. If E has as many roots as F and $\mathbb{K} = \mathbb{C}$, each root of F can be reached by following a path from a solution of $E = 0$. Solutions at infinity of $F = 0$ are classically detected by the use of some homogenization of components of H.

A rough upper bound to the number of roots of F is given by the well known Bezout number. When polynomials of F are sparse, this number over-estimates from far the number of roots of F, and too much paths are followed. [11] exploits an homogeneous structure on H to reduce the number of roots of the start system, and [12] presents a method that uses Newton polytopes to compute a smaller upper bound to the number of roots of F. Path-following is usually performed by a prediction–correction method, that is robust when based on interval arithmetic (see [13,14]). However, these methods become very time-consuming when the number of equations of F grows, due to the number of paths to follow, and to the path-tracking method itself that needs to compute the inverse of the Jacobian matrix of H.

3.2. Outline of the method

Consider the 3D GCSP $G = C[X, A]$ shown in Fig. 1 that consists of constructing a pentahedron with 6 vertices, knowing lengths of its 9 edges and coplanarity constraints for 3 faces including 4 points each. Points P_i are associated with coordinates $\sigma(P_i)$, and lengths h_i with real numbers $\sigma(h_i)$. Solutions $\sigma(X)$ are sought with respect to reference $P_0P_1P_2$ where $\sigma(P_0) = (0, 0, 0)$, $\sigma(P_1) = (h_0, 0, 0)$ and $\sigma(P_2)$ lie in plane $z = 0$. Constraint coplanar(P_0, P_1, P_2, P_3) can be satisfied and deleted from the GCSP by setting the last coordinate of P_3 to 0. So only 11 constraints remain.

There are 11 coordinates to find in order to determine $X = \{P_1, \ldots, P_5\}$. If there is no ambiguity, we assimilate the set X of geometric objects with the set of all free coordinates of its elements. Thus, one can easily define a numerical function $F : \mathbb{R}^{11} \to \mathbb{R}^{11}$ involving X (*e.g.* the set of coordinates of its objects) whose components are functions associated with constraints.

For instance, with P_iP_j meaning the Euclidean distance between points P_i and P_j, component corresponding to constraint distance$(P_2, P_4) = h_8$ is

$$F_8(X) = P_2P_4 - h_8,$$

and the one associated with coplanar(P_0, P_3, P_4, P_5) is

$$F_9(X) = \det(\overrightarrow{P_0P_3}, \overrightarrow{P_0P_4}, \overrightarrow{P_0P_5}).$$

$\sigma(X)$ is a solution of G if and only if $F(\sigma(X)) = 0$. Note that function $F(X)$ also depends on parameters. When parameters of set A must explicitly appear, we note it $F(X, A)$.

Coordinates of points in sketch of Fig. 1 are given by a user and are denoted by $\sigma_{sk}(X)$. Figure $\sigma_{sk}(X)$ is a solution for values $\sigma_{sk}(A)$ of parameters, that are values of lengths read on the sketch.

As in [4], the sketch is used to produce an initial system to solve G by a continuation method. Values $\sigma_{sk}(A)$ are interpolated to values $\sigma_{so}(A)$ for which solutions are guessed in order to define an homotopy function H depending on (X, t). So, function H is defined by $H(X, t) = F(X, d(t))$ where $d(t)$ continuously transforms $\sigma_{sk}(A)$ when $t = 0$ to $\sigma_{so}(A)$ when $t = 1$. For instance, we can choose the linear interpolation $d(t) = (1 - t)\sigma_{sk}(A) + t\sigma_{so}(A)$.

Our method performs the following steps.

`Continuation solving`
`Input`: GCSP $C[X, A]$, valuation $\sigma_{so}(A)$, sketch $\sigma_{sk}(X)$
`Output`: list of solutions $\mathcal{L}_{sol}$

1. Read values $\sigma_{sk}(A)$ from sketch $\sigma_{sk}(X)$.
2. Build homotopy function $H(X, t) = F(X, d(t))$ s.t. $H(\sigma_{sk}(X), 0) = 0$ and d interpolates values of parameters $\sigma_{sk}(A)$ to $\sigma_{so}(A)$. Here $t \in \mathbb{R}$.
3. Follow path of $H^{-1}(0)$ from $(\sigma_{sk}(X), 0)$ until it loops on $(\sigma_{sk}(X), 0)$, while checking if:
   – it converges to a solution at infinity, then stop;
   – it converges to a non-derivable figure, then stop;
   – it crosses hyperplane $\mathcal{P}_1$, then append the solution to $\mathcal{L}_{sol}$.
4. For each solution found, use a construction plan to generate new sketches, call Continuation solving for them.

The checking step will be detailed later. Moreover the algorithm does not mention that a construction plan is used for detecting a solution at infinity.

Unlike classical homotopy methods, here only one solution $(\sigma_{sk}(X), 0)$ is known, and only one path can be tracked. However, this path can cross several times $\mathcal{P}_1$, and we track it entirely, not only for t in [0, 1]. In this sense, our approach is close to the one called global homotopy in p. 81 of [15].

Section 4.2 presents a characterization of homotopy paths: basically they are either diffeomorphic to circles or to open intervals of $\mathbb{R}$. The case where the path to which belongs the sketch is diffeomorphic to a circle is the most beneficial in the two following ways: first, the same path can contain several solutions, then the tracking process can be stopped when the path overlaps. When a path is diffeomorphic to an open interval, it can have infinite length, and its tracking can lead to an infinite computation. The different kinds of paths that are followed by our method are presented in the right part of Fig. 2.

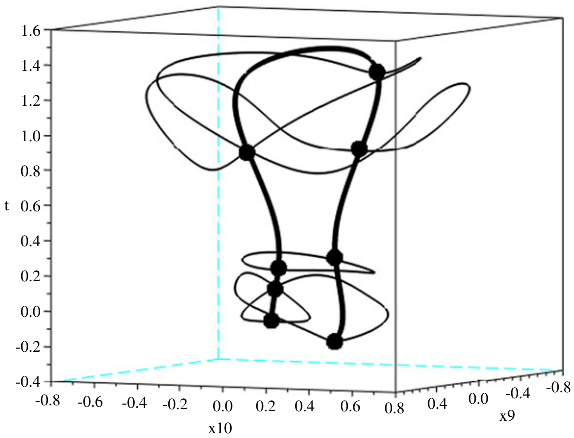

Fig. 3. A projection of a connected component of $H^{-1}(0)$ for the pentahedron problem: H does not meet the regularity condition. The black circles represent points that are not regular since they are crossing points of paths.

Paths are tracked by a classical prediction–correction method: knowing a point of the followed curve, a new point is predicted along the tangent of the curve on this point; this prediction is corrected by using Newton's method.

This characterization of paths holds when homotopy function meets a regularity condition. Its Jacobian matrix has to be on full rank on all points of the path. When this condition is not satisfied, connected components of $H^{-1}(0)$ are no longer 1-dimensional manifolds. It can consist of 1-dimensional sets that cross other similar sets or objects of higher dimensions. Fig. 3 shows one connected component of $H^{-1}(0)$ for the pentahedron problem. The bold line crosses some 1-dimensional sets at irregular points. It cannot be tracked by a classical path-tracking method since several paths can be followed from these points. We have slightly adapted the path-tracking method to take not regular points into account, as presented in Section 7.1. This issue seems to be very rare when solving systems by classical homotopy methods, and easily addressed by a probabilistic argument. However, such situations arise when solving geometric constraint problems in real space containing Boolean constraints such as coplanarity. They correspond to special geometric configurations such as two planes that become coincident during the tracking. Those cases are considered as degeneracies.

It will be first assumed, in order to characterize homotopy paths, that H meets the regularity condition. Section 5 is dedicated to cases where this assumption does not reflect the reality.

Once the path containing the sketch has been entirely tracked, one or more solutions of the GCSP to solve are known. Section 6.1 presents a way of obtaining new sketches from these solutions by using construction plans of slightly modified GCSP. Thus new homotopy functions are defined, and new homotopy paths are followed. Each new path can provide several new solutions. This process can be iterated until no new solution is obtained.

4. Homotopy paths

4.1. Homotopy function and parameter interpolation

Each constraint $c_i \in C$ is naturally associated to a real function f_i of X. If c_i depends on a parameter h_i, f_i depends on its value $\sigma(h_i)$. If this is not the case, for the sake of simplicity, we consider that f_i depends on a null value $\sigma(h_i)$.

Definition 1. We call interpolation function from $\alpha \in \mathbb{R}$ to $\beta \in \mathbb{R}$ a C^∞ function $d : \mathbb{R} \to \mathbb{R}$ that satisfies $d(0) = \alpha$ and $d(1) = \beta$. Let $A = \{a_0, \dots, a_{p-1}\}$ be a set of parameters. We say that $d : \mathbb{R} \to \mathbb{R}^p$ is an interpolation function from $\sigma_1(A)$ to $\sigma_2(A)$ if, for $0 \le j \le p-1$, its component d_j is an interpolation function from $\sigma_1(a_j)$ to $\sigma_2(a_j)$.

If m is the number of free coordinates of X, and a an interpolation function from $\sigma_{sk}(A)$ to $\sigma_{so}(A)$, we define the homotopy function $H : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ as:

$$H(X, t) = \begin{pmatrix} f_0(X, a_0(t)) \\ \dots \\ f_{m-1}(X, a_{m-1}(t)) \end{pmatrix} \quad (1)$$

and we assume that H is C^∞ over an $m + 1$-dimensional manifold $\mathcal{D} \subseteq \mathbb{R}^m \times \mathbb{R}$, without any border.

In the case of the pentahedron problem, the homotopy function is $H : \mathbb{R}^{12} \to \mathbb{R}^{11}$. Its 8-th component that corresponds to constraint distance$(P_2, P_4) = h_8$ is $H_8(X, t) = P_2P_4 - a_8(t)$, where a_8 is an interpolation function from $\sigma_{sk}(h_8)$ to $\sigma_{so}(h_8)$. H_8 is defined over $\mathbb{R}^{12}$, but is not differentiable when $\sigma(P_2) = \sigma(P_4)$, that is an hyperplane of dimension $11-3$ in the space of coordinates of X. The 9-th component of H, corresponding to a coplanarity constraint, is C^∞ over $\mathbb{R}^m \times \mathbb{R}$. Thus, since constraints of the pentahedron problem are either distance or coplanarity constraints, H is C^∞ over $(\mathbb{R}^{11} \setminus \mathcal{H}) \times \mathbb{R}$, where $\mathcal{H}$ is a reunion of hyperplanes of dimensions $11 - 3$ (one by distance constraint).

4.2. Characterizing homotopy paths

In order to characterize the set $H^{-1}(0)$, we introduce the notion of regularity of values of a smooth function.

Definition 2. Let $f : \mathcal{C} \subset \mathbb{R}^p \to \mathbb{R}^q$ be a C^∞ map, and $Jf(x)$ its Jacobian matrix on $x \in \mathcal{C}$. If rank$(Jf(x)) < q$, we call x a critical point of f and $f(x)$ a critical value of f.

Let $H : \mathcal{D} \subseteq \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ be a homotopy function C^∞ over an $m + 1$-dimensional manifold $\mathcal{D}$ without any border. As a consequence of two lemmas that can be found on p. 11 and 13 of [16], we state the following proposition:

Proposition 3. *If* $0 \in \mathbb{R}^m$ *is a regular value of* H*, connected components of* $H^{-1}(0)$ *are* 1*-dimensional closed smooth submanifolds of* $\mathcal{D}$*, without any border.*

A connected component of $H^{-1}(0)$ is called a homotopy path of H. The theorem of classification of 1-dimensional manifolds (see [16, p. 54]) states that any connected such manifold without boundary is diffeomorphic either to a circle or to any open interval of $\mathbb{R}$ (including $\mathbb{R}$ itself). If a connected component is diffeomorphic to an open interval, it can have infinite length. Proposition 5 below states that under a simple condition on the interpolation functions, such a component is either a solution at infinity, or converges to a point outside $\mathcal{D}$, or both (one side of the path is a solution at infinity, the other side converges). Such a case is shown in the right part of Fig. 2 by the middle path. It is diffeomorphic to any open interval of $\mathbb{R}$, and one of its extremities converges to a point outside of $\mathcal{D}$. Considering it as a subset of $\mathbb{R}^{m+1}$ leads to seeing it as neither an open, nor closed set. However, it is closed in $\mathcal{D}$.

Definition 4. If d is an interpolation function between $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, we call positive support of d, and note it Supp(d) the set $\{x \in \mathbb{R} | d(x) \ge 0\}$.

For a tuple $x \in \mathcal{D}$, we note $\pi_t(x) \in \mathbb{R}$ the projection on its last component. Let H_i be a component of H which corresponds to a distance constraint with interpolation function a_i: since a distance cannot be negative, it is obvious that we have $\forall x \in H^{-1}(0),\ \pi_t(x) \in$ Supp(a_i).

Unknowns:
point $P_0, \ldots, P_5$
Parameters:
length $h_0, h_1, h_5 \ldots, h_9$
angle a_0, a_1, a_2
Constraints:
$distance(P_0, P_1) = h_0$
...
$distance(P_2, P_0) = h_9$
$angle(P_2, P_3, P_4) = a_0$
$angle(P_2, P_3, P_0) = a_1$
$angle(P_1, P_0, P_3) = a_2$
$coplanar(P_0, P_1, P_2, P_3)$
$coplanar(P_1, P_2, P_4, P_5)$

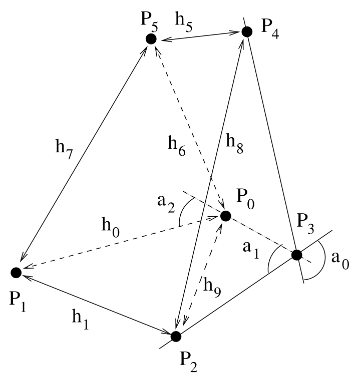

Fig. 4. A statement and a dimensioned sketch of the GCSP 6p7d3a2c. Oriented hyper-edge represents the coplanarity constraint, straight double arrows are distance constraints of parameters h_i and arcs of circle between two lines mark angle constraints of parameters a_i.

Proposition 5. *Let $\mathcal{S}$ be a connected component of $H^{-1}(0)$. Suppose that H_0 corresponds to a constraint of distance, and that* $\mathrm{Supp}(a_0)$ *is compact. Assume that $0 \in \mathbb{R}^m$ is a regular value of H. If $\mathcal{S}$ is diffeomorphic to an open interval, at least one of the two following situations arises:*

(i) $\exists t_l \in \mathrm{Supp}(a_0)$ *s.t. elements of $\mathcal{S}$ are solutions at infinity of the system $H(X, t_l) = 0$,*
(ii) $\exists(\sigma_l(X), t_l) \in (\mathbb{R}^m \times \mathbb{R}) \setminus \mathcal{D}$ *s.t. $(\sigma_l(X), t_l)$ is a point of closure of $\mathcal{S}$.*

The proof of this assertion is postponed to Section 4.3. We illustrate it on a new example, called 6p7d3a2c, of which the statement and a dimensioned sketch are given in Fig. 4. This problem is similar to the pentahedron problem: one coplanarity (on the face (P_0, P_3, P_4, P_5)) and three distance constraints (those of parameters h_2, h_3 and h_4) have been removed and replaced by three constraints on angles $\widehat{P_2P_3P_4}$, $\widehat{P_2P_3P_0}$, $\widehat{P_1P_0P_3}$ and the distance constraint of parameter h_9. Let $G = C[X, A]$ be the GCSP associated with this problem, and $\sigma_{sk}(X)$ a sketch. We solve it with three different valuations $\sigma_1(A), \sigma_2(A), \sigma_3(A)$ that straightforwardly highlight the conclusion of Proposition 5.

Except for the angle measures, values of the initial system equals values for which solutions are sought: $\sigma_i(A \setminus \{a_j\}) = \sigma_{sk}(A \setminus \{a_j\})$. Except h_0, the distance between P_0 and P_1, all parameters are linearly interpolated. Thus the values of parameters h_j, with $j \neq 0$, do not vary with t. h_0 is interpolated by a quadratic function with a negative leading coefficient that has a compact positive support. The interpolation functions of a_j are denoted by d_j.

When solving G for σ_2, the imposed value for a_2 is the same as in the sketch (d_2 is constant). Suppose that there exists a value $t_i \in \mathrm{Supp}(d)$ such that $d_0(t_i) = d_1(t_i) = 0$. Then a figure where $\widehat{P_2P_3P_4} = \widehat{P_2P_3P_0} = 0$ is a solution of G. On such a figure, since the three lines passing by P_3 are not equal, P_3 is infinitely far from other points, and has at least one of its coordinates that is not bounded. The curve in dotted line on Fig. 5 is the projection of a homotopy path of H for which this situation arises, with $t_i \simeq 1.25$. This path is diffeomorphic to $]0, 1[$, and its points are solutions at infinity of the system $H(X, t_i) = 0$.

When the targeted valuation is σ_3, there exists t_1 such that $d_2(t_1)$ (value of $\widehat{P_1P_0P_3}$) is equal to the value of $\widehat{P_1P_2P_0}$, and $d_1(t_1) \in]0, \pi[$. Then P_3 is either equal to P_0, or equal to P_2, and angle $\widehat{P_1P_0P_3}$ has no more sense. In this case, a homotopy path converges to a point where H is not defined. The curve in the dotted line (one long, two shorts) on Fig. 5 illustrates such a path, with $t_1 \simeq 0.7$.

With imposed valuation σ_1, the connected component of $H^{-1}(0)$ to which belongs the sketch is diffeomorphic to a circle (see path in the solid line of Fig. 5). Notice that four solutions for this valuation are found when the homotopy path is entirely tracked, since it crosses four times $\mathcal{P}_1$.

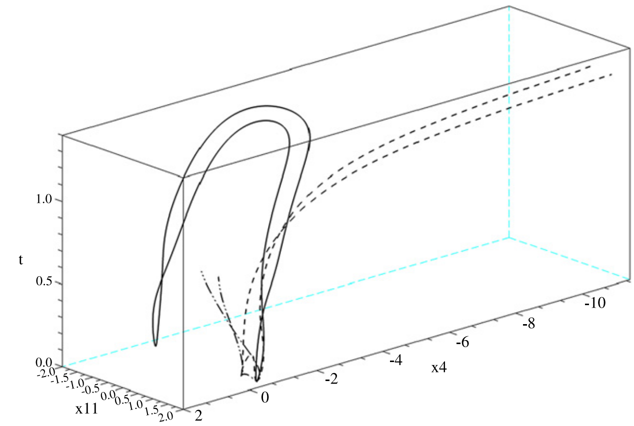

Fig. 5. A projection in $\mathbb{R}^3$ of three homotopy paths for the problem 6p7d3a2c for targeted valuations of parameters σ_1 (solid line), σ_2 (dotted line) and σ_3 (dotted line: one long, two shorts).

4.3. Proof of Proposition 5

Let $\mathcal{S}$ be a homotopy path of function H, such that there is a diffeomorphism $\varphi :]0, 1[\to \mathcal{S}$. To avoid confusions, we denote by x the vector corresponding to the set of free coordinates of objects of X. We first show the lemma:

Lemma 6. *Let $t \in \mathbb{R}$, and $H_t : \mathbb{R}^m \to \mathbb{R}^m$ be the function defined by $H_t(x) = H(x, t)$. If 0 is a regular value of H, vectors $x \in \mathbb{R}^m$ s.t. $H_t(x) = 0$ are isolated, and in finite number.*

Indeed, according to the lemma that can be found on p. 11 of [16], $H_t^{-1}(0)$ is a manifold of dimension 0, that consists of isolated points of $\mathbb{R}^m$. Another way of proving this fact is to remark that vectors $x \in \mathbb{R}^m$ s.t. $H_t(x) = 0$ are the solutions of a polynomial system, that only admits a finite number of isolated solutions.

Next, we prove that there exists t_l with $\lim_{z->1} \pi_t(\varphi(z)) = t_l$. Since $\mathrm{Supp}(a_0)$ is compact, there exists $t_0 \in \mathbb{R}_+$ with

$$\forall z \in]0, 1[, \quad -t_0 < \pi_t(\varphi(z)) < t_0.$$

Let $t_1 \in]-t_0, t_0[$. From Lemma 6, it exists $M_1 \in]0, 1[$ s.t.

$$\forall z > M_1, \quad \pi_t(\varphi(z)) < t_1 \quad \text{or} \quad \pi_t(\varphi(z)) > t_1$$

(otherwise there would be an infinity of x on $\mathcal{S}$ such that $\pi_t(x) = t_1$ and $H_{t_1}(x) = 0$). Then, for every real number z greater than M_1, we have:

(i) $\pi_t(\varphi(z)) \in]-t_0, t_1[$, or
(ii) $\pi_t(\varphi(z)) \in]t_1, t_0[$.

Note $\mathcal{I}_1 =]-t_0, t_1[$ if (i) holds, $\mathcal{I}_1 =]t_1, t_0[$ otherwise. Let t_2 be the middle of interval $\mathcal{I}_1$. From Lemma 6, there exists $M_2 \in]M_1, 1[$ s.t.

$$\forall z > M_2, \quad \pi_t(\varphi(z)) < t_2 \quad \text{or} \quad \pi_t(\varphi(z)) > t_2.$$

Note $\mathcal{I}_2 =]\inf(\mathcal{I}_1), t_2[$ if $\pi_t(\varphi(z)) \in]\inf(\mathcal{I}_1), t_2[$, $\mathcal{I}_2 =]t_2, \sup(\mathcal{I}_1)[$ otherwise, where $\inf(\mathcal{I})$ (respectively $\sup(\mathcal{I})$) is the biggest (resp. smallest) real number less (resp. greater) than all elements of an interval $\mathcal{I}$. Applying recursively this reasoning, we construct a sequence $(\mathcal{I}_n)_{n\in\mathbb{N}}$. If we note $\mathrm{midd}(\mathcal{I})$ the middle of $\mathcal{I}$, it is easy to show that the three sequences $(\inf(\mathcal{I}_n))_{n\in\mathbb{N}}$, $(\mathrm{midd}(\mathcal{I}_n))_{n\in\mathbb{N}}$ and $(\sup(\mathcal{I}_n))_{n\in\mathbb{N}}$ converge to the same limit $t_l \in \mathrm{Supp}(a_0)$, and that $\lim_{z\to 1} \pi_t(\varphi(z)) = t_l$.

To conclude the proof of Proposition 5, remark that we can distinguish two cases: either $\mathcal{S}$ has its closure $(\mathcal{S})^\circ$ entirely in $\mathcal{D}$, or there are a vector $(\sigma_l(X), t_l) \in (\mathbb{R}^m \times \mathbb{R}) \setminus \mathcal{D}$ and a sequence of vectors of $\mathcal{S}$ converging to it.

If $(\mathcal{S})^\circ \subset \mathcal{D}$, $\mathcal{S}$ is closed in $\mathbb{R}^m \times \mathbb{R}$. Since $\mathcal{S}$ is not compact, it is not bounded, and $\forall M \in \mathbb{R}, \exists s_1, s_2 \in \mathcal{S}$ s.t. $\|s_2 - s_1\| > M$. Then $\lim_{z\to 1} \|\varphi(z)\| = +\infty$, and elements of $\mathcal{S}$ are solutions at infinity of the system $H_{t_l}(x) = 0$.

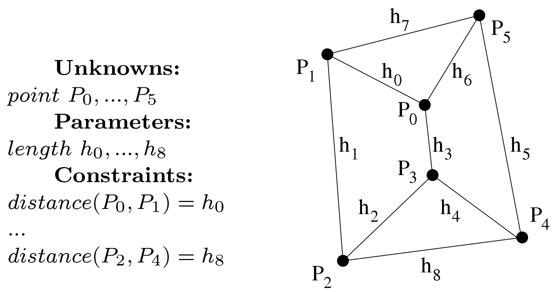

Fig. 6. A statement and a dimensioned sketch of a 2D GCSP, that can be seen as the flattened pentahedron problem.

5. Degenerated cases

In this section, we focus on the hypothesis of regularity of value 0 for homotopy function H. When solving problems that only involve constraints with parameters (*e.g.* distances, angles), cases where H does not satisfy this hypothesis seem very rare, and we never find them. It is justified by Sard's Theorem (see among others p. 77 of [15]) adapted here to our notations: the set of valuations $\sigma_{sk}(A)$, $\sigma_{so}(A)$ for which 0 is not a regular value of H is negligible in the sense of Lebesgue measure. In classical homotopy methods, a complex perturbation of parameters of homotopy function guarantees regularity of 0. When solving systems in real space, [17] shows that with randomly chosen parameters values, critical points are in a finite number, and do not affect path-tracking in a real space.

When Boolean constraints (*e.g.* coplanarity) are also considered, critical points are no longer finite in number, and correspond to degenerated configurations, for instance three collinear points, or two planes that coincide. When these critical points are also solutions of the GSCP, they could belong to a set of solutions of strictly positive dimension. We adapt the homotopy function in order to cross these sets and continue the tracking.

5.1. A simple example

Let $G = C[X, A]$ be the 3D GCSP of the pentahedron problem (see Fig. 1), $\sigma_{sk}(X)$ be a sketch, $\sigma_{sk}(A)$ be the values of parameters read on it, and $\sigma_{so}(A)$ be those that have to be fulfilled by solutions. We call $H(X, t)$ the homotopy function constructed with interpolation function d, and $\mathcal{S}$ the connected component of $H^{-1}(0)$ to which belongs $(\sigma_{sk}(X), 0)$.

Consider now the GCSP given in Fig. 6, that consists of constructing 6 points in a plane knowing 9 distances between some of these points. We call it $G_2 = C_2[X_2, A_2]$. Remark that $X_2 = X, A_2 = A$, and $C_2 \subset C$. Actually, the sole constraints that are not in G_2 concern coplanarity and are automatically fulfilled in 2D. It is obvious that for any valuation $\sigma(A)$, a solution $\sigma(X)$ of G_2 can be straightforwardly translated into a solution of G by embedding it in 3D. In other words, for any valuation $\sigma(A)$, there is at least one "degenerated" planar solution among all the solutions.

Now, when parameter t is varying, all planar solutions belong to a particular collection of 1-dimensional sets corresponding to homotopy paths we could have by considering the homotopy function associated with G_2. It could happen that when tracking a 3D solutions along a path $\mathcal{S}$, a solution $(\sigma_2(X), t_2) \in \mathcal{S}$ that lies on a plane is encountered. $(\sigma_2(X), t_2)$ also belongs to a path, say $\mathcal{S}_2$, containing only planar solutions; it is a critical point.

If objects of the sketch $\sigma_{sk}(X)$ do not all lie in the same plane, $\sigma_{sk}(X)$ is not a solution of G_2, and $\mathcal{S}_2 \neq \mathcal{S}$. Thus $\mathcal{S}_2$ is strictly included in $\mathcal{S}$, and $\mathcal{S}$ is not a 1-dimensional manifold and the hypothesis of regularity is not satisfied; $\mathcal{S}$ contains critical points.

Fig. 3 shows a projection on $\mathbb{R}^3$ of the connected component $\mathcal{S} \subset H^{-1}(0)$. Dimensions on axes are x_9 and x_{10}, the two last coordinates of point P_5, and t on the vertical axes. The bold curve, of which points satisfy $x_{10} = 0$, is $\mathcal{S}_2$, and figures of this curves are solutions of G_2. When following $\mathcal{S}$ from $(\sigma_{sk}(X), 0)$, $\mathcal{S}_2$ is crossed in critical points, depicted by black circles.

We end the discussion on this example by remarking that having all the points on the same plane on a solution of G is not a coincidence. Firstly, on a solution $\sigma(X)$ of G on which P_0, P_3, P_4 and P_1, P_2, P_5 are not collinear, $\sigma(P_4)$ lies on the plane Pl_0 passing by $\sigma(P_0), \dots, \sigma(P_3)$, if and only if f $\sigma(P_5)$ lies likewise on Pl_0. Then all points are coplanar if and only if the third coordinate of P_4 or P_5 vanishes, which corresponds to an hyperplane in the total space $\mathbb{R}^{12}$, that separates this space. Configurations on which P_0, P_3, P_4 and P_1, P_2, P_5 are collinear correspond in the space of configurations to the intersection of two sets of dimensions $\mathbb{R}^{10}$ that does not exist in generic cases.

5.2. Tracks to manage degeneracies

In the previous example, critical points are roughly speaking, intersections of two 1-dimensional sets. More complicated situations can arise when problems are generically under-constrained. In such cases, the followed path crosses a manifold of higher dimension.

We propose here to substitute Boolean constraints by metric constraints with non-identically null parameters values. For instance, a coplanarity constraint coplanar(P_0, P_3, P_4, P_5) can be substituted by the constraint of volume $\det(\overrightarrow{P_0P_3}, \overrightarrow{P_0P_4}, \overrightarrow{P_0P_5}) = h$ with parameter h. When $\sigma(h) = 0$, these two constraints are equivalent.

Remark this substitution also allows the user to draw a sketch that does not satisfy Boolean constraints.

Practically speaking, for Boolean constraints $c_m, \dots, c_n$ of parameters $h_m, \dots, h_n$ (that were introduced for simplicity), we use interpolation functions $d_m(t), \dots, d_n(t)$ satisfying

(i) $(d_m(0), \dots, d_n(0)) = \sigma_{sk}(h_m, \dots, h_n)$,
(ii) $(d_m(1), \dots, d_n(1)) = (0, \dots, 0)$,
(iii) $t \in \mathbb{R} \setminus \{0, 1\} \Rightarrow d_i(t) \neq 0$ for $i \in \{m, \dots, n\}$.

We assume that with such interpolation functions, for almost all valuations $\sigma_{sk}(A)$, $\sigma_{so}(A)$ and $t \in \mathbb{R} \setminus \{0, 1\}$, homotopy function H has either 0 as a regular value, or admits only a finite number of critical points. We justify this assumption by the following. Let $F(X, A, B)$ be the numerical function associated to a problem, depending on X, the unknowns, A the parameters of metric constraints, and B the parameters added to make Boolean constraints metric. Then if 0 is a regular value of F, Sard's theorem states that for almost all $\sigma(B)$, 0 is a regular value of $F_{|\sigma(B)}(X, A) = F(X, A, \sigma(B))$. If $\sigma(B)$ is such that $F_{|\sigma(B)}(X, A)$ does not admit 0 as a regular value, whose appendage in shown examples with $\sigma(B) = (0, \dots, 0)$, then for almost all $\sigma'(B)$, $F_{|\sigma(B)+\sigma'(B)}(X, A)$ admits 0 as a regular value. Intuitively, the interpolation function d provides a little perturbation.

In the case of the pentahedron problem, consider that for at least one coplanarity the interpolation function d_i satisfies above conditions. If $t \neq 0, t \neq 1$ and $(\sigma(X), t) \in \mathcal{S}$ ($\mathcal{S}$ is the followed path), points of $\sigma(X)$ are not all coplanar, and $\sigma(X)$ does not satisfy G_2, for almost all valuation of parameters. If $t = 1$ and if $\sigma(X)$ is solution of G_2, let $\mathcal{S}_2$ be the homotopy path for G_2 with $(\sigma(X), t) \in \mathcal{S}_2$. Since G_2 admits only a finite number of solutions for valuation $\sigma_{so}(A)$, $\mathcal{S}_2$ intersects only in a finite number of points the hyperplane $\mathcal{P}_1$, and $\mathcal{S} \cap \mathcal{S}_2$ consists of isolated points.

Another example is given Fig. 7. It consists of yielding an octahedron (6 quadrangular faces, 12 edges, 8 vertices) knowing lengths of its 12 edges. The associated GCSP $G = C[X, A]$ involves

Unknowns:
point $P_0, ..., P_7$
Parameters:
length $h_0, ..., h_{11}$
Constraints:
$distance(P_0, P_1) = h_0$
...
$distance(P_4, P_5) = h_{11}$
$coplanar(P_0, P_1, P_2, P_3)$
$coplanar(P_1, P_2, P_4, P_7)$
$coplanar(P_2, P_3, P_5, P_4)$
$coplanar(P_3, P_0, P_6, P_5)$
$coplanar(P_0, P_1, P_7, P_6)$
$coplanar(P_4, P_5, P_6, P_7)$

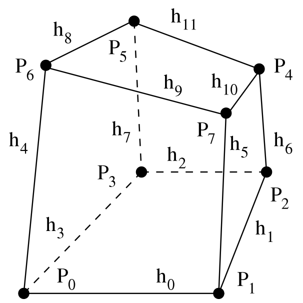

Fig. 7. A statement and a dimensioned sketch for the octahedron problem.

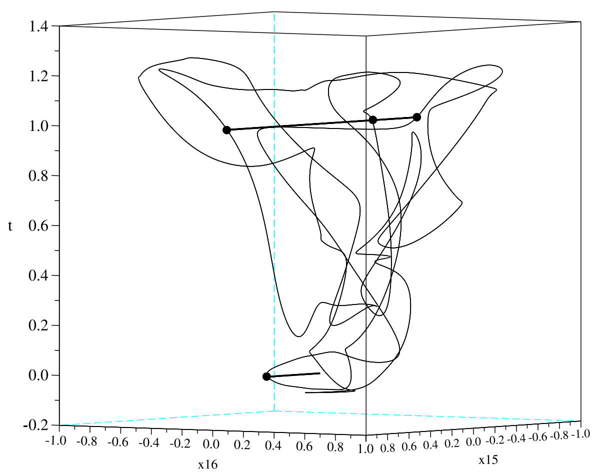

Fig. 8. A projection of a connected component of $H^{-1}(0)$ for the hexahedron problem. x_{15}, x_{16} (dimensions on axes) are the two last coordinates of P_7.

12 constraints of distance between points, and 6 coplanarity constraints. As in the case of the pentahedron, one vertex, one edge and one face are chosen as reference, and the homotopy function is $H : \mathbb{R}^{18} \to \mathbb{R}^{17}$, constructed with interpolation function d. For certain parameter values, it can happen that a solution is flattened with all points in the same plane. Such a solution is also a solution of a 2D GCSP $G_2 = C_2[X, A]$ of 8 points and 12 distances. But in 2D, this problem is under-constrained; indeed, a necessary condition for a 8-points problem to be well-constrained is that it has 13 constraints.

When following homotopy path $\mathcal{S}$ of function H, if a point $(\sigma_2(X), t_2)$ such as figure $\sigma_2(X)$ is planar is encountered, $\sigma_2(X)$ is a solution of G_2. Applying the same reasoning as in the case of the pentahedron problem yields that $\mathcal{S}$ crosses a continuum of solutions of G_2 embedded in a 3D space. In this case, this continuum is a two dimensional manifold. Indeed, associated homotopy function H_2 has 2 more unknowns than equations. In what follows, we consider that H_2 is the homotopy function of G_2 embedded in 3D, thus $H_2 : \mathbb{R}^{18} \to \mathbb{R}^{16}$.

If parameters of coplanarity constraints are interpolated with interpolation such that (i)–(iii) hold, a solution of G_2 satisfies G for valuation $d(t)$ if and only if $t = 1$ or $t = 0$ if the sketch satisfies coplanarity constraints. Bold lines on Fig. 8 are projections of a part of $(H_2^{-1}(0)) \cap (H^{-1}(0))$ for the hexahedron problem.

Consider now the set $\mathcal{S}' = \mathcal{S} \setminus H_2^{-1}(0)$, that contains elements of $\mathcal{S}$ that are not solutions of $H_2(.) = (0)$, and the set $\mathcal{S}'' = \{(x_2, t_2) \in H_2^{-1}(0) | (x_2, t_2) = \lim_{t \to \{0,1\}} (x, t), (x, t) \in \mathcal{S}\}$. Black circles on Fig. 8 represent some points of $\mathcal{S}''$. We make the assumptions that $\mathcal{S}''$ consists of a finite number of points, and that a connected component of $\mathcal{S}' \cup \mathcal{S}''$ is diffeomorphic either to a circle, or to a line going to a solution at infinity. Fig. 8 shows the projection of a connected component of $\mathcal{S}' \cup \mathcal{S}''$, that can be entirely tracked from an initial points (the sketch). This homotopy path crosses $H_2^{-1}(0)$ (bold lines) on a finite number of points (black circles) corresponding to solutions of G_2, the flattened problem. Intersections of $\mathcal{S}'$ with $\mathcal{P}_1$ are then solutions of G.

6. Construction plans as guides

When solving GCSP $G = C[X, A]$, we use a construction plan P for X. GCSP considered are not solvable geometrically and P is a plan for a geometrically solvable problem close to G. The only thing that matters is that this plan covers the same unknowns.

Plan P helps to find other solutions to G (see Section 6.1). On the other hand, when following an homotopy path, P can be used to detect if this path goes to a solution at infinity (see Section 6.2). In that case, the following process of the path can be stopped, and no solutions are missed.

A simple method to get a construction plan is the Locus Intersection Method (LIM [6]). It transposes a set of constraints into loci and constructs objects by intersecting loci. LIM is very efficient when it works, but solves very few problems in 3D. For instance, the pentahedron problem cannot be translated into a construction plan by LIM, since points P_i are each involved in 5 constraints and should be at the intersection of 5 loci (one sphere by distance constraint, one plane by coplanarity constraint), which cannot hold in generic cases. When system G is not solvable by LIM, an idea picked-up from [6] is to slightly modify G by changing some constraints to obtain a new GCSP G' solvable by LIM. For generic valuations of parameters of the modified system G', P yields figures that fulfill all constraints of G but the removed ones. An important issue lies in efficiently finding values for added parameters such that figures resulting of the CP evaluation also fulfill removed constraints (see for instance [18]). This method is called reparametrization in [19]. We use this method to obtain a construction plan which almost solve the GCSP, but, on the contrary to [6] or [19], this construction plan is not used to numerically catch the removed constraints.

6.1. Obtaining new solutions

Let $G = C[X, A]$ be a GCSP. Using the GCSP syntax, we note $P = C'[X, A']$ a CP for the set X with parameters A'. Here, C' denotes an ordered list of terms that constructs geometric objects from those previously constructed, and from parameters (reference or measure). For instance, in:

$$C'[X, A'] = \{\ldots, S_0 = \text{mkSphere}(P_0, h_0), \\ S_1 = \text{mkSphere}(P_1, h_1), S_2 = \text{mkSphere}(P_2, h_2), \\ P_3 = \text{intersect}(S_0, S_1, S_2), \ldots\}$$

h_i are parameters and P_0, P_1, P_2 are either parameters, or constructed by previous instructions. Notice that all leaves of the CP do not correspond to a figure, particularly when for the valuation of parameters, an instruction cannot be realized. Such a case arises in the CP given above when S_0, S_1 and S_2 do not have any intersections. Knowing a figure $\sigma_0(X)$, one can easily determine the pair $(\sigma_0(A'), b_0)$ such that evaluating P on branch b_0 with parameters $\sigma_0(A')$ constructs this figure. In above example, for finding h_0, one can evaluate on $\sigma_0(X)$ the distance between P_0 and P_3, then $\sigma_0(P_3)$ is compared with the two intersections of the spheres to determine the branch.

Suppose that when solving G with our method, after having followed the homotopy path containing the initial sketch, the set of solutions $S^0 = \{\sigma_0(X), \ldots, \sigma_{i_0}(X)\}$ has been obtained. For each solution $\sigma_j(X)$, one can find the pair $(\sigma_j(A'), b_j)$ such that the evaluation of P constructs $\sigma_j(X)$. When evaluating P on other branches than b_j, other figures can be obtained. These figures are thus used as new sketches to define new homotopy functions as described in Section 4, and to follow new homotopy paths. A set S of solutions is obtained. Let $S^1 = \{\sigma_{i_0+1}(X), \ldots, \sigma_{i_1}(X)\}$ be the set of solutions $\sigma_j(X) \in S$, such that $\sigma_j(X) \notin S^0$. The process can be iterated since the set S^l obtained at iteration l is not empty.

Remark that in contrast to classical homotopy methods, where the initial system is constructed in such a way that all its solutions are known, here we only know one solution of the start system. By generating new sketches, we follow paths that are new in the sense that they are paths of a new homotopy function, but it cannot be ensured that they lead to new solutions.

We conclude this part by observing that this step can be very long, and more precisely can take a time that grows exponentially with the number of constraints in G: assuming that $|C|$ is proportional to $|C'|$, $P = C'[X', A']$ has at worst $2^{|C'|}$ branches. Indeed, intersections between lines, planes, spheres are at most two objects in general.

6.2. *Detecting solutions at infinity*

In Section 4.2, we have established that a homotopy path followed by our method can have infinite length and lead to a solution at infinity. Here, we show that such solutions at infinity correspond to specific geometric configurations that can be detected with the construction plan P associated to the solved GCSP G.

If P is a construction plan, one can associate to P a system of equations $\mathcal{P}$, of which unknowns are elements of X, such that $\sigma(X)$ is a solution of $\mathcal{P}$ if and only if it exists a branch b and a valuation $\sigma(A')$ for which $\sigma(X)$ is the figure provided by the evaluation of P. For instance, to the CP given in Section 6.1 we associate the system:

$$\mathcal{P} : \begin{cases} \ldots \\ \mathcal{P}_j : P_0P_3 - h_0 = 0 \\ \mathcal{P}_{j+1} : P_1P_3 - h_1 = 0 \\ \mathcal{P}_{j+2} : P_2P_3 - h_2 = 0 \\ \ldots . \end{cases} \quad (2)$$

In $\mathcal{P}$, we call a *block* a subset of equations for loci that determine an object. In (2), $\{\mathcal{P}_j, \mathcal{P}_{j+1}, \mathcal{P}_{j+2}\}$ is a block, since it determines object P_3. Remark that the set of solutions of $\mathcal{P}$ depends on the valuation of parameters A'. We note $\mathcal{P}_\sigma$ the system for the valuation $\sigma(A')$ of A'.

Let us now use the same notations as in Section 4: H is the homotopy function and $\mathcal{S}$ the homotopy path followed from the sketch. Let $(\sigma_0(X), t_0)$ be an element of $\mathcal{S}$. One can read on figure $\sigma_0(X)$ values $\sigma_0(A')$ of parameters A' such that $\sigma_0(X)$ is a solution of $\mathcal{P}_{\sigma_0}$.

Suppose now that $\mathcal{S}$ is diffeomorphic to an open interval of infinite length. We can extract from $\mathcal{S}$ a sequence $(\sigma_n(X), t_n)_{n\in\mathbb{N}}$, with $\lim_{n\to+\infty} \|(\sigma_n(X), t_n)\| = +\infty$, and t_n is bounded $\forall n \in \mathbb{N}$. By reordering elements of X with their order of appearance in $\mathcal{P}$, we write $X = \{x_0, \ldots, x_m\}$. Thus, there exists a smallest index $i \in \{0, \ldots, m\}$ such that $\lim_{n\to+\infty} \|\sigma_n(x_i)\| = +\infty$, and $\sigma_n(x_j)$, with $j < i$, is bounded $\forall n \in \mathbb{N}$. Then, elements of the sequence $(\sigma_n(\{x_0, \ldots, x_{j-1}\}))_{n\in\mathbb{N}}$ are not solutions at infinity of the sub-systems formed by the $j-1$ first blocks of $\mathcal{P}_{\sigma_n}$, and elements of $(\sigma_n(\{x_0, \ldots, x_j\}))_{n\in\mathbb{N}}$ are solutions at infinity of the sub-systems formed by the j first blocks of $\mathcal{P}_{\sigma_n}$. It follows that elements of $(\sigma_n(x_j))_{n\in\mathbb{N}}$ are solutions at infinity of the block j of systems $\mathcal{P}_{\sigma_n}$ for parameters $(\sigma_n(\{x_0, \ldots, x_{j-1}\}))_{n\in\mathbb{N}}$.

Then a solution at infinity of H is necessarily a solution at infinity of a block of $\mathcal{P}$, and one can easily show the reciprocity of this relation. Now, blocks of $\mathcal{P}$ correspond to intersection of geometric loci, and it is possible to enumerate, for each block, configurations that lead to a solution at infinity. For instance, the solution of a block corresponding to the intersection of two planes goes to infinity iff the two planes are parallel, and not equal.

By enumerating these configurations, it can be detected during the following of a path $\mathcal{S}$ if one of them holds. It is then guaranteed that no solution lies in the section of $\mathcal{S}$ that is not tracked if the tracking step is stopped, because the detected configuration arises if and only if the path is converging to a solution at infinity. Obviously, the path has to be tracked again from the sketch towards the opposite direction.

7. Results

We present here quantitative results of our method on four problems. Two of them involve only distance constraints, the two others involve also coplanarity constraints. Usual combinatorial decomposition strategies fail for each of them. In particular, they are not solvable by LIM, and construction plans are obtained with the method given in [6]. Paths are tracked with a classic prediction–correction method, that is adapted in the case where degeneracies are encountered. No strategy of prediction step adaptation is used. Section 7.1 describes this method.

For each of these problems Table 1 gives: the number of solutions found on the first path and the time in seconds needed to track it entirely (line first path), the number of solutions found after having computed all the paths given by new sketches, and the overall computing time (line "whole process"). When it is possible (*e.g.* the problem has not too many constraints), we also give the number of solutions found and the duration process when solving the problem with the software HOM4PS-2.0 [20] that implements polyhedral homotopy described in [12]. Among several free-softwares implementing homotopy for polynomials like PhcPack, Bertini, *etc.*, we choose HOM4PS for its speed and robustness. Indeed, obtained solutions are generally the same from one run to another. For the pentahedron and the hexahedron problems, the number of degenerated solutions among all the sets of solutions is indicated. Times are given for an Intel(R) Core(TM) i5 CPU 750 @ 2.67 GHz. Details about each problem are given below.

7.1. *Path tracking algorithm*

A path $\mathcal{S}$ is tracked from the sketch $(\sigma_{sk}(X), 0)$ with a classic prediction–correction method. Suppose that a point on the path $(\sigma_0(X), t_0) \in \mathcal{S}$ is known. The Jacobian matrix $JH(\sigma_0(X), t_0)$ is numerically computed with the finite difference method, and the 1-dimensional kernel T of the linear application associated with $JH(\sigma_0(X), t_0)$ is obtained with basic linear algebra tools. A prediction $(\sigma_0'(X), t_0')$ is then computed in the direction of T, with prediction step $\delta \in \mathbb{R}$. This prediction is then corrected with Newton's method in $(\sigma_1(X), t_1) \in \mathcal{S}$ such that $(\sigma_1(X), t_1) \in T_p$, where T_p is the hyperplane perpendicular to T. When $(\sigma_0(X), t_0)$ and the prediction $(\sigma_0'(X), t_0')$ are such that $(t_0 - 1) \cdot (t_0' - 1) < 0$ (*i.e.* hyperplane $\mathcal{P}_1$ lies between these two points), a new prediction $(\sigma_0''(X), t_0'')$ such that $t_0'' = 1$ is made, and $(\sigma_0''(X), t_0'')$ is corrected in $(\sigma_*(X), 1)$ to precise the solution. The left part of Fig. 9 illustrates this process.

When degeneracies are encountered, for instance when tracking a path for the hexahedron problem, this process is adapted in the following way. Recall (see Section 5.2) that the homotopy function H is constructed such that degeneracies arise only if $t = 0$ or $t = 1$. When approaching hyperplane $t = 1$ (the case $t = 0$ is similar), if degeneracies hold, a prediction–correction step can fail:

(i) if $t_0 = 1$, the Jacobian matrix does not have full rank and T is no longer a 1-dimensional linear space; the prediction has no more meaning,

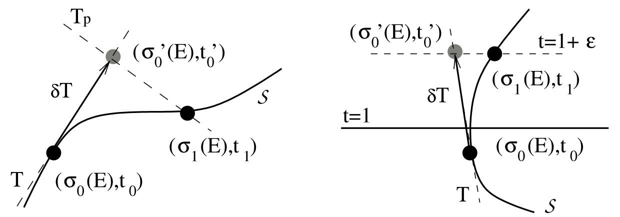

Fig. 9. The classical prediction–correction method (left), and its adaptation to cases where degeneracies can be encountered (right).

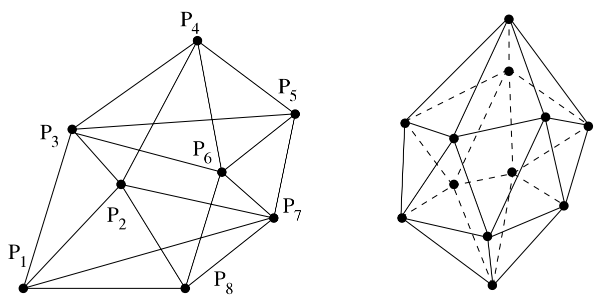

Fig. 10. Left: the disulfide problem. Right: the icosahedron problem. Edges are distance constraints.

(ii) if $t'_0 = 1$, the Jacobian matrix does not have full rank and Newton's method fails,
(iii) if $(t_0 - 1) \cdot (t'_0 - 1) < 0$ and $t_1 = 1$, $(\sigma_1(X), t_1)$ is a degenerated solution, and the next step of prediction–correction fails.

We prevent these failures by applying the two following rules: (1) if case (iii) or (ii) occurs, prediction step δ is chosen such that $t'_0 = 1 + \epsilon$ if $t'_0 < 1$, $t'_0 = 1 - \epsilon$ otherwise, with $\epsilon > 0$, and $(\sigma'_0(X), t'_0)$ is corrected in $(\sigma_1(X), t_1)$ such that $t_1 = t'_0$; (2) if case (i) happens, $(\sigma_0(X), t_0)$ is set as the previous known point on curve $\mathcal{S}$, and rule (1) is applied. The right part of Fig. 9 shows rule (1).

The prediction–correction process is iterated until one of the following situations holds:

- $\mathcal{P}_0$ is crossed, and current figure $\sigma_0(X)$ is near from the sketch (*i.e.* $\|\sigma_0(X) - \sigma_{sk}(X)\| < \epsilon$, with $\epsilon > 0$),
- figure $\sigma_0(X)$ is near of a configuration where H is not C^∞,
- figure $\sigma_0(X)$ is near of a configuration of a solution at infinity.

In the two last cases, the path is tracked again from the sketch, but towards the opposite direction. Proposition 5 justifies that this process terminates, and that all solutions on $\mathcal{S}$ are found.

7.2. Applications

The first problem comes from molecular chemistry and is picked up from [21]. It consists of constructing 8 points in 3D space knowing 18 pairwise distances. It corresponds to a disulfide molecule (see left part of Fig. 10).

A valuation of parameters is exhibited in [21] that leads to 18 solutions all found by a branch-and-prune algorithm in more than 10 min. When solving this problem with our method, the homotopy function is a map depending on a vector of $\mathbb{R}^{18} \times \mathbb{R}$. The homotopy path containing the sketch is entirely computed in about 3 s. This first step provides 8 solutions, since the path intersects 8 times $\mathcal{P}_1$.

When applying the step described in Section 6.1, 124 different new sketches are obtained at the first iteration. Corresponding paths are followed and 5 new solutions are found. Iterating this step does not provide any new solution, and the process stops after 108 s. Notice that HOM4PS2 yields all the solutions in 6129 s.

Table 1
Running times for an Intel(R) Core(TM) i5 CPU 750 @ 2.67 GHz.

	Disulfide	Pentahedron	Hexahedron	Icosahedron
HOM4PS2:				
nb sol	18	9 (6 flat)	16	–
Times	6129 s	23 s	12 800 s	–
First path:				
nb sol	8	9 (6 flat)	7 (3 flat)	28
times	3 s	0.4 s	1.6 s	9 s
Whole process:				
nb sol	13	9 (6 flat)	251 (235 flat)	–
Times	108 s	1.8 s	136 s	–

The pentahedron problem, with valuations:

$$h_0 = 1.3, \quad h_1 = 2.9, \quad h_2 = 2.6,$$
$$h_3 = 1.35, \quad h_4 = 1.9, \quad h_5 = 1.5,$$
$$h_6 = 0.7, \quad h_7 = 1.1, \quad h_8 = 2.6,$$

admits 9 solutions, that are found by HOM4PS2 in about 23 s. Among them, 6 are flattened. Our method finds these 9 solutions when following the first path in about 0.4 s. Evaluating associated construction plan provides 24 new sketches. Since no new solution can be found in paths associated with these sketches, the process stops. Overall the process takes about 1.8 s.

Fig. 7 illustrates the hexahedron problem. With valuations:

$$h_0 = 1.3, \quad h_1 = 1.3, \quad h_2 = 1.3, \quad h_3 = 0.5,$$
$$h_4 = 0.7, \quad h_5 = 0.9, \quad h_6 = 0.8, \quad h_7 = 0.8,$$
$$h_8 = 0.5, \quad h_9 = 1.9, \quad h_{10} = 1.8, \quad h_{11} = 1.5,$$

it admits 16 non degenerated solutions, i.e. 16 solutions that are not flattened. These solutions are found in 12 800 s by HOM4PS. Degenerated solutions are not found by classical homotopy methods, since they ensure by slightly perturbing parameters that followed paths do not contain critical points. Our method crosses degenerated configurations, and provides some of them as solutions. They are part of continuous sets of solutions, that are not explored. Following the first path provides 4 not flattened solutions and 3 flat solutions in 1.6 s. The construction plan is only evaluated on the non-flat solutions. 114 paths are followed before the end of the process, after which the 16 non-flat solutions are found in addition to 235 flat solutions.

An icosahedron is a solid with 12 vertices, 30 edges and 20 triangular faces. The icosahedron problem consists of constructing its 12 vertices, knowing the lengths of its 30 edges. The right part of Fig. 10 presents a sketch for this problem. Solving it with our method leads to define an homotopy function $H : \mathbb{R}^{31} \to \mathbb{R}^{30}$ of which each component is a distance function. With generic values for parameters, no degeneracies hold. This problem has too many constraints to be solved by HOM4PS2 (the process never ends). With our method, the path of the sketch is entirely tracked in about 9 s, giving 28 solutions. We did not apply all the process, since the number of sketches obtained by the construction plan is exponential: from the 28 solutions, 4393 sketches are provided. In about 8 h, all the corresponding paths are followed, yielding 308 new solutions.

8. Conclusion

In this paper, we describe an adaptation of a general method based on homotopy to GCSP. This specialization makes the solving process more efficient than classical methods and allows to take degenerated cases into account.

Our method consists of entirely tracking the paths starting from solutions associated to a geometric sketch. This way we can find several solutions of a GCSP instead of a single one with the classical method used in CAD. Tracking complete paths is made possible by their characterization in our framework: we showed that under some regularity conditions they are 1-dimensional manifolds that are diffeomorphic either to circles or to open intervals.

We use geometric knowledges to lead the continuation method, or in more detail:

- to change the nature of problematic paths by modifying the interpolation functions,
- to detect paths going to infinity,
- to deal with degenerated cases,
- and, finally, to yield different sketches giving birth to new paths to follow and to new solutions.

We think that this semantic based approach can bring a lot to numerical methods used to solve CAD GCSP.

Some problematic points remain and there is of course room for improvements. Here are some tracks we will explore in the future:

- We saw in the paper that 3D problems sometimes have total or partial flattened solutions. It could be interesting to warn the designer that this phenomenon occurs with his/her dimensioned sketch. The question is then: "Given a 3D sketch, say a polyhedron, how do we detect all the possible degeneracies of the associated GCSP?".
- We do not guarantee that all the solutions can be found with our method, we just explore the similar ones by tracking complete paths and by producing more or less similar sketches to iterate the process. It would be interesting to design some tools to organize the traversal of the solution space in order to present the solutions to the user according to their pertinence. It could be necessary then to be able to iteratively yield all the solutions which is a difficult problem.
- Finally, our method can solve problems with about 40 constraints. In order to make it usable in an industrial product, it has to be combined with a decomposition method to tackle bigger problems.

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