BDDC and FETI–DP algorithms for mortar finite element methods

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Outline

1. Mortar discretization
2. BDDC and FETI–DP algorithms
3. Extensions
4. Numerical results
5. Conclusion
Mortar Discretization

• A model elliptic problem

\[-\Delta u = f \text{ in } \Omega,\]
\[u = 0 \text{ on } \partial\Omega.\]  

(1)

• Finite element space of nonmatching triangulation

Figure 1: Matching grids(left) and nonmatching grids(right)

– Adaptivity: singular points, discontinuous coefficients
– Mesh generation, Multi-physics simulation
– Nonconforming $\not\subseteq H^1(\Omega)$
• **Mortar matching condition**

![Diagram](image)

Figure 2: mortar, nonmortar sides

1. decide nonmortar, mortar
2. build Lagrange multiplier space $M_{ij}$ based on the nonmortar, $(1 \in M_{ij}, \text{d.o.fs})$
3. mortar matching condition

$$
\int_{\Gamma_{ij}} (v_i - v_j) \lambda \, ds = 0 \quad \forall \lambda \in M_{ij}.
$$

(2)
• Mortar discretization is to approximate the solution $u$ in the finite element space satisfying the mortar matching condition.
Matrix Representation of Mortar Discretization

Figure 3: spaces \( W = \prod_{i=1}^{N} W_i \) (left), \( \hat{W} \) (center), \( \tilde{W} \) (right)

- **Finite element Spaces**

\[ X_i : \text{finite element space in } \Omega_i, \quad W_i : \text{trace space of } X_i, \quad X_i(\partial \Omega_i), \]

\[ W = \prod_{i=1}^{N} W_i \quad \text{discontinuous across the interface}, \]

\[ \hat{W} \subset W \quad \text{with mortar matching condition}, \]

\[ \tilde{W} \subset W \quad \text{with primal constraints}. \]
Matrix Representation of Mortar Discretization (cont.)

• Primal constraints
  The primal constraints are selected from the mortar matching condition so that \( \hat{W} \subset \tilde{W} \);
  For example, continuity at corners or averages on edges/faces.

In the following, we simply consider continuity at corners as primal constraints.
Matrix Representation of Mortar Discretization (cont.)

- Equations of the mortar matching condition on $\tilde{W}$

  Separate unknowns $w \in \tilde{W}$ into

  $$w_n \text{ (nonmortar)}, \quad w_m \text{ (mortar)}, \quad w_c \text{ (corners)}.$$ 

  The Eqns. of the mortar matching condition is

  $B_n w_n + B_m w_m + B_c w_c = 0,$

  $w_n = -B_n^{-1}(B_m w_m + B_c w_c).$ \hfill (3)

- The space $W_G$ of unknowns $(w_m, w_c)$.

- Mortar finite element space $\hat{W}$ is represented by $W_G$,

\[
\begin{pmatrix}
  w_n \\
  w_m \\
  w_c
\end{pmatrix} = R^t \begin{pmatrix}
  w_m \\
  w_c
\end{pmatrix} = \begin{pmatrix}
  -B_n^{-1}B_m & -B_n^{-1}B_c \\
  I & 0 \\
  0 & I
\end{pmatrix} \begin{pmatrix}
  w_m \\
  w_c
\end{pmatrix}, \quad \forall (w_m, w_c) \in W_G.
\]
Matrix Representation of Mortar Discretization (cont.)

- Local Schur complement matrix

\[ S^{(i)} = \begin{pmatrix} S_{rr}^{(i)} & (S_{cr}^{(i)})^t \\ S_{rc}^{(i)} & S_{cc}^{(i)} \end{pmatrix}, \quad c: \text{corners}, \ r: \text{remaining}. \]  

(4)

- Subassembly of \( S^{(i)} \), (gluing unknowns at corners)

\[ \tilde{S} = \begin{pmatrix} S_{rr} & S_{cr}^t \\ S_{cr} & S_{cc} \end{pmatrix}, \]  

(5)

\[ S_{rr} = \text{diag}(S_{rr}^{(i)}), \]

\[ S_{cr} = \begin{pmatrix} (R_{c}^{(1)})^t S_{cr}^{(1)} & \cdots & (R_{c}^{(N)})^t S_{cr}^{(N)} \end{pmatrix}, \]

\[ S_{cc} = \sum_{i=1}^{N} (R_{c}^{(i)})^t S_{cc}^{(i)} R_{c}^{(i)}, \]

\( R_{c}^{(i)} \) restriction of the primal unknowns to subdomain \( \Omega_i \).
Matrix Representation of Mortar Discretization (cont.)

- The Eqns. of the mortar discretization is
  \[ R \tilde{S} R^t \begin{pmatrix} w_m \\ w_c \end{pmatrix} = R \begin{pmatrix} g_m \\ g_c \end{pmatrix}, \quad R^t = \begin{pmatrix} -B_n^{-1}B_m & -B_n^{-1}B_c \\ I & 0 \\ 0 & I \end{pmatrix}. \tag{6} \]

- Equivalent dual problem
  The above problem (6) is equivalent to
  \[ \max_\lambda \min_{w \in W} \left\{ \frac{1}{2} \langle \tilde{S}w, w \rangle + \langle \tilde{g}, w \rangle + \langle Bw, \lambda \rangle \right\}, \tag{7} \]
  where
  \[ B = \begin{pmatrix} B_n & B_m & B_c \end{pmatrix}. \]
  After eliminating unknowns other than \( \lambda \), the equations on dual variables \( \lambda \) follow,
  \[ B \tilde{S}^{-1}B^t \lambda = B \tilde{S}^{-1}\tilde{g}, \tag{8} \]
BDDC and FETI–DP algorithms

- **BDDC algorithm**
  
  \[
  R\tilde{S}R^t \begin{pmatrix} w_m \\ w_c \end{pmatrix} = R \begin{pmatrix} g_m \\ g_c \end{pmatrix}, \text{ using } RD\tilde{S}^{-1}DR^t. \tag{9}
  \]

- **FETI–DP algorithm**
  
  \[
  B\tilde{S}^{-1}B^t\lambda = B\tilde{S}^{-1}\tilde{g}, \text{ using } B\Sigma\tilde{S}\Sigma B^t. \tag{10}
  \]

- The weights \( D \) and \( \Sigma \) are chosen to give the optimal condition number bound,

  \[
  \kappa(B_{DDC}), \kappa(F_{DP}) \simeq C(1 + \log \frac{H}{h})^2, \quad \frac{H}{h} : \text{ the size of the local problem}
  \]

  \[
  B_{DDC} = (RD\tilde{S}^{-1}DR^t)R\tilde{S}R^t, \quad F_{DP} = (B\Sigma\tilde{S}\Sigma B^t)B\tilde{S}^{-1}B^t.
  \]
BDDC and FETI–DP algorithms (cont.)

- FETI-DP algorithm for mortar (By Kim and Lee)
  The FETI-DP algorithm solves iteratively
  \[
  B\tilde{S}^{-1}B^t\lambda = B\tilde{S}^{-1}\tilde{g},
  \]
  with the Neumann-Dirichlet preconditioner
  \[
  B\Sigma\tilde{S}\Sigma B^t,
  \]
  of the weights,
  \[
  \Sigma = \begin{pmatrix}
  \Sigma_{nn} & 0 \\
  0 & \Sigma_{mm} \\
  0 & 0
  \end{pmatrix}, \quad \Sigma_{nn} = (B_n^tB_n)^{-1}, \Sigma_{mm} = 0, \Sigma_{cc} = 0.
  \]
  The optimal condition number bound has been shown,
  \[
  \kappa(F_{DP}) \simeq C (1 + \log(H/h))^2 \text{ (indep. of coefficient jumps).}
  \]
BDDC and FETI–DP algorithms (cont.)

- **BDDC algorithm for mortar**
  
  We solve for the primal unknowns \((w_m, w_c)\) iteratively

\[
R\tilde{S}R^t \begin{pmatrix} w_m \\ w_c \end{pmatrix} = R \begin{pmatrix} g_m \\ g_c \end{pmatrix}
\]  

using a preconditioner of the form

\[
RD\tilde{S}^{-1}DR^t.
\]

Our aim is to find weights \(D\) for the BDDC algorithm that has the same spectra as the FETI-DP algorithm.

So that we obtain the optimal converge as well

\[
\kappa(B_{DDC}) \simeq C (1 + \log(H/h))^2.
\]
Connection between BDDC and FETI-DP for mortar

We recall two algorithms

\[
F_{DP} = (B\Sigma\tilde{S}\Sigma B^t)B\tilde{S}^{-1}B^t, \quad B_{DDC} = (RD\tilde{S}^{-1}DR^t)R\tilde{S}R^t.
\]

We define

\[
P_{\Sigma} = \Sigma B^tB \text{ (jump)}, \quad E_D = R^tRD \text{ (average)}. \tag{16}
\]

**Theorem 1 (Li and Widlund)** If they satisfy

\[
P_{\Sigma} + E_D = I
\]

\[
E_D^2 = E_D, \quad P_{\Sigma}^2 = P_{\Sigma},
\]

\[
E_D P_{\Sigma} = P_{\Sigma} E_D = 0,
\]

then the operators \( B_{DDC} \) and \( F_{DP} \) have the same spectra except the eigenvalue 1.
BDDC and FETI–DP algorithms (cont.)

The weights

\[
D = \begin{pmatrix}
D_{nn} & \\
D_{mm} & \\
& D_{cc}
\end{pmatrix}, \quad D_{nn} = 0, \ D_{mm} = I, \ D_{cc} = I
\]

satisfy the above properties when \( \Sigma \) is given by

\[
\Sigma = \begin{pmatrix}
\Sigma_{nn} & 0 \\
0 & \Sigma_{mm} \\
0 & 0 & \Sigma_{cc}
\end{pmatrix}, \quad \Sigma_{nn} = (B_n^t B_n)^{-1}, \ \Sigma_{mm} = 0, \ \Sigma_{cc} = 0.
\]

Note: The pair \((\Sigma, D)\) is unique that satisfy the above properties with the optimal condition number bound.
Extensions

▷ Primal constraints other than continuity at corners

3D elliptic problems, 3D elasticity problems
(need richer primal constraints than corners)

For example,

\[
\int_{F_{ij}} (v_i - v_j) \, ds = 0, \quad \text{(Note : } 1 \in M_{ij} \text{ )},
\]

\[
\int_{F_{ij}} (v_i - v_j) \cdot P_{ij} \mathbf{r} \, ds = 0, \quad \text{(} \mathbf{r} : \text{rigid body motions } \text{)},
\]

\[P_{ij} : \text{proj. onto Lagrange multiplier space.}\]
Extensions (cont.)

▷ Geometrically nonconforming partition.

Figure 4: Geometrically conforming (left) and geometrically non-conforming partitions (right).

A nonmortar $F \subset \partial \Omega_i$ partitioned by its mortar neighbors $\Omega_j$,

$$F = \bigcup_j F_{ij}, \quad F_{ij} = \partial \Omega_i \cap \partial \Omega_j,$$

a function $\phi$ from its mortar neighbors by

$$\phi = w_j \text{ on } F_{ij}. \quad (17)$$
Mortar matching condition is
\[ \int_F (w_i - \phi) \psi \, ds = 0, \quad \forall \psi \in M(F). \]  
\[ (18) \]

\( M(F) \) : Lagrange multiplier space
 Extensions (cont.)

• Primal constraints depending on each $F_{ij}$

$$
\int_{F_{ij}} (w_i - w_j) \psi_{ij} \, ds = 0,
$$

$\psi_{ij}$ is **the sum of Lagrange multipliers bases** supported in $F_{ij}$.

• Primal constraints depending on $F$ (elasticity)

$$
\int_F w_i \, ds = \int_F \phi \, ds, \\
\frac{1}{|F|} \int_F w_i \, ds = \sum_j a_{ij} \frac{1}{|F_{ij}|} \int_{F_{ij}} w_j \, ds, \quad a_{ij} = \frac{|F_{ij}|}{|F|},
$$

$$
\overline{w}_i^F = \sum_j a_{ij} \overline{w}_j^{F_{ij}}. \tag{19}
$$

• Slightly weaker condition number bound $(1 + \log(H/h))^3$. 

Numerical Results

• Comparison of BDDC and FETI-DP algorithms

Model problem

\[-\Delta u(x, y) = f(x, y) \text{ in } \Omega = [0, 1] \times [0, 1]\]

\[u(x, y) = 0 \text{ on } \partial \Omega\]

Exact solution: \[u(x, y) = y(1 - y) \sin \pi x\]

CGM: stopping criterion \(\rightarrow\) relative residual \(\leq 1.0e-6\)

\(N:\) the number of subdomains

\(n:\) the number of nodes on the subdomain edge including end points
Figure 5: Partition of subdomains when $N = 4 \times 4$

Figure 6: Matching grids (left) and nonmatching grids (right) when $n = 5$
Table 1: (Non-matching grids) Comparison of FETI–DP and BDDC algorithms

<table>
<thead>
<tr>
<th>$N = 4 \times 4$</th>
<th>$F_{DP}$</th>
<th>$B_{DDC}$</th>
<th>$n = 5$</th>
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<th>$B_{DDC}$</th>
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<td>$\lambda_{\text{max}}$</td>
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</tr>
</tbody>
</table>
Conclusion

1. FETI-DP with the Neumann-Dirichlet preconditioner
   - Elliptic problems in $2D$, $3D$
   - Stokes problem in $2D$
   - $3D$ compressible elasticity
   - The most efficient for the problems with coefficient jumps

2. A BDDC algorithm well connected to FETI-DP with ND-preconditioner
   - Extended to elliptic problems in both $2D$ and $3D$
   - $3D$ compressible elasticity
   - Geometrically nonconforming subdomain partitions.