Harmonic Analysis – Homework Set 4

1. Let $\varphi \in L^2(\mathbb{R})$.

   (a) Show that the family $\{e^{2\pi imx}\varphi(x)\}_{m\in\mathbb{Z}}$ is an orthonormal system if and only if

   $$\sum_{n\in\mathbb{Z}} |\varphi(x-n)|^2 = 1 \qquad a.e.\ x.$$

   (Hint: Think of an alternative description of the *periodic* function on the left hand side above.)

   (b) Show that the family $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ is an orthonormal system if and only if

   $$\sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\xi-m)|^2 = 1 \qquad a.e.\ \xi.$$

   (c) Show that the family $\{e^{2\pi imx}\varphi(x-n)\}_{(m,n)\in\mathbb{Z}^2}$ is an orthonormal system if and only if

   $$\sum_{n\in\mathbb{Z}} \varphi(x-n-l)\overline{\varphi(x-n)} = \begin{cases} 1, & l=0, \\ 0, & l\in\mathbb{Z}\setminus\{0\}, \end{cases} \qquad a.e.\ x.$$

   (Hint: Same as in 1(a).)

2. Let $\varphi \in L^2(\mathbb{R})$ such that $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ forms an orthonormal system. Assume also that φ satisfies the regularity property $|\varphi(x)| \le \mu(|x|)$ for some $\mu : [0,\infty) \to [0,\infty)$ which is monotonic decreasing, integrable, and $\mu(0) < \infty$ (which is actually implied). Show that there is a finite positive constant C_μ such that

   $$C_\mu^{-1}\|(a_n)\|_{\ell^p} \le \left\| \sum_{n\in\mathbb{Z}} a_n\varphi(\cdot - n) \right\|_{L^p} \le C_\mu\|(a_n)\|_{\ell^p}$$

   for all $1 \le p \le \infty$ and all sequences (a_n).
   (Hint: The upper bound doesn't depend on orthogonality, but the lower bound takes advantage of it. Also, the solution doesn't really have to do with the Fourier characterization given in 1(b); think in terms of simple inequalities you know.)