Harmonic Analysis – Homework Set 3

1. Recall the Riesz transforms R_j on $\mathbb{R}^n$ which we defined via the singular kernels $K_j(x) = x_j/|x|^{n+1}$, $j = 1, \ldots, n$.

   (a) I had claimed that $\widehat{K}_j(\xi) = c_n \xi_j/|\xi|$ for a fixed constant c_n. Prove this statement. (Hence $\partial_j \partial_k$ is a constant multiple of the operator $R_j R_k \Delta$, which we used to prove $\|\partial_j \partial_k f\|_p \lesssim_{p,n} \|\Delta f\|_p$.)

   (b) Show that if a function K on $\mathbb{R}^n$ is homogeneous of order $-n$, and C^1 away from the origin, then $|\nabla K| \lesssim |x|^{-n-1}$.

2. Consider any step function m on $\hat{\mathbb{R}}$ given by

$$m(\xi) = \sum_{i=1}^{n} c_i \chi_{[\alpha_i, \alpha_{i+1})}(\xi),$$

   where $-\infty = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \infty$.

   (a) Show that the operator defined by $T_m f = (m\hat{f})^\vee$ satisfies

$$\|T_m\|_{p \to p} \lesssim_p \|m\|_\infty + |m|_{TV}, \qquad 1 < p < \infty,$$

   where $|\cdot|_{TV}$ is the total variation.
   (Hint: Rewrite m in terms of its jumps and use the Hilbert transform.)

   (b) Extend the above result to show that any $m \in BV(\hat{\mathbb{R}})$ is a Fourier multiplier on L^p, $1 < p < \infty$.

3. The point of this exercise is to characterize fractional Lipschitz spaces by means of dyadic frequency decompositions. Recall that $f \in \mathrm{Lip}^\alpha(\mathbb{R})$ means $\|f\|_{\mathrm{Lip}^\alpha} := \|f\|_\infty + |f|_{\mathrm{Lip}^\alpha} < \infty$, where $|f|_{\mathrm{Lip}^\alpha} := \sup_{x \neq y} |f(x) - f(y)|/|x-y|^\alpha$. For this definition, we have $0 < \alpha \leq 1$.

   Assume $f \in \mathrm{Lip}^\alpha \cap L^1 \cap L^2$ and consider the Littlewood-Paley type decomposition $f = \sum S_j f$ obtained from a smooth dyadic frequency decomposition. That is, $S_j f = f * \psi_j^\vee$ where $\psi_j(\xi) = \psi(\xi/2^j)$ with support around $|\xi| \asymp 2^j$ and $\sum \psi_j = 1$. (Adjust the smoothness of ψ as needed.)

   (a) Show that $\|S_j f\|_\infty \lesssim 2^{-j\alpha} |f|_{\mathrm{Lip}^\alpha}$.

   (b) Now assume $\|S_j f\|_\infty \leq C 2^{-j\alpha}$, where $0 < \alpha < 1$. Show that $|f|_{\mathrm{Lip}^\alpha} \lesssim C$. (This direction will be slightly more difficult: First, notice that the decomposition $f = \sum S_j f$ is "near-orthogonal" in the sense that $\langle S_j f, S_k f \rangle = 0$ if $|j - k| > C$ where C is a fixed constant. ($C = 1$ works for our construction in class.) Hence we can effectively write $f = \sum_j \sum_{|l| \leq 1} S_j S_{j+l} f$. Treat each piece $\sum_j S_j S_{j+l} f$ separately, noting that $|l| \leq 1$.)

   (c) Hence we have, for $0 < \alpha < 1$, the norm equivalence $\|f\|_{\mathrm{Lip}^\alpha} \asymp \|f\|_\infty + \sup_j \; 2^{j\alpha} \|S_j f\|_\infty$. Compare this with the lack of necessary and sufficient decay conditions on $|\widehat{f}(\xi)|$ for f to belong to Lip^α.

   (d) Compare (c) with 5(a) in Homework 1. Can you guess a similar statement for $f \in \mathrm{Lip}^\alpha(\mathbb{T})$?

   (Hint for (a) and (b): Carry out your estimates staying in the space domain.)