Harmonic Analysis – Homework Set 2

1. Justify each of the following steps to show that the range of p in the Hausdorff-Young inequality (which states that the Fourier transform is a bounded operator from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$ for all $1 \leq p \leq 2$) cannot be extended beyond 2.

   (a) It suffices to check that the Fourier transform is not bounded as an operator from $(\mathcal{S}, \|\cdot\|_p)$ to $(\mathcal{S}, \|\cdot\|_q)$ for any q if $p > 2$.

   (b) Proving (a) is equivalent to proving that for $p < 2$, there is no uniform constant $C_p > 0$ such that $\|\widehat{f}\|_{p'} \geq C_p \|f\|_p$ for all $f \in \mathcal{S}$, that is, the direction of the Hausdorff-Young inequality cannot be reversed. (Note that for $p = 2$ we have Plancherel.)

   (c) There is an absolute constant C such that for all a and b

   $$\left| \int_a^b e^{-ix^2} dx \right| \leq C.$$

   (d) Let $f_\lambda(x) = e^{-\pi x^2} e^{-\pi i \lambda x^2}$. Using the above result, and after an appropriate change of variables and integration by parts, show that $\|\widehat{f_\lambda}\|_\infty \leq C\lambda^{-1/2}$ for some absolute constant C.

   (e) Let $p < 2$. Prove the chain of inequalities

   $$\left\| \widehat{f_\lambda} \right\|_{p'} \leq \left\| \widehat{f_\lambda} \right\|_2^{2/p'} \left\| \widehat{f_\lambda} \right\|_\infty^{1-2/p'} \leq C\lambda^{1/p'-1/2}$$

   and finish the proof of the result.

2. (a) Show that if $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ is such that

   $$\sum_{k\in\mathbb{Z}} |\hat{f}(k)| < \infty,$$

   then the Poisson summation formula (which we showed for $f \in \mathcal{S}$)

   $$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$$

   holds for a.e. x.

   (b) Show that the Fourier transform of the Cauchy-Poisson kernel $k_\lambda(x) := \frac{\lambda}{\pi(\lambda^2+x^2)}$ is given by $e^{-2\pi\lambda|\xi|}$. Show also that for this function the Poisson summation formula above holds for all x.

   (c) Use (a) and (b) together to evaluate

   $$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

   (d) (Bonus) The equality in (a) need not hold for all x. Set $\phi_m(x) = (1-|2x|)^m - (1-|2x|)^{m+1}$, for $m \geq 1$ and $|x| \leq 1/2$, and $f(x) = \sum_{m=1}^{\infty} \phi_m(x-m)\chi_{[m-1/2,m+1/2]}(x)$. Show that f satisfies the hypothesis of (a). Show also that $\sum f(n) = 0$ but $\sum \hat{f}(k) = 1$, so that the equality in (a) fails for $x = 0$.

(HW continued on page 2)

3. (a) Show that a bounded non-zero complex-valued homomorphism of a group is necessarily unimodular, i.e., takes values on the unit circle.

   (b) Show that the algebra (with respect to pointwise addition and multiplication) of n-times continuously differentiable functions $C^n(\mathbb{T})$ is a Banach algebra for the norm

$$\|f\| := \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!}.$$

   (c) Show that the norm

$$\|P\| := \sum_{k=0}^{n} |a_k|, \text{ where } P(x) = \sum_{k=0}^{n} a_k x^k,$$

   is submultiplicative in the algebra of polynomials (with respect to pointwise addition and multiplication) over the complex field. Is this space complete? If not, can you identify the completion?

4. Let $G = \mathbb{Z}_2^n = \{0,1\}^n$ with its Haar measure normalized so that $\mu(G) = 1$. Identify each point x in G (and in $\widehat{G}$) with a subset A of $\{1,\ldots,n\}$ where $x_i = 1$ iff $i \in A$. Let f be a Boolean function on G, that is, $f(x)$ takes values 0 or 1. Treat f also as a random variable.

   (a) Show that $\widehat{f}(\emptyset) = \text{Prob}\{f = 1\}$.

   (b) For each $i \in \{1,\ldots,n\}$, let δ_i be the Boolean function defined by $\delta_i(x) = x_i$. Show that $\widehat{f}(\{i\}) = \frac{1}{2} - \text{Prob}\{f = \delta_i\}$.

   (c) More generally, for each $A \subset \{1,\ldots,n\}$ and binary string $x \in G$, let $\pi_A(x)$ be the parity of those x_i such that $i \in A$, that is, $\pi_A(x) = \sum_{i\in A} x_i \pmod 2$. Show that $\widehat{f}(A) = \frac{1}{2} - \text{Prob}\{f = \pi_A\}$ for each $A \neq \emptyset$.

   (d) Argue that for a typical subset A of $\{1,\ldots,n\}$ one has $\text{Prob}\{f = \pi_A\} \approx \frac{1}{2}$. Quantify this statement by showing that

$$\#\left\{A \neq \emptyset : \left|\frac{1}{2} - \text{Prob}\{f = \pi_A\}\right| \geq \delta\right\} \leq \frac{1}{4\delta^2}.$$

   (Hint: Plancherel)