Harmonic Analysis – Homework Set 1

Recall that the L^2-Sobolev spaces $H^s(\mathbb{T})$, $s > 0$, are defined via the norms

$$\|f\|_{H^s(\mathbb{T})} := \|f\|_{L^2(\mathbb{T})} + \left(\sum_{n\in\mathbb{Z}} |n|^{2s} |\hat{f}(n)|^2 \right)^{1/2}.$$

It is easy to see that functions in $H^{\frac{1}{2}+\epsilon}(\mathbb{T})$ have absolutely convergent Fourier series for any $\epsilon > 0$, but not necessarily for $\epsilon = 0$. The question of uniform convergence of $S_N f$ for $f \in H^{\frac{1}{2}}(\mathbb{T})$ is more subtle.

1. The following (non-absolutely convergent) series define functions in $H^{\frac{1}{2}}(\mathbb{T})$.

$$f_S(x) := \sum_{n=2}^{\infty} \frac{\sin(2\pi n x)}{n \log n}, \qquad f_C(x) := \sum_{n=2}^{\infty} \frac{\cos(2\pi n x)}{n \log n}$$

   Show that the first series converges uniformly (hence $f_S \in C(\mathbb{T})$), but the second does not. In fact, show that $f_C(x) \geq c \log\log \frac{1}{|x|}$ as $x \to 0$.
   (Hint: Summation by parts.)

2. Suppose $f \in H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T})$. Show that $S_N f \to f$ uniformly.
   (Hint: Study $S_N f - \sigma_N f$.)

3. (Pr. 4.8 in Katznelson) Let $0 < \alpha < 1$. Show that the function

$$F(x) = \sum_{m=1}^{\infty} \frac{1}{3^{m\alpha}} \cos(2\pi 3^m x)$$

   belongs to $\mathrm{Lip}_\alpha(\mathbb{T})$. Hence the exponent α in the decay estimate $\hat{f}(n) = O(|n|^{-\alpha})$ for functions $f \in \mathrm{Lip}_\alpha(\mathbb{T})$ cannot be improved in general.

4. Draw a Venn diagram that contains all the sets below:

$$C(\mathbb{T}), \quad A(\mathbb{T}), \quad \mathrm{Lip}_{2/3}(\mathbb{T}), \quad H^{1/2}(\mathbb{T}), \quad U(\mathbb{T}) := \{f : S_N f \to f \text{ uniformly}\}.$$

   Justify your diagram. In addition, your diagram should not have any redundancy or ambiguity, i.e., if $A \cap B = \emptyset$, $A \subset B$, or $A \neq B$, this should be visible and indicated. Give an example (or show the existence) of a function in each region of intersection.

5. Let $\mathcal{T}_n$ denote the linear space of trigonometric polynomials of degree up to n and

$$E_n(f) := \inf_{P\in\mathcal{T}_n} \|f - P\|_2 = \|f - S_N f\|_2 = \left(\sum_{|k|>n} |\widehat{f}(k)|^2 \right)^{1/2}.$$

   (a) Let $0 < \alpha < 1$. Show that $E_n(f) \lesssim n^{-\alpha}$ if and only if $f \in \mathrm{Lip}_{\alpha,L^2}(\mathbb{T})$.
   (Hint: The "if" part is the same as in the proof of Bernstein's theorem that we saw in class. For the "only if" part, it will again help to consider a dyadic splitting of the frequencies.)

   (b) Show that $H^\alpha(\mathbb{T})$ is strictly contained in $\mathrm{Lip}_{\alpha,L^2}(\mathbb{T})$.