Figure 1: real tree.

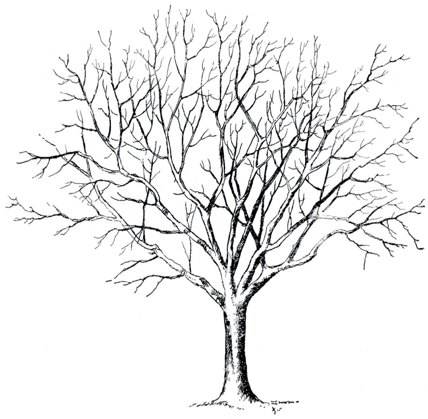


Figure 2: schematic tree.

Trees: Reality and Imagination

Misha Gromov

April 2, 2019

Marvellous branching patterns of trees repeatedly come again and again everywhere in nature.

In arrangements of leaves and of veins in the leaves of trees.

Figure 3: fern.

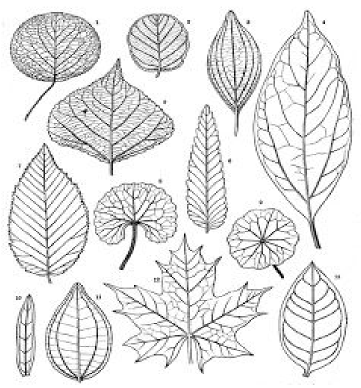


Figure 4: veins in leaves.

In the neurones and in the lungs in your body.

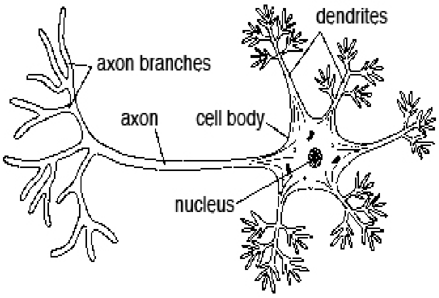


Figure 5: schematic neuron.

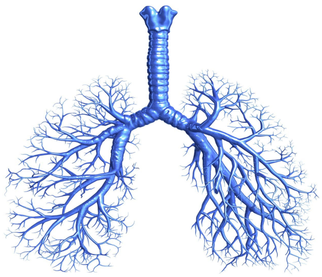


Figure 6: schematic lungs.

On many scales in your vascular, lymphatic and nervous systems.

Figure 7: Blood Vessels

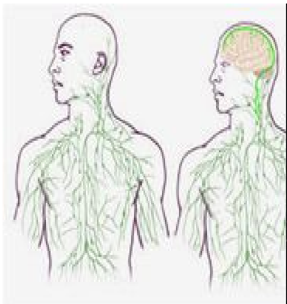


Figure 8: lymphatic system

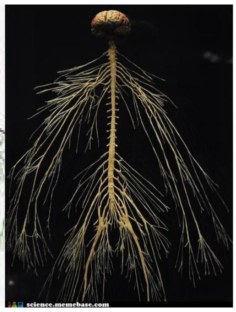


Figure 9: nervous system

Also in non-living things: in growing crystals, in branching of electrical discharges, in the drainage systems of rivers.

Figure 10: fractuals in agate

Figure 11: electrical discharge

Figure 12: drying Desert River

For a mathematician, trees are particular (connected) *networks* also called (connected) *graphs*, namely the ones *without cycles*.

It takes a long and involved discussion, which goes well beyond what mathematicians call *graph theory*, to fully and truly explain why "no cycles" makes trees so special.

And even if you have no idea why this is so, why trees are special, you can predict this by following an amazingly effective mathematician's motto:

If you don't understand – COUNT!

Indeed, simple – this is simple if you are a mathematician – counting shows that there are by for more networks in the world (of mathematics) than of trees which are bare of cycles:

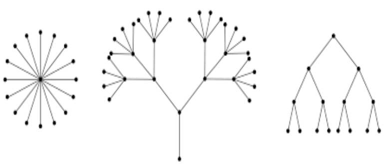


Figure 13: trees

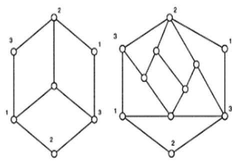


Figure 14: networks with cycles

the number of different graphic trees with N nodes is of the same order of magnitude as the number 2^N of binary sequences of length N, while the number of general graphs with N nodes is of order $2^{\frac{N^2}{2}}$ – this is incomparably greater than 2^N.

To appreciate the difference between these numbers think of something large, say of the synaptic memory system of you brain, or of the electronic memory of a powerful computer. However huge, they hardly contain more than $2^{50} \approx 10^{15}$(million billions) units – each unit is capable to store a single bit of information. But this makes only a tiny fraction of the number $2^{25 \times 50} = 2^{1250}$.

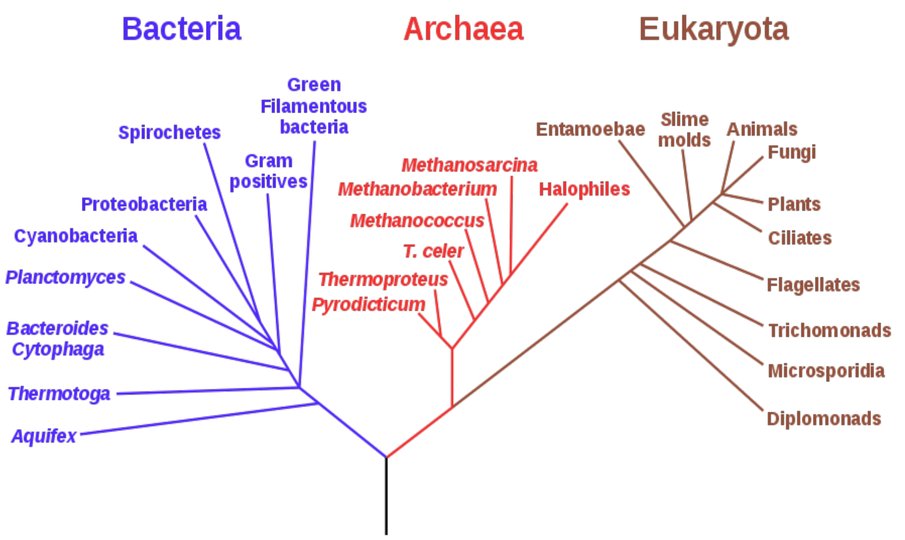
(Even the number of atoms in the entire (observable) universe, something which is evaluated close to 2^{300}, is dwarfed by the enormity of this $2^{1250} \approx 10^{375}$.)

Why does then Nature enjoy expressing yourself in the images of trees, why does she do it so consistently, so much unproportionally frequently with respect to the modest share of the trees among general networks?

The apparent reason for this is one-dimensionality of time versus three-dimensionality of space: the trees in the above pictures are still images of time dependent processes of spacial growth and development cast in unmovable space.

But more often than not, when the ancestral patterns are not physically preserved and only the final stage of a developmental process is recorded by Nature, the beauty of trees is still there, but now visible only with an imaginative mathematician's eye.

The most famous is the phylogenetic tree of Life worth seeng first on the grand scale



and then looked at closer at the branches of our own "family trees".

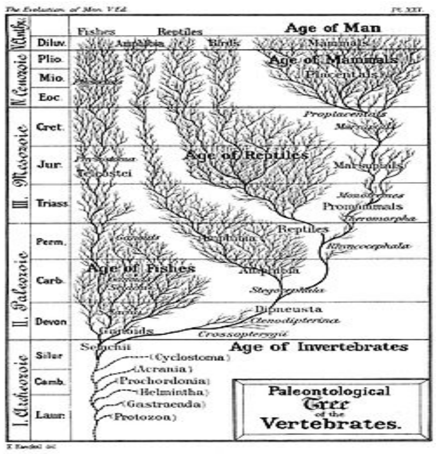


Figure 15:

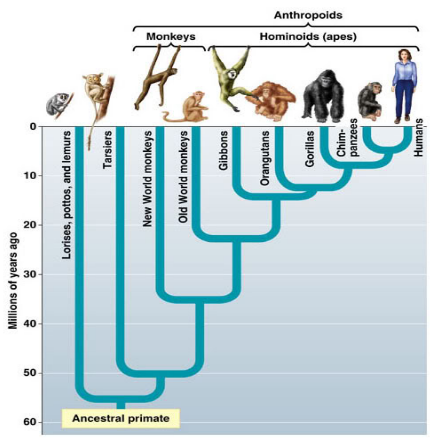
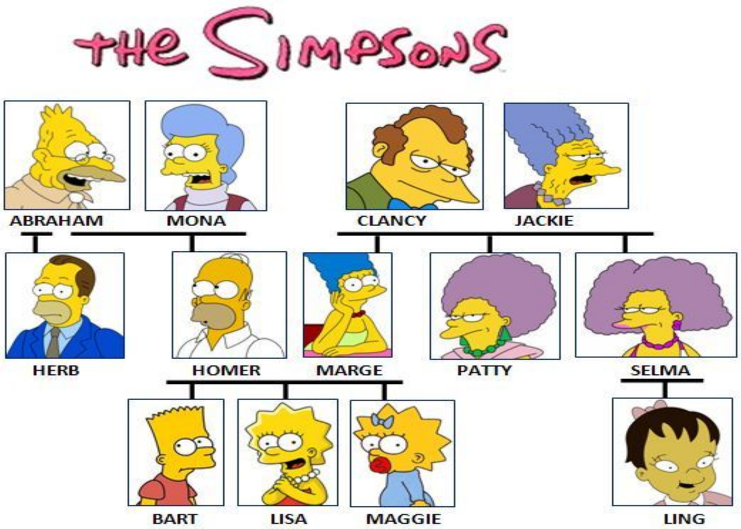


Figure 16:

And if, myopically, you focus on yourself, you'll see your personal family tree which looks, at least for a mathematician's eye, quite different: the directionality is reversed and the tree branches in both time directions.

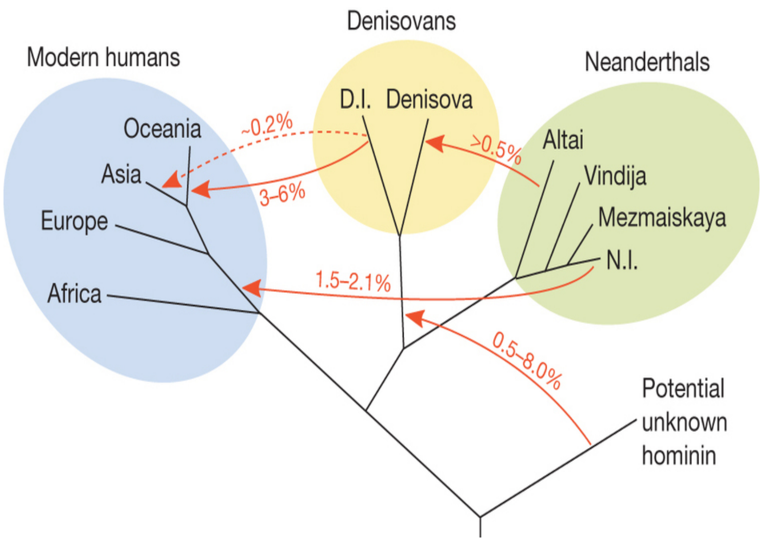


Besides, due to numerically unavoidable inbreeding, there are lots of cycles in this "tree" on relatively short time intervals, because, without, say 1000 years long cycles, you would have

2^{40} – more than trillion $\underbrace{\text{grand-grand-grand-grand-grand-...-grandmothers}}_{1000}$

assuming maternal age of 25. This is greater than the number of people who have ever lived on Earth.

And similar interbreeding/hybridisation linkages systematically violate/complement the phylogenetic pure tree structures, as it is clearly visible in the human genomes if you eye is aided with mathematics needed for this.



Most beautiful for a mathematician and most significant for a modern biologist are molecular phylogenetic trees, the nodes of which are marked by sequences of bases in DNA or amino acids in proteins.

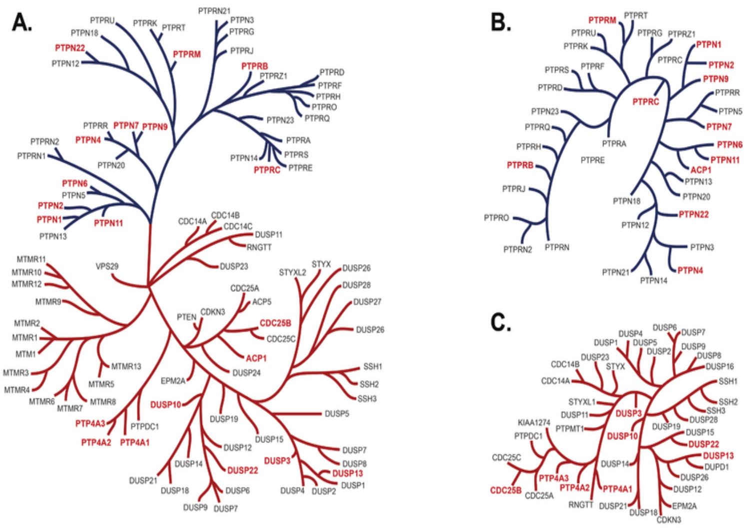


Figure 17: Dendrograms of evolution of protein tyrosine phosphatase and of its active sites

Here again, the pure tree structure may be (often is) sprinkled with horizontal links, due to viruses and to mobile genetic elements which can move around within a genome, or that can be transferred from one species to another.

But there are also "perfect" trees in biology – binary cell lineage trees with exactly two immediate descendants for each nod which pictorially describe the history of cell colonies arising by cell division.

(Even here "perfect treeness" is disrupted/complemented, e.g. by exchange of genetic material in unicellular organisms, such as bacterial conjugation and mating of yeast.)

You whole body – a colony of 30-40 trillions cells, or rather the history of its development and growth, can be described by such a tree rooted in a single cell, called *zygote*, where cells in different parts of the body can be traced back to the zygote by different numbers of steps of cell divisions, which is regulated by sill elusive control mechanisms of the numbers of cells constituting particular types of tissues in the body.

Less obviously, trees make the main ingredients in organisation of the energy landscape levels of (large) molecules, e.g. (folded and unfolded) proteins, and in the fitness landscapes of evolution.

Here beware: the word "landscape" misdirects you imagination: what is commonly seen as "landscape" has rather primitive tree structure in it – mountains don't branch much.

It takes a bit of mathematical thinking to see trees in the multidimensional energy and/or fitness landscapes and to appreciate the significance of this treeness.

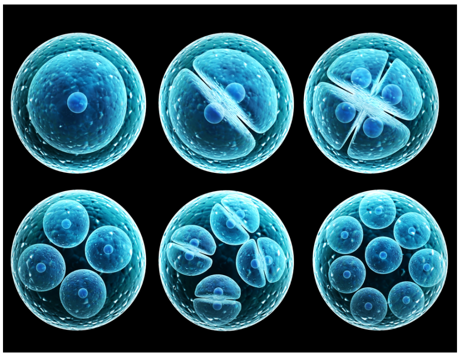


Figure 18: First three rounds of embryonic cell division

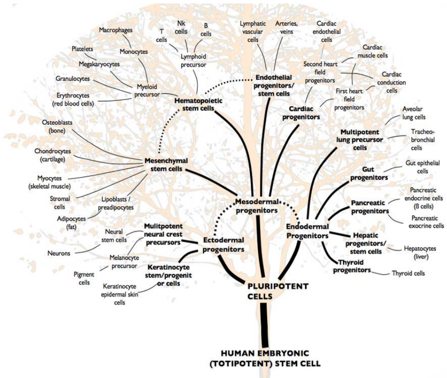


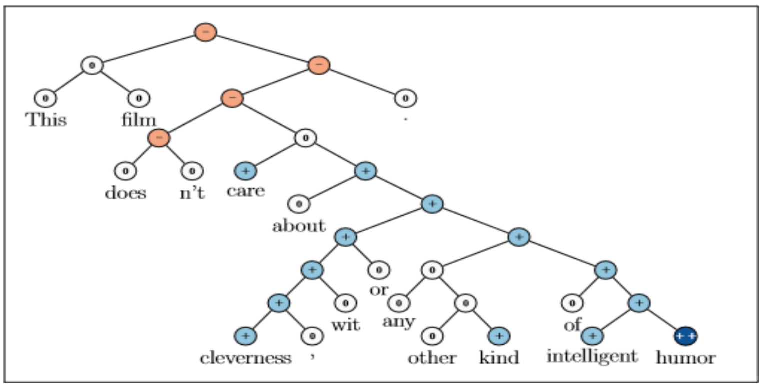
Figure 19: Cellular Differentiation Tree

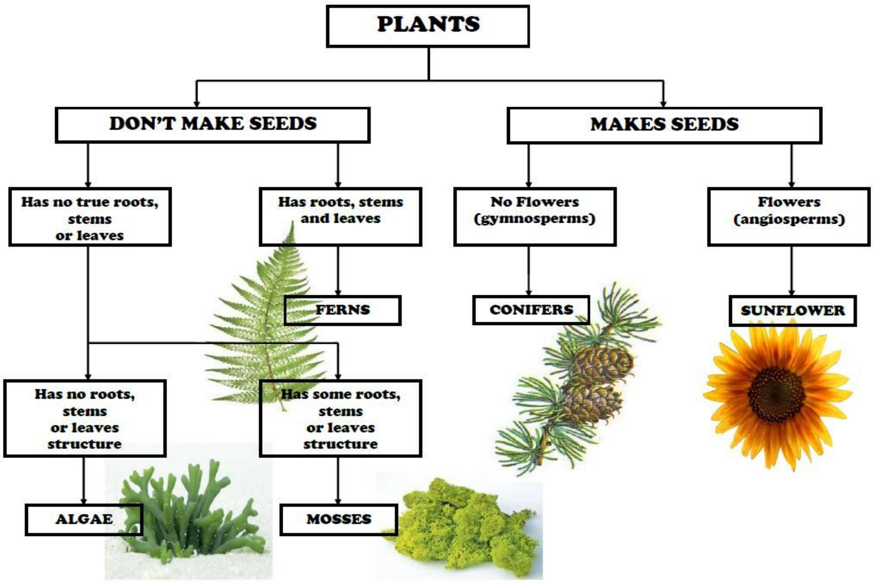
Also, tree-like patterns are persistent in products of the human mind.

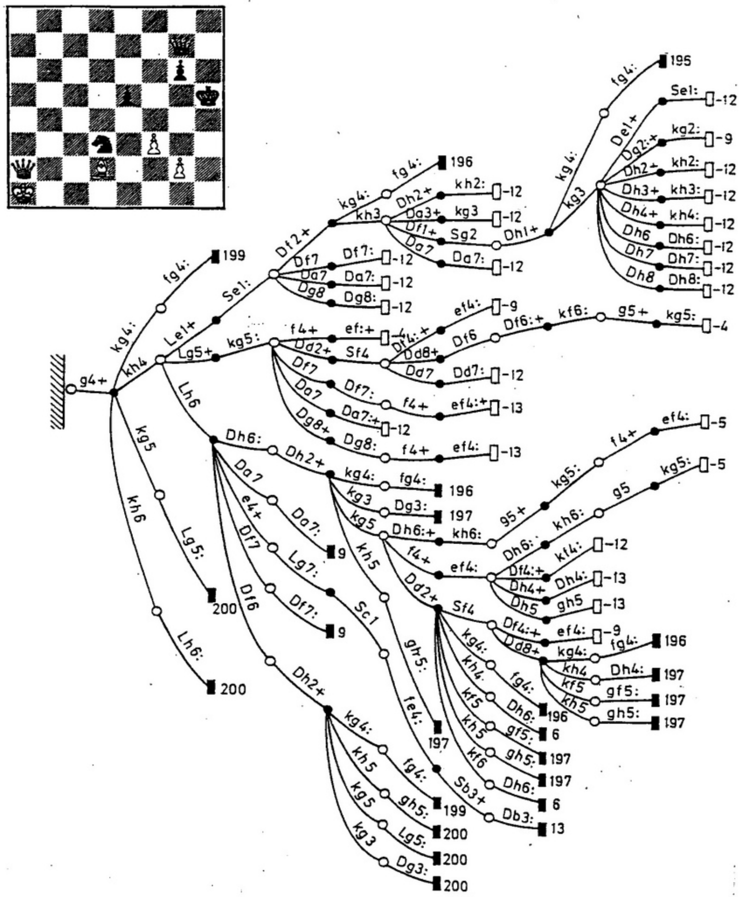
Thus, a mathematical grammarian assigns parse tree structures to nested arrangements of phrases in sentences in the human languages, while the semantic meaning of a word or of a short phrase is reflected in the branching pattern of the tree of longer phrases containing it.

And the tree organisation of itemised ideas is ubiquitous in all kind of classification schemes, both within and without the individual human mind, be these ideas about objets, e.g. plants or about moves in a chess game.

Probably, all of what is going on in the human mind can be depicted as a conglomeration of several overlapping trees.


This
film
does
n't
care
about
cleverness
,
wit
or
any
other
kind
of
intelligent
humor
.


PLANTS
DON'T MAKE SEEDS
MAKES SEEDS
Has no true roots, stems or leaves
Has roots, stems and leaves
No Flowers (gymnosperms)
Flowers (angiosperms)
FERNS
CONIFERS
SUNFLOWER
Has no roots, stems or leaves structure
Has some roots, stems or leaves structure
ALGAE
MOSSES



g4+
kg4:
fg4:
199
kh4
Le1+
Se1:
Df2+
kg4:
fg4:
196
kh3
Dh2+
kh2:
-12
Da3+
kg3
-12
Df1+
Sg2
Dh1+
kg3
kg4:
fg4:
195
De1+
Se1:
-12
Dg2:+
kg2:
-9
Dh2+
kh2:
-12
Dh3+
kh3:
-12
Dh4+
kh4:
-12
Dh6
Dh6:
-12
Dh7
Dh7:
-12
Dh8
Dh8:
-12
Da7
Da7:
-12
Df7
Df7:
-12
Da7
Da7:
-12
Dg8
Dg8:
-12
Lg5+
kg5:
f4+
ef:+
-4
Dd2+
Sf4
Df4:+
ef4:
-9
Dd8+
Df6
Df6:+
kf6:
g5+
kg5:
-4
Dd7
Dd7:
-12
Df7
Df7:
f4+
ef4:+
-13
Da7
Da7:+
-12
Dg8+
Dg8:
f4+
ef4:
-13
Lh6
Dh6:
Dh2+
kg4:
fg4:
196
kg3
Dg3:
197
kg5
Dh6:+
kh6:
Dh6:
g5+
kg5:
f4+
ef4:
-5
kh6:
g5
kg5:
-5
f4+
ef4:
Df4:+
kf4:
-12
Dh4+
Dh4:
-13
Dh5
gh5
-13
Dd2+
Sf4
Df4:+
ef4:
-9
Dd8+
kg4:
fg4:
196
kh4
Dh4:
197
kf5
gf5:
197
kh5
gh5:
197
kg4:
fg4:
196
kh4:
Dh6:
6
kf5
gf5:
197
kh5
gh5:
197
kf6
Dh6:
6
kh5
gh5:
fe4:
197
Sc1
Sb3+
Db3:
13
Da7
Da7:
9
e4+
Lg7:
Df7
Df7:
9
Df6
Dh2+
kg4:
fg4:
199
kh5
gh5:
200
kg5
Lg5:
200
kg3
Dg3:
200
kg5
Lg5:
200
kh6
Lh6:
200