Probability/Topology – Synopsis of lecture $2\frac{1}{2}$

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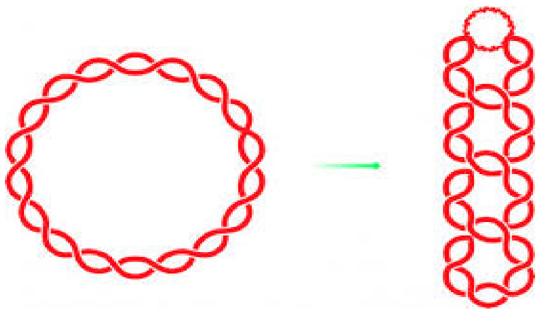

Figure 1: Bacterial DNA: Linking and supercoiling .

Derrick, A volume-diameter inequality for n-cube

(Besicovitch?) Derrick Theorem. If the distances between opposite $(n-1)$-faces of a "curved cube" $\tilde{\square}^n$ are $\geq d_i$, $i = 1, ..., n$, then

$$vol(\tilde{\square}^n) \geq \times_i d_i.$$

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Averaging sets: A generalization

of mean values and spherical designs Seymour-Zaslavsky.

Given: subset $X \subset \mathbb{R}^n$, which contains n-paths, $\phi_i : [0,1] \to X$, such that the n vectors $\phi_i(1) - \phi_i(0) \in \mathbb{R}^n$ span $\mathbb{R}^n$.

Then, for each point $y \in inter.conv(X) \subset \mathbb{R}^n$, there exists a finite subset $Z = \{x_i\}_{i\in I} \subset X$, such that

$$centr.mass(Z) = \frac{1}{card(I)} \sum_{i\in I} x_i.$$

Lemmas. (a)Let $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ be n-simplices spanned by subsets $\{x_j\}, \{y_j\} \subset \mathbb{R}^n$, $j \in J = \{0, 1, ...n\}$.

Let $f_\varepsilon : \Delta_1 \to \mathbb{R}^n$ be a continuous map such that, for all $K \subset J$ the image of the K-face from Δ_2 is contained in the ε-neighbourhood of the K-face in Δ_2.

If $y \in \Delta_2$ lies far from the boundary,

$$dist(y, \partial\Delta_2) > C_n\varepsilon,$$

($C_n = 1$ will do.)

then $\exists x \in \Delta_1$, such that $f(x) = y$.

(b) Given a bounded domain $\Omega\mathbb{R}^n$ and a continuous map $f : \omega \to \mathbb{F}^n$ such that f and a homotopy f_t, $t \in [0.1]$, to the identity map, e.g. $(x,t) \mapsto (1-t)x + tf(x)$.

If at no t the image $f_t(\Omega)$ contains a given point $x \in \Omega$ partial then $x \in f(\omega)$

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Non-homological Definition of Degree. The degree of a proper generic map between connected oriented manifolds, $f : X^n \to Y^n$, is d if the image of the fundamental cycle of X is equal d times the fundamental cycle of X.

Theorem: This degree is a homotopy invariant of f. Moreover is f_1 and f_2 are homologically equivalelent, then $deg(f_1) - deg(f_2)$.

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Thom Isomorphism. Given:

$p : V \to X$: a *fiber-wise oriented* smooth (which is unnecessary) $\mathbb{R}^N$-bundle over X,

$X \subset V$ is embedded as the zero section,

$V_\bullet$ be Thom space of V, i.e. (if X is compact) is one point compactification of V

Intersection $\cap : H_{i+N}(V_\bullet) \to H_i(X)$ is defined by intersecting generic $(i+N)$-cycles in $V_\bullet$ with X.

Thom Suspension $S_\bullet : H_i(X) \to H_{i+N}(V_\bullet)$: every cycle $C \subset X$ goes to the Thom space of the restriction of V to C, i.e. $C \mapsto (p^{-1}(C))_\bullet \subset V_\bullet$.

These $\cap$ and $S_\bullet$ are mutually reciprocal. Indeed $(\cap \circ S_\bullet)(C) = C$ for all $C \subset X$ and also $(S_\bullet \circ \cap)(C') \sim C'$ for all cycles C' in $V_\bullet$ where the homology is established by the fiberwise radial homotopy of C' in $V_\bullet \supset V$, which fixes $\bullet$ and move each $v \in V$ by $v \mapsto tv$. Clearly, $tC' \to (S_\bullet \circ \cap)(C')$ as $t \to \infty$ for all generic cycles C' in $V_\bullet$.

Thom isomorphism:

$$H_i(X) \leftrightarrow H_{i+N}(V_\bullet).$$

The Thom space of every $\mathbb{R}^N$-bundle $V \to X$ is $(N-1)$-connected, i.e. $\pi_j(V_\bullet) = 0$ for $j = 1, 2, \ldots N-1$, since a generic j-sphere $S^j \to V_\bullet$ with $j < N$ does not intersect $X \subset V$, where X is embedded into V by the zero section. Therefore, this sphere radially (in the fibers of V) contracts to $\bullet \in V_\bullet$.

Euler Number of a fibration $f : X \to B$ with $\mathbb{R}^{2k}$-fibers over a smooth closed oriented manifold B. enoted $e[B]$ is self intersection index of B in X.

Since the intersection pairing is symmetric on H_{2k} the sign of the Euler number does not depend on the orientation of B, but it does depend on the orientation of X.

If X equals the tangent bundle $T(B)$ then X is canonically oriented (even if B is non-orientable) and the Euler number is non-ambiguously defined and it equals the self-intersection number of the diagonal $X_{diag} \subset X \times X$.

Poincaré-Hopf Formula. *The Euler number* e *of the tangent bundle* $T(B)$ *of every closed oriented* $2k$*-manifold* B *satisfies*

$$e = \chi(B) = \sum_{i=0,1,\ldots 2k} rank(H_i(B;\mathbb{Q})).$$

(If $n = dim(B)$ is *odd*, then $\sum_{i=0,1,\ldots n} rank(H_i$
0 by the Poincaré duality.)

It is hard the believe this may be true! A single cycle (let it be the *fundamental* one) knows something about all of the homology of B.

The most transparent proof of this formula is, probably, via the Morse theory (known to Poincaré) and it hardly can be called "trivial".

A more algebraic proof follows from the Künneth formula (see below) and an expression of the class $[X_{diag}] \in H_{2k}(X \times X)$ in terms of the intersection ring structure in $H_*(X)$.

The Euler number can be also defined for connected *non-orientable* B as follows. Take the canonical oriented double covering $\tilde{B} \to B$, where each point $\tilde{b} \in \tilde{B}$ over $b \in B$ is represented as b + an orientation of B near b. Let the bundle $\tilde{X} \to \tilde{B}$ be induced from X by the covering map $\tilde{B} \to B$, i.e. this $\tilde{X}$ is the obvious double covering of X corresponding to $\tilde{B} \to B$. Finally, set $e(X) = e(\tilde{X})/2$.

The Poincaré-Hopf formula for non-orientable $2k$-manifolds B follows from the orientable case by the *mul-*

tiplicativity of the Euler characteristic χ which is valid for all compact triangulated spaces B,

an l-sheeted covering $\tilde{B} \to B$ has $\chi(\tilde{B}) = l \cdot \chi(B)$.

If the homology is defined via a triangulation of B, then $\chi(B)$ equals the alternating sum $\sum_i (-1)^i N(\Delta^i)$ of the numbers of i-simplices by straightforward linear algebra and the multiplicativity follows. But this is not so easy with our geometric cycles. (If B is a closed manifold, this also follows from the Poincaré-Hopf formula and the obvious multiplicativity of the Euler number for covering maps.)