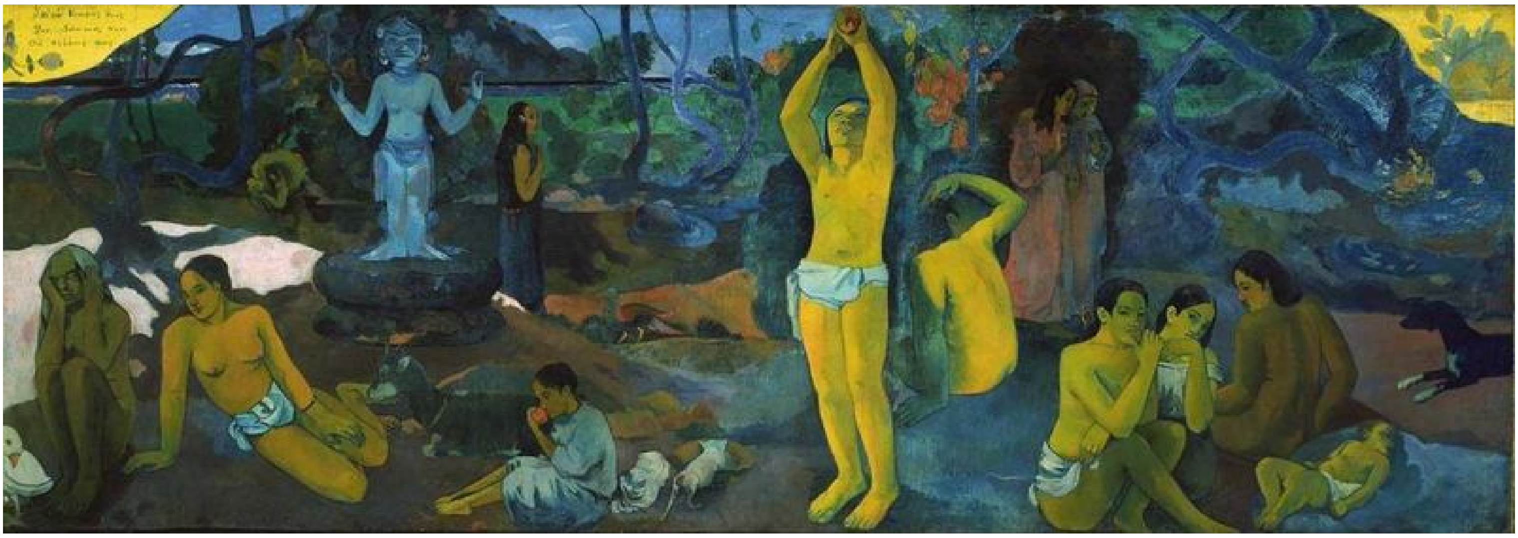

Manifolds: Where Do We Come From? What Are We? Where Are We Going

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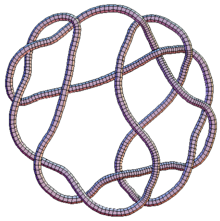
Contents



Abstract

Descendants of algebraic kingdoms of high dimensions, enchanted by the magic of Thurston and Donaldson, lost in the whirlpools of the Ricci flow, topologists dream of an ideal land of manifolds – perfect crystals of mathematical structure which would capture our vague mental images of geometric spaces. We browse through the ideas inherited from the past hoping to penetrate through the fog which conceals the future.

1 Ideas and Definitions.

We are fascinated by knots and links. Where does this feeling of beauty and mystery come from? To get a glimpse at the answer let us move by 25 million years in time.

25×10^6 is, roughly, what separates us from orangutans: 12 million years to our common ancestor on the phylogenetic tree and then 12 million years back by another branch of the tree to the present day orangutans.

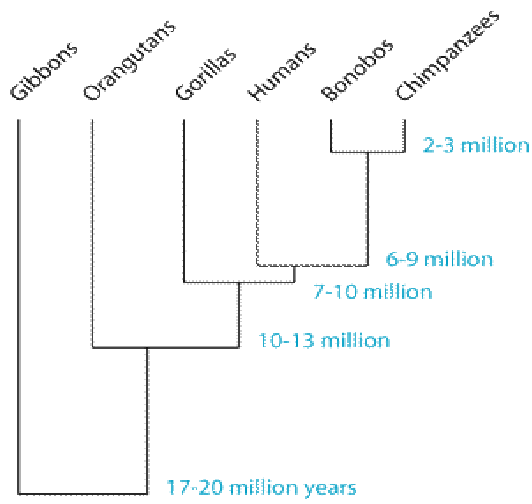
But are there topologists among orangutans?

Yes, there definitely are: many orangutans are good at "proving" the triviality of elaborate knots, e.g. they fast master the art of untying boats from their mooring when they fancy taking rides downstream in a river, much to the annoyance of people making these knots with a different purpose in mind.

A more amazing observation was made by a zoo-psychologist Anne Russon in mid 90's at Wanariset Orangutan Reintroduction Project (see p. 114 in [69]).

"... Kinoi [a juvenile male orangutan], when he was in a possession of a hose, invested every second in making giant hoops, carefully inserting one end of his hose into the other and jamming it in tight. Once he'd made his hoop, he passed various parts of himself back and forth through it – an arm, his head, his feet, his whole torso – as if completely fascinated with idea of going through the hole."

Playing with hoops and knots, where there is no visible goal or any practical gain – be it an ape or a $3D$-topologist – appears fully "non-intelligent" to a practically minded observer. But we, geometers, feel thrilled at seeing an animal whose space perception is so similar to ours.



It is unlikely, however, that Kinoi would formulate his ideas the way we do and that, unlike our students, he could be easily intimidated into accepting "equivalence classes of atlases" and "ringed spaces" as appropriate definitions of his topological playground. (Despite such display of disobedience, we would enjoy the company of young orangutans; they are charmingly playful creatures, unlike the aggressive and reckless chimpanzees – our nearest evolutionary neighbors.)

Apart from topology, orangutans do not rush to accept another human definition, namely that of "tools", as of

"external *detached* objects (to exclude a branch used for climbing a tree) employed for reaching specific goals".

(The use of tools is often taken by zoo-psychologists for a measure of "intelligence" of an animal.)

Being imaginative arboreal creatures, orangutans prefer a broader definition: For example (see [69]):

• they bunch up leaves to make wipers to clean their bodies without detaching the leaves from a tree;

• they often break branches but deliberately leave them attached to trees when it suits their purposes – these could not have been achieved if orangutans were bound by the "detached" definition.

Morale. Our best definitions, e.g. that of a manifold, tower as prominent landmarks over our former insights. Yet, we should not be hypnotized by definitions. After all, they are remnants of the past and tend to misguide us when we try to probe the future.

Remark. There is a non-trivial similarity between the neurological structures underlying the behaviour of playful animals and that of working mathematicians (see [31]).

2 Homotopies and Obstructions.

For more than half a century, starting from Poincaré, topologists have been laboriously stripping their beloved science of its geometric garments.

"Naked topology", reinforced by homological algebra, reached its to-day breathtakingly high plateau with the following

Serre $[S^{n+N} \to S^N]$-Finiteness Theorem. (1951) *There are at most finitely many homotopy classes of maps between spheres $S^{n+N} \to S^N$ but for the two exceptions:*

• *equivi-dimensional case where $n = 0$ $\pi_N(S^N) = \mathbb{Z}$; the homotopy class of a map $S^N \to S^N$ in this case is determined by an integer that is the degree of a map.*

(Brouwer 1912, Hopf 1926. We define degree in section 4.) This is expressed in the standard notation by writing

$$\pi_N(S^N) = \mathbb{Z}.$$

• *Hopf case, where N is even and $n = 2N - 1$. In this case $\pi_{2N-1}(S^N)$ contains a subgroup of finite index isomorphic to $\mathbb{Z}$.*

It follows that

the homotopy groups $\pi_{n+N}(S^N)$ are finite for $N >> n$,

where, by the *Freudenthal suspension theorem* of 1928 (this is easy),

the groups $\pi_{n+N}(S^N)$ for $N \geq n$ do not depend on N.

These are called *the stable homotopy groups of spheres* and are denoted π_n^{st}.

H. Hopf proved in 1931 that the map $f : S^3 \to S^2 = S^3/\mathbb{T}$, for the group $\mathbb{T} \subset \mathbb{C}$ of the complex numbers with norm one which act on $S^3 \subset \mathbb{C}^2$ by $(z_1, z_2) \mapsto (tz_1, tz_2)$, is *non-contractible.*

In general, the unit tangent bundle $X = UT(S^{2k}) \to S^{2k}$ has finite homology $H_i(X)$ for $0 < i < 4k - 1$. By Serre's theorem, there exists a map $S^{4k-1} \to X$ of positive degree and the composed map $S^{4k-1} \to X \to S^{2k}$ generates an *infinite* cyclic group of *finite index* in $\pi_{4k-1}(S^{2k})$.

The proof by Serre – a geometer's nightmare – consists in tracking a multitude of linear-algebraic relations between the homology and homotopy groups of infinite dimensional spaces of maps between spheres and it tells you next to nothing about the geometry of these maps. (See [59] for a "semi-geometric" proof of the finiteness of the stable homotopy groups of spheres and section 5 of this article for a related discussion. Also, the construction in [23] may be relevant.)

Recall that the set of the homotopy classes of maps of a sphere S^M to a connected space X makes *a group* denoted $\pi_M(X)$, (π is for Poincaré who defined *the fundamental group* π_1) where the definition of the group structure depends on distinguished points $x_0 \in X$ and $s_0 \in S^M$. (The groups π_M defined with different x_0 are mutually isomorphic, and if X is *simply connected*, i.e. $\pi_1(X) = 1$, then they are *canonically* isomorphic.)

This point in S^M may be chosen with the representation of S^M as the one point compactification of the Euclidean space $\mathbb{R}^M$, denoted $\mathbb{R}^M_\bullet$, where this infinity point $\bullet$ is taken for s_0. It is convenient, instead of maps $S^m = \mathbb{R}^m_\bullet \to (X, x_0)$, to deal with maps $f : \mathbb{R}^M \to X$ "with compact supports", where the *support* of an f is the closure of the (open) subset $supp(f) = supp_{x_0}(f) \subset \mathbb{R}^m$ which consists of the points $s \in \mathbb{R}^m$ such that $f(s) \neq x_0$.

A pair of maps $f_1, f_2 : \mathbb{R}^M \to X$ with *disjoint* compact supports obviously defines "the joint map" $f : \mathbb{R}^M \to X$, where the homotopy class of f (obviously) depends only on those of f_1, f_2, provided $supp(f_1)$ lies in the left half space $\{s_1 < 0\} \subset \mathbb{R}^m$ and $supp(f_2) \subset \{s_1 > 0\} \subset \mathbb{R}^M$, where s_1 is a non-zero linear function (coordinate) on $\mathbb{R}^M$.

The composition of the homotopy classes of two maps, denoted $[f_1] \cdot [f_2]$, is defined as the homotopy class of the joint of f_1 moved far to the left with f_2 moved far to the right.

Geometry is sacrificed here for the sake of algebraic convenience: first, we break the symmetry of the sphere S^M by choosing a base point, and then we destroy the symmetry of $\mathbb{R}^M$ by the choice of s_1. If $M = 1$, then there are essentially two choices: s_1 and $-s_1$, which correspond to interchanging f_1 with f_2 – nothing wrong with this as the composition is, in general, non-commutative.

In general $M \geq 2$, these $s_1 \neq 0$ are, homotopically speaking, parametrized by the unit sphere $S^{M-1} \subset \mathbb{R}^M$. Since S^{M-1} is connected for $M \geq 2$, the composition is commutative and, accordingly, the composition in π_i for $i \geq 2$ is denoted $[f_1] + [f_2]$. Good for algebra, but the $O(M+1)$-ambiguity seems too great a price for this. (An algebraist would respond to this by pointing out that the ambiguity is resolved in the language of *operads* or something else of this kind.)

But this is, probably, unavoidable. For example, the best you can do for maps $S^M \to S^M$ in a given non-trivial homotopy class is to make them symmetric (i.e. equivariant) under the action of the maximal torus $\mathbb{T}^k$ in the orthogonal group $O(M+1)$, where $k = M/2$ for even M and $k = (M+1)/2$ for M odd.

And if $n \geq 1$, then, with a few exceptions, there are no apparent symmetric representatives in the homotopy classes of maps $S^{n+N} \to S^N$; yet Serre's theorem does carry a geometric message.

If $n \neq 0, N-1$, then every continuous map $f_0 : S^{n+N} \to S^N$ is homotopic to a map $f_1 : S^{n+N} \to S^N$ of dilation bounded by a constant,

$$dil(f_1) =_{def} \sup_{s_1 \neq s_2 \in S^{n+N}} \frac{dist(f(s_1), f(s_2))}{dist(s_1, s_2)} \leq const(n, N).$$

Dilation Questions. (1) What is the asymptotic behaviour of $const(n, N)$ for $n, N \to \infty$?

For all we know the *Serre dilation constant* $const_S(n, N)$ may be bounded for $n \to \infty$ and, say, for $1 \le N \le n-2$, but a bound one can see offhand is that by an exponential tower $(1+c)^{(1+c)^{(1+c)^{\cdots}}}$, of height N, since each geometric implementation of the homotopy lifting property in a Serre fibrations may bring along an exponential dilation. Probably, the (questionably) geometric approach to the Serre theorem via "singular bordisms" (see [76], [23],[1] and section 5) delivers a better estimate.

(2) Let $f : S^{n+N} \to S^N$ be a contractible map of dilation d, e.g. f equals the m-multiple of another map where m is divisible by the order of $\pi_{n+N}(S^N)$.

What is, roughly, the minimum $D_{min} = D(d, n, N)$ of dilations of maps F of the unit ball $B^{n+N+1} \to S^N$ which are equal to f on $\partial(B^{n+N+1}) = S^{n+N}$?

Of course, this dilation is the most naive invariant measuring the "geometric size of a map". Possibly, an interesting answer to these questions needs a more imaginative definition of "geometric size/shape" of a map, e.g. in the spirit of the minimal degrees of polynomials representing such a map.

Serre's theorem and its descendants underlie most of the topology of the high dimensional manifolds. Below are frequently used corollaries which relate homotopy problems concerning general spaces X to the homology groups $H_i(X)$ (see section 4 for definitions) which are much easier to handle.

$[S^{n+N} \to X]$**-Theorems.** Let X be a compact connected triangulated or cellular space, (defined below) or, more generally, a connected space with *finitely generated homology groups* $H_i(X)$, $i = 1, 2, \ldots$. If the space X is *simply connected*, i.e. $\pi_1(X) = 1$, then its homotopy groups have the following properties.

(1) *Finite Generation.* The groups $\pi_m(X)$ are (Abelian!) *finitely generated* for all $m = 2, 3, \ldots$.

(2) *Sphericity.* If $\pi_i(X) = 0$ for $i = 1, 2, N-1$, then the (obvious) Hurewicz homomorphism

$$\pi_N(X) \to H_N(X),$$

which assigns, to a map $S^N \to X$, the N-cycle represented by this N-sphere in X, is an *isomorphism.* (This is elementary, Hurewicz 1935.)

(3) $\mathbb{Q}$-*Sphericity.* If the groups $\pi_i(X)$ are finite for $i = 2, N-1$ (recall that we assume $\pi_1(X) = 1$), then the Hurewicz homomorphism *tensored with rational numbers*,

$$\pi_{N+n}(X) \otimes \mathbb{Q} \to H_{N+n}(X) \otimes \mathbb{Q},$$

is an *isomorphism* for $n = 1, \ldots, N-2$.

Because of the finite generation property, The $\mathbb{Q}$-sphericity is equivalent to

(3') **Serre m-Sphericity Theorem**. Let the groups $\pi_i(X)$ be finite (e.g. trivial) for $i = 1, 2, \ldots, N-1$ and $n \le N-2$. Then

an m-multiple of every $(N+n)$-cycle in X for some $m \ne 0$ is homologous to an $(N+n)$-sphere continuously mapped to X;

every two homologous spheres $S^{N+n} \to X$ become homotopic when composed with a non-contractible i.e. of degree $m \ne 0$, self-mapping $S^{n+N} \to S^{n+N}$. In more algebraic terms, the elements $s_1, s_2 \in \pi_{n+N}(X)$ represented by these spheres satisfy $ms_1 - ms_2 = 0$.

The following is the dual of the m-Sphericity.

Serre $[\to S^N]_{\mathbb{Q}}$- Theorem. Let X be a compact triangulated space of dimension $n+N$, where *either N is odd or $n < N-1$.*

Then a non-zero multiple of every homomorphism $H_N(X) \to H_N(S^N)$ can be realized by a continuous map $X \to S^N$.

If two continuous maps are $f, g : X \to S^N$ are homologous, i.e. if the homology homomorphisms $f_*, g_* : H_N(X) \to H_N(S^N) = \mathbb{Z}$ are equal, then there exists a continuous self-mapping $\sigma : S^N \to S^N$ of non-zero degree such that the composed maps $\sigma \circ f$ and $\sigma \circ f : X \to S^N$ are homotopic.

These $\mathbb{Q}$-theorems follow from the Serre finiteness theorem for maps between spheres by an elementary argument of *induction by skeletons* and rudimentary *obstruction theory* which run, roughly, as follows.

Cellular and Triangulated Spaces. Recall that a *cellular space* is a topological space X with an ascending (finite or infinite) sequence of closed subspaces $X_0 \subset X_1 \subset ... \subset X_i \subset ...$ called the *i-skeleta* of X, such that $\bigcup_i X_i = X$ and such that

X_0 is a discrete finite or countable subset.

every X_i, $i > 0$ is obtained by attaching a countably (or finitely) many i-balls B^i to X_{i-1} by continuous maps of the boundaries $S^{i-1} = \partial(B^i)$ of these balls to X_{i-1}.

For example, if X is a *triangulated* space then it comes with homeomorphic *embeddings* of the i-simplices $\Delta^i \to X_i$ extending their boundary maps, $\partial(\Delta^i) \to X_{i-1} \subset X_i$ where one additionally requires (here the word "simplex", which is, topologically speaking, is indistinguishable from B^i, becomes relevant) that the intersection of two such simplices Δ^i and Δ^j imbedded into X is a simplex Δ^k which is a *face* simplex in $\Delta^i \supset \Delta^k$ and in $\Delta^j \supset \Delta^k$.

If X is a non-simplicial cellular space, we also have continuous maps $B^i \to X_i$ but they are, in general, embeddings *only* on the interiors $B^i \setminus \partial(B^i)$, since the attaching maps $\partial(B^i) \to X_{i-1}$ are not necessarily injective. Nevertheless, the images of B^i in X are called *closed cells*, and denoted $B_i \subset X_i$, where the union of all these i-cells equals X_i.

Observe that the homotopy equivalence class of X_i is determined by that of X_{i-1} and by the homotopy classes of maps from the spheres $S^{i-1} = \partial(B^i)$ to X_{i-1}. We are free to take any maps $S^{i-1} \to X_{i-1}$ we wish in assembling a cellular X which make cells more efficient building blocks of general spaces than simplices.

For example, the sphere S^n can be made of a 0-cell and a single n-cell.

If $X_{i-1} = S^l$ for some $l \le i-1$ (one has $l < i-1$ if there is no cells of dimensions between l and $i-1$) then the homotopy equivalence classes of X_i with a single i-cell one-to-one correspond to the homotopy group $\pi_{i-1}(S^l)$.

On the other hand, every cellular space can be approximated by a homotopy equivalent simplicial one, which is done by induction on skeletons X_i with an approximation of *continuous* attaching maps by *simplicial* maps from $(i-1)$-spheres to X_{i-1}.

Recall that a *homotopy equivalence* between X_1 and X_2 is given by a pair of maps $f_{12} : X_1 \to X_2$ and $f_{21} : X_2 \to X_1$, such that both composed maps $f_{12} \circ f_{21} : X_1 \to X_1$ and $f_{21} \circ f_{12} : X_2 \to X_2$ are homotopic to the identity.

Obstructions and Cohomology. Let Y be a connected space such that $\pi_i(Y) = 0$ for $i = 1, ..., n-1 \ge 1$, let $f : X \to Y$ be a continuous map and let us construct, by induction on $i = 0, 1, ..., n-1$, a map $f_{new} : X \to Y$ which is homotopic to f and which sends X_{n-1} to a point $y_0 \in Y$ as follows.

Assume $f(X_{i-1}) = y_0$. Then the resulting map $B^i \xrightarrow{f} Y$, for each i-cell B^i from X_i, makes an i-sphere in Y, because the boundary $\partial B^i \subset X_{i-1}$ goes to a single point – our to y_0 in Y.

Since $\pi_i(Y) = 0$, this B^i in Y can be contracted to y_0 without disturbing its boundary. We do it all i-cells from X_i and, thus, contract X_i to y_0. (One can not, in general, extend a continuous map from a closed subset $X' \subset X$ to X, but one always can extend a continuous *homotopy* $f'_t : X' \to Y$, $t \in [0,1]$, of a given map $f_0 : X \to Y$, $f_0|X' = f'_0$, to a homotopy $f_t : X \to Y$ for all closed subsets $X' \subset X$, similarly to how one extends $\mathbb{R}$-valued functions from $X' \subset X$ to X.)

The contraction of X to a point in Y can be obstructed on the n-th step, where $\pi_n(Y) \neq 0$, and where each *oriented* n-cell $B^n \subset X$ mapped to Y with $\partial(B^n) \to y_0$ represents an element $c \in \pi_n(Y)$ which may be non-zero. (When we switch an orientation in B^n, then $c \mapsto -c$.)

We assume at this point, that our space X is a triangulated one, switch from B^n to Δ^n and observe that the function $c(\Delta^n)$ is (obviously) an *n-cocycle* in X with values in the group $\pi_n(Y)$, which means (this is what is longer to explain for general cell spaces) that the sum of $c(\Delta^n)$ over the $n+2$ face-simplices $\Delta^n \subset \partial\Delta^{n+1}$ *equals zero*, for all Δ^{n+1} in the triangulation (if we canonically/correctly choose orientations in all Δ^n).

The cohomology class $[c] \in H^n(X; \pi_n(X))$ of this cocycle does not depend (by an easy argument) on how the $(n-1)$-skeleton was contracted. Moreover, every cocycle c' in the class of $[c]$ can be obtained by a homotopy of the map on X_n which is kept constant on X_{n-2}. (Two A-valued n-cocycles c and c', for an abelian group A, are *in the same cohomology class* if there exists an A-valued function $d(\Delta^{n-1})$ on the oriented simplices $\Delta^{n-1} \subset X_{n-1}$, such that $\sum_{\Delta^{n-1} \subset \Delta^n} d(\Delta^{n-1}) = c(\Delta^n) - c'(\Delta^n)$ for all Δ^n. The set of the cohomology classes of n-cocycles with a natural additive structure is called the *cohomology group* $H^n(X; A)$. It can be shown that $H^n(X; A)$ depends only on X but not an a particular choice of a triangulation of X. See section 4 for a lighter geometric definitions of homology and cohomology.)

In particular, if $dim(X) = n$ we, thus, equate the set $[X \to Y]$ of the homotopy classes of maps $X \to Y$ with the cohomology group $H^n(X; \pi_n(X))$. Furthermore, this argument applied to $X = S^n$ shows that $\pi_n(X) = H_n(X)$ and, in general, that

the set of the homotopy classes of maps $X \to Y$ equals the set of homomorphisms $H_n(X) \to H_n(Y)$, provided $\pi_i(Y) = 0$ for $0 < i < dim(X)$.

Finally, when we use this construction for proving the above $\mathbb{Q}$-theorems where one of the spaces is a sphere, we keep composing our maps with self-mappings of this sphere of suitable degree $m \neq 0$ that kills the obstructions by the Serre finiteness theorem.

For example, if X is a finite cellular space without 1-cells, one can define the *homotopy multiple* $l^* X$, for every integer l, by replacing the attaching maps of all $(i+1)$-cells, $S^i \to X_i$, by l^{k_i}-multiples of these maps in $\pi_i(X_i)$ for $k_2 << k_3 << ...$, where this $l^* X$ comes along with a map $l^* X \to X$ which induces isomorphisms on all homotopy groups tensored with $\mathbb{Q}$.

The obstruction theory (developed by Eilenberg in 1940 following Pontryagin's 1938 paper) well displays the logic of algebraic topology: the geometric symmetry of X (if there was any) is broken by an arbitrary triangulation or a cell decomposition and then another kind of symmetry, an Abelian algebraic one, emerges on the (co)homology level.

(See [57] for a comprehensive overview of algebraic and geometric topology.)

Serre's idea is that the homotopy types of finite simply connected cell complexes as well as of finite diagrams of continuous maps between these are *finitary arithmetic objects* which can be encoded by finitely many polynomial equations and non-equalities with integer coefficients, and where the structural organization of the homotopy theory depends on *non-finitary objects* which are *inductive limits of finitary ones*, such as the homotopy types of spaces of continuous maps between finite cell spaces.

3 Generic Pullbacks.

A common zero set of N smooth (i.e. infinitely differentiable) functions $f_i : \mathbb{R}^{n+N} \to \mathbb{R}$, $i = 1, ...N$, may be very nasty even for $N = 1$ – every closed subset in $\mathbb{R}^{n+1}$ can be represented as a zero of a smooth function. However, if the functions f_i are taken *in general position*, then the common zero set is a smooth n-submanifold in $\mathbb{R}^{n+N}$.

Here and below, "f in general position" or "generic f", where f is an element of a topological space F, e.g. of the space of C^∞-maps with the C^∞-topology, means that what we say about f applies to all f in an *open and dense* subset in F. (Sometimes, one allows not only open dense sets in the definition of genericity but also their *countable* intersections.)

Generic smooth (unlike continuous) objects are as nice as we expect them to be; the proofs of this "niceness" are local-analytic and elementary (at least in the cases we need); everything trivially follows from *Sard's theorem + the implicit function theorem.*

The representation of manifolds with functions generalizes as follows..

Generic Pullback Construction (Pontryagin 1938, Thom 1954). Start with a smooth N-manifold V, e.g. $V = \mathbb{R}^N$ or $V = S^N$, and let $X_0 \subset V$ be a smooth submanifold, e.g. $0 \in \mathbb{R}^N$ or a point $x_0 \in S^N$. Let W be a smooth manifold of dimension M, e.g. $M = n + N$.

if $f : W \to V$ is a generic smooth map, then the pullback $X = f^{-1}(X_0) \subset W$ is a smooth submanifold in W with $codim_W(X) = codim_V(X_0)$, i.e. $M - dim(X) = N - dim(X_0)$.

Moreover, if the manifolds W, V and X_0 are oriented, then X comes with a natural orientation.

Furthermore, if W has a boundary then X is a smooth submanifold in W with a boundary $\partial(X) \subset \partial(W)$.

Examples. (a) Let $f : W \subset V \supset X_0$ be a smooth, possibly non-generic, embedding of W into V. Then a small generic perturbation $f' : W \to V$ of f remains an embedding, such that image $W' = f'(W) \subset V$ in V becomes *transversal* (i.e. nowhere tangent) to X_0. One sees with the full geometric clarity (with a picture of two planes in the 3-space which intersect at a line) that the intersection $X = W' \cap X_0 (= (f')^{-1}(X_0))$ is a submanifold in V with $codim_V(X) = codim_V(W) + codim_V(X_0)$.

(b) Let $f : S^3 \to S^2$ be a smooth map and $S_1, S_2 \in S^3$ be the pullbacks of two generic points $s_1, s_2 \in S^2$. These S_i are smooth closed curves; they are naturally oriented, granted orientations in S^2 and in S^3.

Let $D_i \subset B^4 = \partial(S^3)$, $i = 1, 2$, be generic smooth oriented surfaces in the ball $B^4 \supset S^3 = \partial(B^4)$ with their oriented boundaries equal S_i and let $h(f)$ denote the *intersection index* (defined in the next section) between D_i.

Suppose, the map f is homotopic to zero, extend it to a smooth generic map $\varphi : B^4 \to S^2$ and take the φ-pullbacks $D_i^\varphi = \varphi^{-1}(s_i) \subset B^4$ of s_i.

Let S^4 be the 4-sphere obtained from the two copies of B^4 by identifying the boundaries of the balls and let $C_i = D_i \cup D_i^\varphi \subset S^4$.

Since $\partial(D_i) = \partial(D_i^\varphi) = S_i$, these C_i are *closed* surfaces; hence, the intersection index between them equals zero (because they are homologous to zero in S^4, see the next section), and since D_i^φ do not intersect, the intersection index $h(f)$ between D_i is also zero.

It follows that *non-vanishing* of the *Hopf invariant* $h(f)$ implies that f is *non-homotopic to zero.*

For instance, the Hopf map $S^3 \to S^2$ is non-contractible, since every two transversal flat dicks $D_i \subset B^4 \subset \mathbb{C}^2$ bounding equatorial circles $S_i \subset S^3$ intersect at a single point.

The essential point of the seemingly trivial pull-back construction, is that starting from "simple manifolds" $X_0 \subset V$ and W, we produce complicated and more interesting ones by means of "complicated maps" $W \to V$. (It is next to impossible to make an interesting manifold with the "equivalence class of atlases" definition.)

For example, if $V = \mathbb{R}$, and our maps are functions on W, we can generate lots of them by using algebraic and analytic manipulations with functions and then we obtain maps to $\mathbb{R}^N$ by taking N-tuples of functions.

And less obvious (smooth generic) maps, for all kind of V and W, come as smooth generic approximations of continuous maps $W \to V$ delivered by the algebraic topology.

Following Thom (1954) one applies the above to maps into one point compactifications $V_\bullet$ of open manifolds V where one still can speak of generic pullbacks of smooth submanifolds X_0 in $V \subset V_\bullet$ under maps $W \to V_\bullet$

Thom spaces. The Thom space of an N*-vector bundle* $V \to X_0$ over a compact space X_0 (where the pullbacks of all points $x \in X_0$ are Euclidean spaces $\mathbb{R}^N_x = \mathbb{R}^N$) is the one point compactifications $V_\bullet$ of V, where X_0 is canonically embedded into $V \subset V_\bullet$ as the zero section of the bundle (i.e. $x \mapsto 0 \in \mathbb{R}^N_x$).

If $X = X^n \subset W = W^{n+N}$ is a smooth submanifold, then the total space of its *normal bundle* denoted $U^\perp \to X$ is (almost canonically) diffeomorphic to a small (normal) ε-neighbourhood $U(\varepsilon) \subset W$ of X, where every ε-ball $B^N(\varepsilon) = B^N_x(\varepsilon)$ normal to X at $x \in X$ is radially mapped to the fiber $\mathbb{R}^N = \mathbb{R}^N_x$ of $U^\perp \to X$ at x.

Thus the Thom space $U^\perp_\bullet$ is identified with $U(\varepsilon)_\bullet$ and the tautological map $W_\bullet \to U(\varepsilon)_\bullet$, that equals the identity on $U(\varepsilon) \subset W$ and sends the complement $W \setminus U(\varepsilon)$ to $\bullet \in U(\varepsilon)_\bullet$, defines the *Atiyah-Thom map* for all closed smooth submanifold $X \subset W$,

$$A^\perp_\bullet : W_\bullet \to U^\perp_\bullet.$$

Recall that every $\mathbb{R}^N$-bundle over an n-dimensional space with $n < N$, can be induced from the tautological bundle V over the Grassmann manifold $X_0 =$

$Gr_N(\mathbb{R}^{n+N})$ of N-planes (i.e. linear N-subspaces in $\mathbb{R}^{n+N}$) by a continuous map, say $G : X \to X_0 = Gr_N(\mathbb{R}^{n+N})$.

For example, if $X \subset \mathbb{R}^{n+N}$, one can take *the normal Gauss map* for G that sends $x \in X$ to the N-plane $G(x) \in Gr_N(\mathbb{R}^{n+N}) = X_0$ which is parallel to the normal space of X at x.

Since the Thom space construction is, obviously, functorial, every $U^\perp$-bundle inducing map $X \to X_0 = Gr_N(\mathbb{R}^{n+N})$ for $X = X^n \subset W = W^{n+N}$, defines a map $U^\perp_\bullet \to V_\bullet$ and this, composed with $A^\perp_\bullet$, gives us the *Thom map*

$$T_\bullet : W_\bullet \to V_\bullet \text{ for the tautological } N\text{-bundle } V \to X_0 = Gr_N(\mathbb{R}^{n+N}).$$

Since all n-manifolds can be (obviously) embedded (by generic smooth maps) into Euclidean spaces $\mathbb{R}^{n+N}$, $N >> n$, every *closed*, i.e. compact without boundary, n-manifold X comes from the generic pullback construction applied to maps f from $S^{n+N} = \mathbb{R}^{n+N}_\bullet$ to the Thom space $V_\bullet$ of the canonical N-vector bundle $V \to X_0 = Gr_N(\mathbb{R}^{n+N})$,

$$X = f^{-1}(X_0) \text{ for generic } f : S^{n+N} \to V_\bullet \supset X_0 = Gr_N(\mathbb{R}^{n+N}).$$

In a way, Thom has discovered the source of all manifolds in the world and responded to the question "Where are manifolds coming from?" with the following

1954 Answer. *All closed smooth n-manifolds X come as pullbacks of the Grassmannians $X_0 = Gr_N(\mathbb{R}^{n+N})$ in the ambient Thom spaces $V_\bullet \supset X_0$ under generic smooth maps $S^{n+N} \to V_\bullet$.*

The manifolds X obtained with the generic pull-back construction come with a grain of salt: generic maps are abundant but it is hard to put your finger on any one of them – we can not say much about topology and geometry of an individual X. (It seems, one can not put *all* manifolds in one basket without some "random string" attached to it.)

But, empowered with Serre's theorem, this construction unravels an amazing structure in the "space of all manifolds" (Before Serre, Pontryagin and following him Rokhlin proceeded in the reverse direction by applying smooth manifolds to the homotopy theory via the Pontryagin construction.)

Selecting an object X, e.g. a submanifold, from a given collection $\mathcal{X}$ of similar objects, where there is *no distinguished* member $X^\star$ among them, is a notoriously difficult problem which had been known since antiquity and can be traced to De Cael of Aristotle. It reappeared in 14th century as *Buridan's ass problem* and as *Zermelo's choice problem* at the beginning of 20th century.

A geometer/analyst tries to select an X by first finding/constructing a "value function" on $\mathcal{X}$ and then by taking the "optimal" X. For example, one may go for n-submanifolds X of *minimal volumes* in an $(n+N)$-manifold W endowed with a *Riemannian metric.* However, minimal manifolds X are usually singular except for hypersurfaces $X^n \subset W^{n+1}$ where $n \leq 6$ (Simons, 1968).

Picking up a "generic" or a "random" X from $\mathcal{X}$ is a geometer's last resort when all "deterministic" options have failed. This is aggravated in topology, since

• there is no known construction delivering *all* manifolds X in a reasonably controlled manner besides generic pullbacks and their close relatives;

• on the other hand, geometrically interesting manifolds X are not anybody's pullbacks. Often, they are "complicated quotients of simple manifolds", e.g. $X = S/\Gamma$, where S is a symmetric space, e.g. the hyperbolic n-space, and Γ is a discrete isometry group acting on S, possibly, with fixed points.

(It is obvious that every surface X is homeomorphic to such a quotient, and this is also so for compact 3-manifolds by a theorem of Thurston. But if $n \geq 4$, one does not know if every closed smooth manifold X is homeomorphic to such an S/Γ. It is hard to imagine that there are infinitely many non-diffeomorphic but mutually homeomorphic S/Γ for the hyperbolic 4-space S, but this may be a problem with our imagination.)

Starting from another end, one has *ramified covers* $X \to X_0$ of "simple" manifolds X_0, where one wants the ramification locus $\Sigma_0 \subset X_0$ to be a subvariety with "mild singularities" and with an "interesting" fundamental group of the complement $X_0 \setminus \Sigma_0$, but finding such Σ_0 is difficult (see the discussion following (3) in section 7).

And even for simple $\Sigma_0 \subset X_0$, the description of ramified coverings $X \to X_0$ where X are *manifolds* may be hard. For example, this is non-trivial for ramified coverings over the flat n-torus $X_0 = \mathbb{T}^n$ where Σ_0 is the union of several flat $(n-2)$-subtori in general position where these subtori may intersect one another.

4 Duality and the Signature.

Cycles and Homology. If X is a smooth n-manifold X one is inclined to define "geometric i-cycles" C in X, which represent homology classes $[C] \in H_i(X)$, as "compact oriented i-submanifolds $C \subset X$ with singularities of *codimension two*".

This, however, is too restrictive, as it rules out, for example, closed self-intersecting curves in surfaces, and/or the double covering map $S^1 \to S^1$.

Thus, we allow $C \subset X$ which may have singularities of codimension one, and, besides orientation, a locally constant integer valued function on the non-singular locus of C.

First, we define dimension on all closed subsets in smooth manifolds with the usual properties of monotonicity, locality and max-additivity, i.e. $dim(A \cup B) = \max(dim(A), dim(B))$.

Besides we want our dimension to be monotone under *generic* smooth maps of compact subsets A, i.e. $dim(f(A)) \leq dim(A)$ and if $f : X^{m+n} \to Y^n$ is a *generic* map, then $f^{-1}(A) \leq dim(A) + m$.

Then we define the "generic dimension" as the *minimal* function with these properties which coincides with the ordinary dimension on smooth compact submanifolds. This depends, of course, on specifying "generic" at each step, but this never causes any problem in-so-far as we do not start taking limits of maps.

An i-cycle $C \subset X$ is a closed subset in X of dimension i with a $\mathbb{Z}$*-multiplicity function* on C defined below, and with the following set decomposition of C.

$$C = C_{reg} \cup C_{\times} \cup C_{sing},$$

such that

• C_{sing} is a closed subset of dimension$\leq i-2$.

• C_{reg} is an open and dense subset in C and it is a smooth i-submanifold in X.

$C_\times \cup C_{sing}$ is a closed subset of dimension $\leq i-1$. Locally, at every point, $x \in C_\times$ the union $C_{reg} \cup C_\times$ is diffeomorphic to a collection of smooth copies of $\mathbb{R}^i_+$ in X, called *branches*, meeting along their $\mathbb{R}^{i-1}$-boundaries where the basic example is the union of hypersurfaces in general position.

• The $\mathbb{Z}$*-multiplicity structure*, is given by an orientation of C_{reg} and a locally constant *multiplicity/weight* $\mathbb{Z}$-function on C_{reg}, (where for $i=0$ there is only this function and no orientation) such that the sum of these oriented multiplicities over the branches of C at each point $x \in C_\times$ equals zero.

Every C can be modified to C' with empty $C'_\times$ and if $codim(C) \geq 1$, i.e. $dim(X) > dim(C)$, also with weights $= \pm 1$.

For example, if $2l$ oriented branches of C_{reg} with multiplicities 1 meet at $C_\times$, divide them into l pairs with the partners having *opposite* orientations, keep these partners attached as they meet along $C_\times$ and separate them from the other pairs.

No matter how simple, this separation of branches is, say with the total weight $2l$, it can be performed in $l!$ different ways. Poor C' burdened with this ambiguity becomes rather non-efficient.

If X is a closed oriented n-manifold, then it itself makes an n-cycle which represents what is called the *fundamental class* $[X] \in H_n(X)$. Other n-cycles are integer combinations of the oriented connected components of X.

It is convenient to have singular counterparts to manifolds with boundaries. Since "chains" were appropriated by algebraic topologists, we use the word "plaque", where an $(i+1)$*-plaque* D with a boundary $\partial(D) \subset D$ is the same as a cycle, except that there is a subset $\partial(D)_\times \subset D_\times$, where the sums of oriented weights do not cancel, where the closure of $\partial(D)_\times$ equals $\partial(D) \subset D$ and where $dim(\partial(D) \setminus \partial(D)_\times) \leq i-2$.

Geometrically, we impose the local conditions on $D \setminus \partial(D)$ as on $(i+1)$-cycles and add the local i-cycle conditions on (the closed set) $\partial(D)$, where this $\partial(D)$ comes with the canonical weighted orientation induced from D.

(There are *two* opposite canonical induced orientations on the boundary $C = \partial D$, e.g. on the circular boundary of the 2-disc, with no apparent rational for preferring one of the two. We choose the orientation in $\partial(D)$ defined by the frames of the tangent vectors $\tau_1, ..., \tau_i$ such that the orientation given to D by the $(i+1)$-frames $\nu, \tau_1, ..., \tau_i$ agrees with the original orientation, where ν is the *inward* looking normal vector.)

Every plaque can be "subdivided" by enlarging the set $D_\times$ (and/or, less essentially, D_{sing}). We do not care distinguishing such plaques and, more generally, the equality $D_1 = D_2$ means that the two plaques have a common subdivision.

We go further and write $D = 0$ if the weight function on D_{reg} equals zero.

We denote by $-D$ the plaque with the either minus weight function or with the opposite orientation.

We define $D_1 + D_2$ if there is a plaque D containing both D_1 and D_2 as its sub-plaques with the obvious addition rule of the weight functions.

Accordingly, we agree that $D_1 = D_2$ if $D_1 - D_2 = 0$.

On Genericity. We have not used any genericity so far except for the definition of dimension. But from now on we assume all our object to be generic. This is needed, for example, to define $D_1 + D_2$, since the sum of arbitrary plaques *is not* a plaque, but the sum of generic plaques, obviously, *is.*

Also if you are used to genericity, it is obvious to you that

If $D \subset X$ is an i-plaque (i-cycle) then the image $f(D) \subset Y$ under a generic map $f : X \to Y$ is an i-plaque (i-cycle).

Notice that for $dim(Y) = i + 1$ the self-intersection locus of the image $f(D)$ becomes a part of $f(D)_\times$ and if $dim(Y) = i + 1$, then the new part the $\times$-singularity comes from $f(\partial(D))$.

It is even more obvious that

the pullback $f^{-1}(D)$ of an i-plaque $D \subset Y^n$ under a generic map $f : X^{m+n} \to Y^n$ is an $(i+m)$-plaque in X^{m+n}; if D is a cycle and X^{m+n} is a closed manifold (or the map f is proper), then $f^{-1}(D)$ is cycle.

As the last technicality, we extend the above definitions to arbitrary triangulated spaces X, with "smooth generic" substituted by "piecewise smooth generic" or by piecewise linear maps.

Homology. Two i-cycles C_1 and C_2 in X are called *homologous*, written $C_1 \sim C_2$, if there is an $(i+1)$-plaque D in $X \times [0,1]$, such that $\partial(D) = C_1 \times 0 - C_2 \times 1$.

For example *every contractible cycle $C \subset X$ is homologous to zero*, since the cone over C in $Y = X \times [0,1]$ corresponding to a smooth generic homotopy makes a plaque with its boundary equal to C.

Since small subsets in X are contractible, a cycle $C \subset X$ is homologous to zero if and only if it admits a decomposition into a sum of "arbitrarily small cycles", i.e. if, for every locally finite covering $X = \bigcup_i U_i$, there exist cycles $C_i \subset U_i$, such that $C = \sum_i C_i$.

The *homology group* $H_i(X)$ is defined as the Abelian group with generators $[C]$ for all i-cycles C in X and with the relations $[C_1] - [C_2] = 0$ whenever $C_1 \sim C_2$.

Similarly one defines $H_i(X;\mathbb{Q})$, for the field $\mathbb{Q}$ of rational numbers, by allowing C and D with fractional weights.

Examples. Every closed orientable n-manifold X with k connected components has $H_n(X) = \mathbb{Z}^k$, where $H_n(X)$ is generated by the fundamental classes of its components.

This is obvious with our definitions since the only plaques D in $X \times [0,1]$ with $\partial(D) \subset \partial(X \times [0,1]) = X \times 0 \cup X \times 1$ are combination of the connected components of $X \times [0,1]$ and so $H_n(X)$ equals the group of n-cycles in X. Consequently,

every closed orientable manifold X is non-contractible.

The above argument may look suspiciously easy, since it is even hard to prove *non-contractibility of S^n* and issuing from this the *Brouwer fixed point theorem* within the world of *continuous* maps without using generic smooth or combinatorial ones, except for $n = 1$ with the covering map $\mathbb{R} \to S^1$ and for S^2 with the Hopf fibration $S^3 \to S^2$.

The catch is that the difficulty is hidden in the fact that a *generic* image of an $(n+1)$-plaque (e.g. a cone over X) in $X \times [0,1]$ is again an $(n+1)$-plaque.

What is obvious, however without any appeal to genericity is that $H_0(X) = \mathbb{Z}^k$ for every manifold or a triangulated space with k components.

The spheres S^n have $H_i(S^n) = 0$ for $0 < i < n$, since the complement to a point $s_0 \in S^n$ is homeomorphic to $\mathbb{R}^n$ and a generic cycles of dimension $< n$ misses s_0, while $\mathbb{R}^n$, being contractible, has zero homologies in positive dimensions.

It is clear that continuous maps $f : X \to Y$, when generically perturbed, define homomorphisms $f_{*i} : H_i(X) \to H_i(Y)$ for $C \mapsto f(C)$ and that

homotopic maps $f_1, f_2 : X \to Y$ induce equal homomorphisms $H_i(X) \to H_i(Y)$.

Indeed, the cylinders $C \times [0,1]$ generically mapped to $Y \times [0,1]$ by homotopies f_t, $t \in [0,1]$, are plaque D in our sense with $\partial(D) = f_1(C) - f_2(C)$.

It follows, that the

homology is invariant under homotopy equivalences $X \leftrightarrow Y$ for manifolds X, Y as well as for triangulated spaces.

Similarly, if $f : X^{m+n} \to Y^n$ is a proper (pullbacks of compact sets are compact) smooth generic map between *manifolds* where Y has no boundary, then the pullbacks of cycles define homomorphism, denoted, $f^! : H_i(Y) \to H_{i+m}(X)$, which is invariant under proper homotopies of maps.

The homology groups are much easier do deal with than the homotopy groups, since the definition of an i-cycle in X is purely local, while "spheres in X" can not be recognized by looking at them point by point. (Holistic philosophers must feel triumphant upon learning this.)

Homologically speaking, a space *is* the sum of its parts: the locality allows an effective computation of homology of spaces X assembled of simpler pieces, such as cells, for example.

The locality+additivity is satisfied by the *generalized homology functors* that are defined, following Sullivan, by limiting possible singularities of cycles and plaques [6]. Some of these, e.g. *bordisms* we meet in the next section.

Degree of a Map. Let $f : X \to Y$ be a smooth (or piece-wise smooth) generic map between closed connected oriented equidimensional manifolds

Then the degree $deg(f)$ can be (obviously) equivalently defined either as the image $f_*[X] \in \mathbb{Z} = H_n(Y)$ or as the $f^!$-image of the generator $[\bullet] \in H_0(Y) \in \mathbb{Z} = H_0(X)$. For, example, l-sheeted covering maps $X \to Y$ have degrees l. Similarly, one sees that

finite covering maps between arbitrary spaces are surjective on the rational homology groups.

To understand the local geometry behind the definition of degree, look closer at our f where X (still assumed compact) is allowed a non-empty boundary and observe that the f-pullback $\tilde{U}_y \subset X$ of some (small) open neighbourhood $U_y \subset Y$ of a generic point $y \in Y$ consists of finitely many connected components $\tilde{U}_i \subset \tilde{U}$, such that the map $f : \tilde{U}_i \to U_y$ is a diffeomorphism for all $\tilde{U}_i$.

Thus, every $\tilde{U}_i$ carries two orientations: one induced from X and the second from Y via f. The sum of $+1$ assigned to $\tilde{U}_i$ where the two orientation agree and of -1 when they disagree is called the *local degree $deg_y(f)$*.

If two generic points $y_1, y_2 \in Y$ can be joined by a path in Y which does not cross the f-image $f(\partial(X)) \subset Y$ of the boundary of X, then $deg_{y_1}(f) = deg_{y_2}(f)$ since the f-pullback of this path, (which can be assumed generic) consists, besides possible closed curves, of several segments in Y, joining ± 1-degree points in $f^{-1}(y_1) \subset \tilde{U}_{y_1} \subset X$ with ∓ 1-points in $f^{-1}(y_2) \subset \tilde{U}_{y_2}$.

Consequently, the local degree does not depend on y if X has no boundary. Then, clearly, it coincides with the homologically defined degree.

Similarly, one sees in this picture (without any reference to homology) that the local degree is invariant under generic homotopies $F : X \times [0,1] \to Y$, where the smooth (typically disconnected) pull-back curve $F^{-1}(y) \subset X \times [0,1]$ joins ± 1-points in $F(x,0)^{-1}(y) \subset X = X \times 0$ with ∓ 1-points in $F(x,1)^{-1}(y) \subset X = X \times 1$.

Geometric Versus Algebraic Cycles. Let us explain how the geometric definition matches the algebraic one for triangulated spaces X.

Recall that the homology of a triangulated space is algebraically defined with $\mathbb{Z}$-*cycles* which are $\mathbb{Z}$-chains, i.e. formal linear combinations $C_{alg} = \sum_s k_s \Delta_s^i$ of oriented i-simplices Δ_s^i with integer coefficients k_s, where, by the definition of "algebraic cycle" , these sums have zero algebraic boundaries, which is equivalent to $c(C_{alg}) = 0$ for every $\mathbb{Z}$-cocycle c cohomologous to zero (see chapter 2).

But this is exactly the same as our generic cycles C_{geo} in the i-skeleton X_i of X and, tautologically, $C_{alg} \overset{taut}{\mapsto} C_{geo}$ gives us a homomorphism from the algebraic homology to our geometric one.

On the other hand, an $(i+j)$-simplex minus its center can be radially homotoped to its boundary. Then the obvious reverse induction on skeleta of the triangulation shows that the space X minus a subset $\Sigma \subset X$ of codimension $i+1$ can be homotoped to the i-skeleton $X_i \subset X$.

Since every generic i-cycle C misses Σ it can be homotoped to X_i where the resulting map, say $f : C \to X_i$, sends C to an algebraic cycle.

At this point, the equivalence of the two definitions becomes apparent, where, observe, the argument applies to all cellular spaces X with piece-wise linear attaching maps.

The usual definition of homology of such an X amounts to working with all i-cycles contained in X_i and with $(i+1)$-plaques in X_{i+1}. In this case the group of i-cycles becomes a subspace of the group spanned by the i-cells, which shows, for example, that the rank of $H_i(X)$ does not exceed the number of i-cells in X_i.

We return to generic geometric cycles and observe that if X is a *non-compact* manifold, one may drop "compact" in the definition of these cycles. The resulting group is denoted $H_1(X, \partial_\infty)$. If X is compact with boundary, then this group of the interior of X is called the *relative homology group* $H_i(X, \partial(X))$. (The ordinary homology groups of this interior are canonically isomorphic to those of X.)

Intersection Ring. The intersection of cycles in general position in a smooth manifold X defines a multiplicative structure on the homology of an n-manifold X, denoted

$$[C_1] \cdot [C_2] = [C_1] \cap [C_2] = [C_1 \cap C_2] \in H_{n-(i+j)}(X)$$

$$\text{for } [C_1] \in H_{n-i}(X) \text{ and } [C_2] \in H_{n-j}(X),$$

where $[C] \cap [C]$ is defined by intersecting $C \subset X$ with its small generic perturbation $C' \subset X$.

(Here genericity is most useful: intersection is painful for simplicial cycles confined to their respective skeleta of a triangulation. On the other hand, if X

is a *not* a manifold one may adjust the definition of cycles to the local topology of the singular part of X and arrive at what is called the *intersection homology.*)

It is obvious that the intersection is respected by $f^!$ for proper maps f, but not for f_*. The former implies. in particular, that this product is invariant under oriented (i.e. of degrees +1) homotopy equivalences between *closed equidimensional* manifolds. (But $X \times \mathbb{R}$, which is homotopy equivalent to X has trivial intersection ring, whichever is the ring of X.)

Also notice that the intersection of cycles of *odd* codimensions is *anti-commutative* and if one of the two has *even* codimension it is *commutative.*

The intersection of two cycles of complementary dimensions is a 0-cycle, the total $\mathbb{Z}$-weight of which makes sense if X is oriented; it is called *the intersection index of the cycles.*

Also observe that the intersection between C_1 and C_2 equals the intersection of $C_1 \times C_2$ with the diagonal $X_{diag} \subset X \times X$.

Examples. (a) The intersection ring of the complex projective space $\mathbb{C}P^k$ is multiplicatively generated by the homology class of the hyperplane, $[\mathbb{C}P^{k-1}] \in H_{2k-2}(\mathbb{C}P^k)$, with the only relation $[\mathbb{C}P^{k-1}]^{k+1} = 0$ and where, obviously, $[\mathbb{C}P^{k-i}] \cdot [\mathbb{C}P^{k-j}] = [\mathbb{C}P^{k-(i+j)}]$.

The only point which needs checking here is that the homology class $[\mathbb{C}P^i]$ *(additively) generates* $H_i(\mathbb{C}P^k)$, which is seen by observing that $\mathbb{C}P^{i+1} \setminus \mathbb{C}P^i$, $i = 0, 1, ..., k-1$, is an open $(2i+2)$-cell, i.e. the open topological ball B^{2i+2}_{op} (where the cell attaching map $\partial(B^{2i+2}) = S^{2i+1} \to \mathbb{C}P^i$ is the quotient map $S^{2i+1} \to S^{2i+1}/\mathbb{T} = \mathbb{C}P^{i+1}$ for the obvious action of the multiplicative group $\mathbb{T}$ of the complex numbers with norm 1 on $S^{2i+1} \subset \mathbb{C}^{2i+1}$).

(b) The intersection ring of the n-torus is isomorphic to the exterior algebra on n-generators, i.e. the only relations between the multiplicative generators $h_i \in H_{n-1}(\mathbb{T}^n)$ are $h_i h_j = -h_j h_i$, where h_i are the homology classes of the n coordinate subtori $\mathbb{T}^{n-1}_i \subset \mathbb{T}^n$.

This follows from the Künneth formula below, but can be also proved directly with the obvious cell decomposition of $\mathbb{T}^n$ into 2^n cells.

The intersection ring structure immensely enriches homology. Additively, $H_* = \oplus_i H_i$ is just a graded Abelian group – the most primitive algebraic object (if finitely generated) – fully characterized by simple numerical invariants: the rank and the orders of their cyclic factors.

But the ring structure, say on H_{n-2} of an n-manifold X, for $n = 2d$ defines a symmetric d-form, on $H_{n-2} = H_{n-2}(X)$ which is, a polynomial of degree d in r variables with integer coefficients for $r = rank(H_{n-2})$. All number theory in the world can not classify these for $d \geq 3$ (to be certain, for $d \geq 4$).

One can also intersect non-compact cycles, where an intersection of a compact C_1 with a non-compact C_2 is compact; this defines the *intersection pairing*

$$H_{n-i}(X) \otimes H_{n-j}(X, \partial_\infty) \xrightarrow{\cap} H_{n-(i+j)}(X).$$

Finally notice that generic 0 cycles C in X are finite sets of points $x \in X$ with the "orientation" signs ± 1 attached to each x in C, where the sum of these ± 1 is called the *index of* C. If X is *connected*, then $ind(C) = 0$ if and only if $[C] = 0$.

Thom Isomorphism. Let $p : V \to X$ be a *fiber-wise oriented* smooth (which is unnecessary) $\mathbb{R}^N$-bundle over X, where $X \subset V$ is embedded as the zero section and let $V_\bullet$ be Thom space of V. Then there are two natural homology homomorphisms.

Intersection $\cap : H_{i+N}(V_\bullet) \to H_i(X)$. This is defined by intersecting generic $(i+N)$-cycles in $V_\bullet$ with X.

Thom Suspension $S_\bullet : H_i(X) \to H_i(V_\bullet)$, where every cycle $C \subset X$ goes to the Thom space of the restriction of V to C, i.e. $C \mapsto (p^{-1}(C))_\bullet \subset V_\bullet$.

These $\cap$ and $S_\bullet$ are mutually reciprocal. Indeed $(\cap \circ S_\bullet)(C) = C$ for all $C \subset X$ and also $(S_\bullet \circ \cap)(C') \sim C'$ for all cycles C' in $V_\bullet$ where the homology is established by the fiberwise radial homotopy of C' in $V_\bullet \supset V$, which fixes $\bullet$ and move each $v \in V$ by $v \mapsto tv$. Clearly, $tC' \to (S_\bullet \circ \cap)(C')$ as $t \to \infty$ for all generic cycles C' in $V_\bullet$.

Thus we arrive at the Thom isomorphism

$$H_i(X) \leftrightarrow H_{i+N}(V_\bullet).$$

Similarly we see that

The Thom space of every $\mathbb{R}^N$-bundle $V \to X$ is $(N-1)$-connected, i.e. $\pi_j(V_\bullet) = 0$ for $j = 1, 2, ...N-1$.

Indeed, a generic j-sphere $S^j \to V_\bullet$ with $j < N$ does not intersect $X \subset V$, where X is embedded into V by the zero section. Therefore, this sphere radially (in the fibers of V) contracts to $\bullet \in V_\bullet$.

Euler Class. Let $f : X \to B$ be a fibration with $\mathbb{R}^{2k}$-fibers over a smooth closed oriented manifold B. Then the intersection indices of $2k$-cycles in B with $B \subset X$, embedded as the zero section, defines an *integer cohomology class*, i.e. a *homomorphism* (additive map) $e : H_{2k}(B) \to \mathbb{Z} \subset \mathbb{Q}$, called the *Euler class* of the fibration. (In fact, one does not need B to be a manifold for this definition.)

Observe that the Euler number *vanishes* if and only if the homology projection homomorphism ${}_0f_{*2k} : H_{2k}(V \setminus B; \mathbb{Q}) \to H_{2k}(B; \mathbb{Q})$ is *surjective*, where $B \subset X$ is embedded by the zero section $b \mapsto 0_b \in \mathbb{R}^k_b$ and ${}_0f : V \setminus B \to B$ is the restriction of the map (projection) f to $V \setminus B$.

Moreover, it is easy to see that the *ideal* in $H^*(B)$ generated by the Euler class (for the $\smile$*-ring structure on cohomology* defined later in this section) equals the kernel of the cohomology homomorphism ${}_0f^* : H^*(B) \to H^*(V \setminus B)$.

If B is a closed connected oriented manifold, then $e[B]$ is called the *Euler number* of $X \to B$ also denoted e.

In other words, the number e equals the self-intersection index of $B \subset X$. Since the intersection pairing is symmetric on H_{2k} the sign of the Euler number does not depend on the orientation of B, but it does depend on the orientation of X.

Also notice that if X is embedded into a larger $4k$-manifold $X' \supset X$ then the self-intersection index of B in X' equals that in X.

If X equals the tangent bundle $T(B)$ then X is canonically oriented (even if B is non-orientable) and the Euler number is non-ambiguously defined and it equals the self-intersection number of the diagonal $X_{diag} \subset X \times X$.

Poincaré-Hopf Formula. *The Euler number e of the tangent bundle $T(B)$ of every closed oriented $2k$-manifold B satisfies*

$$e = \chi(B) = \sum_{i=0,1,...2k} rank(H_i(B; \mathbb{Q})).$$

(If $n = dim(B)$ is *odd*, then $\sum_{i=0,1,...n} rank(H_i(B;\mathbb{Q})) = 0$ by the Poincaré duality.)

It is hard the believe this may be true! A single cycle (let it be the *fundamental* one) knows something about all of the homology of B.

The most transparent proof of this formula is, probably, via the Morse theory (known to Poincaré) and it hardly can be called "trivial".

A more algebraic proof follows from the Künneth formula (see below) and an expression of the class $[X_{diag}] \in H_{2k}(X \times X)$ in terms of the intersection ring structure in $H_*(X)$.

The Euler number can be also defined for connected *non-orientable* B as follows. Take the canonical oriented double covering $\tilde{B} \to B$, where each point $\tilde{b} \in \tilde{B}$ over $b \in B$ is represented as b + an orientation of B near b. Let the bundle $\tilde{X} \to \tilde{B}$ be induced from X by the covering map $\tilde{B} \to B$, i.e. this $\tilde{X}$ is the obvious double covering of X corresponding to $\tilde{B} \to B$. Finally, set $e(X) = e(\tilde{X})/2$.

The Poincaré-Hopf formula for non-orientable $2k$-manifolds B follows from the orientable case by the *multiplicativity* of the Euler characteristic χ which is valid for all compact triangulated spaces B,

an l-sheeted covering $\tilde{B} \to B$ has $\chi(\tilde{B}) = l \cdot \chi(B)$.

If the homology is defined via a triangulation of B, then $\chi(B)$ equals the alternating sum $\sum_i (-1)^i N(\Delta^i)$ of the numbers of i-simplices by straightforward linear algebra and the multiplicativity follows. But this is not so easy with our geometric cycles. (If B is a closed manifold, this also follows from the Poincaré-Hopf formula and the obvious multiplicativity of the Euler number for covering maps.)

Künneth Theorem. *The rational homology of the Cartesian product of two spaces equals the graded tensor product of the homologies of the factors. In fact, the natural homomorphism*

$$\bigoplus_{i+j=k} H_i(X_1;\mathbb{Q}) \otimes H_j(X_2;\mathbb{Q}) \to H_k(X_1 \times X_2;\mathbb{Q}),\ k = 0,1,2,...$$

is an isomorphism. Moreover, if X_1 and X_2 are closed oriented manifolds, this homomorphism is compatible (if you say it right) with the intersection product.

This is obvious if X_1 and X_2 have cell decompositions such that the numbers of i-cells in each of them equals the ranks of their respective H_i. In the general case, the proof is cumbersome unless you pass to the language of chain complexes where the difficulty dissolves in linear algebra. (Yet, keeping track of geometric cycles may be sometimes necessary, e.g. in the algebraic geometry, in the geometry of foliated cycles and in evaluating the so called *filling profiles* of products of Riemannian manifolds.)

Poincaré $\mathbb{Q}$-Duality. Let X be a connected oriented n-manifold.

The intersection index establishes a linear duality between homologies of complementary dimensions:

$$H_i(X;\mathbb{Q}) \textit{ equals the } \mathbb{Q}\textit{-linear dual of } H_{n-i}(X,\partial_\infty;\mathbb{Q}).$$

In other words, the intersection pairing

$$H_i(X) \otimes H_{n-i}(X,\partial_\infty) \xrightarrow{\cap} H_0(X) = \mathbb{Z}$$

is $\mathbb{Q}$-faithful: a multiple of a compact i-cycle C is homologous to zero if and only if its intersection index with every non-compact $(n-i)$-cycle in general position equals zero.

Furthermore, if X equals the interior of a compact manifolds with a boundary, then a multiple of a non-compact cycle is homologous to zero if and only if its intersection index with every compact generic cycle of the complementary dimension equals zero.

Proof of $[H_i \leftrightarrow H^{n-i}]$ for Closed Manifolds X. We, regretfully, break the symmetry by choosing some *smooth* triangulation T of X which means this T is locally as good as a triangulation of $\mathbb{R}^n$ by affine simplices (see below).

Granted T, assign to each generic i-cycle $C \subset X$ the intersection index of C with every oriented Δ^{n-i} of T and observe that the resulting function $c^\perp : \Delta^{n-i} \mapsto ind(\Delta^{n-i} \cap C)$ is a $\mathbb{Z}$-valued cocycle (see section 2), since the intersection index of C with every $(n-i)$-sphere $\partial(\Delta^{n-i+1})$ equals zero, because these spheres are homologous to zero in X.

Conversely, given a $\mathbb{Z}$-cocycle $c(\Delta^{n-i})$ construct an i-cycle $C_\perp \subset X$ as follows. Start with $(n-i+1)$-simplices Δ^{n-i+1} and take in each of them a smooth oriented curve S with the boundary points located at the centers of the $(n-i)$-faces of Δ^{n-i+1}, where S is normal to a face Δ^{n-i} whenever it meets one and such that the intersection index of the (slightly extended across Δ^{n-i}) curve S with Δ^{n-i} equals $c(\Delta^{n-i})$. Such a curve, (obviously) exists because the function c is a *cocycle*. Observe, that the union of these S over all $(n-i+1)$-simplices in the boundary sphere $S^{n-i+1} = \partial\Delta^{n-i+2}$ of every $(n-i+2)$-simplex in T is a *closed* (disconnected) curve in S^{n-i+1}, the intersection index of which with every $(n-i)$-simplex $\Delta^{n-i} \subset S^{n-i+1}$ equals $c(\Delta^{n-i})$ (where this intersection index is evaluated in S^{n-i+1} but *not* in X).

Then construct by induction on j the (future) intersection $C^j_\perp$ of $C_\perp$ with the $(n-i+j)$-skeleton T_{n-i+j} of our triangulation by taking the cone from the center of each simplex $\Delta^{n-i+j} \subset T_{n-i+j}$ over the intersection of $C^j_\perp$ with the boundary sphere $\partial(\Delta^{n-i+j})$.

It is easy to see that the resulting $C_\perp$ is an i-cycle and that the composed maps $C \to c^\perp \to C_\perp$ and $c \to C_\perp \to c^\perp$ define identity homomorphisms $H_i(X) \to H_i(X)$ and $H^{n-i}(X;\mathbb{Z}) \to H^{n-i}(X;\mathbb{Z})$ correspondingly and we arrive at the Poincaré $\mathbb{Z}$-isomorphism,

$$H_i(X) \leftrightarrow H^{n-i}(X;\mathbb{Z}).$$

To complete the proof of the $\mathbb{Q}$-duality one needs to show that $H^j(X;\mathbb{Z}) \otimes \mathbb{Q}$ equals the $\mathbb{Q}$-linear dual of $H_j(X;\mathbb{Q})$.

To do this we represent $H_i(X)$ by algebraic $\mathbb{Z}$-cycles $\sum_j k_j \Delta^i$ and now, in the realm of algebra, appeal to the linear duality between homologies of the chain and cochain complexes of T:

the natural pairing between classes $h \in H_i(X)$ and $c \in H^i(X;\mathbb{Z})$, which we denote $(h,c) \mapsto c(h) \in \mathbb{Z}$, establishes, when tensored with $\mathbb{Q}$, an isomorphism between $H^i(X;\mathbb{Q})$ and the $\mathbb{Q}$-linear dual of $H_i(X;\mathbb{Q})$

$$H^i(X;\mathbb{Q}) \leftrightarrow Hom[H_i(X;\mathbb{Q})] \to Q]$$

for all compact triangulated spaces X. QED.

Corollaries (a) The non-obvious part of the Poincaré duality is the claim that, for ever $\mathbb{Q}$-*homologically non-trivial* cycle C, there is a cycle C' of the complementary dimension, such that the intersection index between C and C' *does not vanish.*

But the easy part of the duality is also useful, as it allows one to give a *lower* bound on the homology by producing sufficiently many *non-trivially* intersecting cycles of complementary dimensions.

For example it shows that closed manifolds are non-contractible (where it reduces to the degree argument). Also it implies that the Künneth pairing $H_*(X;\mathbb{Q})\otimes H_*(Y;\mathbb{Q}) \to H_*(X\times Y;\mathbb{Q})$ is injective for closed orientable manifolds X.

(b) Let $f : X^{m+n} \to Y^n$ be a smooth map between closed orientable manifolds such the homology class of the pullback of a generic point is not homologous to zero, i.e. $0 \neq [f^{-1}(y_0)] \in H_m(X)$. Then the homomorphisms $f^{!i} : H_i(Y;\mathbb{Q}) \to H_{i+m}(X;\mathbb{Q})$ are *injective* for all i.

Indeed, every $h \in H_i(Y;\mathbb{Q})$ different from zero comes with an $h' \in H_{n-i}(Y)$ such that the intersection index d between the two is $\neq 0$. Since the intersection of $f^!(h)$ and $f^!(h')$ equals $d[f^{-1}(y_0)]$ none of $f^!(h)$ and $f^!(h')$ equals zero.

Consequently/similarly *all* $f_{*j} : H_j(X) \to H_j(Y)$ *are surjective.*

For example,

(b1) *Equidimensional maps f of positive degrees between closed oriented manifolds are surjective on rational homology.*

(b2) Let $f : X \to Y$ be a smooth fibration where the fiber is a closed oriented manifold with *non-zero Euler characteristic*, e.g. homeomorphic to S^{2k}. Then the fiber is *non-homologous to zero*, since the Euler class e of the fiberwise tangent bundle, which defined on *all* of X, does not vanish on $f^{-1}(y_0)$; hence, f_* *is surjective.*

Recall that the unit tangent bundle fibration $X = UT(S^{2k}) \to S^{2k} = Y$ with S^{2k-1}-fibers has $H_i(X;\mathbb{Q}) = 0$ for $1 \leq i \leq 4k-1$, since the Euler class of $T(S^{2k})$ *does not* vanish; hence; f_* *vanishes on all* $H_i(X;\mathbb{Q})$, $i > 0$.

Geometric Cocycles. We gave only a combinatorial definition of cohomology, but this can be defined more invariantly with geometric i-cocycles c being "generically locally constant" functions on oriented plaques D such that $c(D) = -c(-D)$ for reversing the orientation in D, where $c(D_1 + D_2) = c(D_1) + c(D_2)$ and where the final cocycle condition reads $c(C) = 0$ for all *i-cycles* C which are *homologous to zero.* Since every $C \sim 0$ decomposes into a sum of small cycles, the condition $c(C) = 0$ needs to be verified only for (arbitrarily) "small cycles" C.

Cocycles are as good as Poincaré's dual cycles for detecting non-triviality of geometric cycles C: if $c(C) \neq 0$, then, C is non-homologous to zero and also c is not cohomologous to zero.

If we work with $H^*(X;\mathbb{R})$, these cocycles $c(D)$ can be averaged over measures on the space of smooth self-mapping $X \to X$ homotopic to the identity. (The averaged cocycles are kind of duals of generic cycles.) Eventually, they can be reduced to differential forms invariant under a given compact connected automorphism group of X, that let cohomology return to geometry by the back door.

On Integrality of Cohomology. In view of the above, the rational cohomology classes $c \in H^i(X;\mathbb{Q})$ can be *defined* as homomorphisms $c : H_i(X) \to \mathbb{Q}$. Such a c is called *integer* if its image is contained in $\mathbb{Z} \subset \mathbb{Q}$. (Non-integrality of certain classes underlies the existence of *nonstandard smooth structures* on topological spheres discovered by Milnor, see section 6.)

The $\mathbb{Q}$-duality does not tell you the whole story. For example, the following simple property of *closed n-manifolds X* depends on the full homological duality:

Connectedness/Contractibility. *If X is a closed k-connected n-manifold, i.e.* $\pi_i(X) = 0$ *for* $i = 1, ..., k$, *then the complement to a point,* $X \setminus \{x_0\}$, *is* $(n-k-1)$*-contractible, i.e. there is a homotopy* f_t *of the identity map* $X \setminus \{x_0\} \to X \setminus \{x_0\}$ *with* $P = f_1(X \setminus \{x_0\})$ *being a smooth triangulated subspace* $P \subset X \setminus \{x_0\}$ *with* $codim(P) \geq k+1$.

For example, if $\pi_i(X) = 0$ *for* $1 \leq i \leq n/2$, *then X is homotopy equivalent to* S^n.

Smooth triangulations. Recall, that "smoothness" of a triangulated subset in a smooth n-manifold, say $P \in X$, means that, for every closed i-simplex $\Delta \subset P$, there exist

an open subset $U \subset X$ which contains Δ,

an affine triangulation P' of $\mathbb{R}^n$, $n = dim(X)$,

a diffeomorphism $U \to U' \subset \mathbb{R}^n$ which sends Δ onto an i-simplex Δ' in P'.

Accordingly, one defines the notion of a *smooth triangulation T* of a smooth manifold X, where one also says that the *smooth structure in X is compatible* with T.

Every smooth manifold X can be given a smooth triangulation, e.g. as follows.

Let S be an affine (i.e. by affine simplices) triangulation of $\mathbb{R}^M$ which is invariant under the action of a lattice $\Gamma = \mathbb{Z}^M \subset \mathbb{R}^M$ (i.e. S is induced from a triangulation of the M-torus $\mathbb{R}^M/\Gamma$) and let $X \subset_f \mathbb{R}^M$ be a smoothly embedded (or immersed) closed n-submanifold. Then there (obviously) exist

- an arbitrarily small positive constant $\delta_0 = \delta_0(S) > 0$,
- an arbitrarily large constant $\lambda \geq \lambda_0(X, f, \delta_0) > 0$,
- δ-small moves of the vertices of S for $\delta \leq \delta_0$, where these moves themselves depend on the embedding f of X into $\mathbb{R}^M$ and on λ, such that the simplices of the correspondingly moved triangulation, say $S' = S'_\delta = S'(X, f)$ are δ'*-transversal* to the λ-scaled X, i.e. to $\lambda X = X \subset_{\lambda f} \mathbb{R}^M$, where

the δ'*-transversality* of an affine simplex $\Delta' \subset \mathbb{R}^M$ to $\lambda X \subset \mathbb{R}^M$ means that the affine simplices Δ'' obtained from Δ' by arbitrary δ'-moves of the vertices of Δ' for some $\delta' = \delta'(S, \delta) > 0$ are transversal to λX. In particular, the intersection "angles" between λX and the i-simplices, $i = 0, 1, ..., M-1$, in S' are all $\geq \delta'$.

If λ is sufficiently large (and hence, $\lambda X \subset \mathbb{R}^M$ is nearly flat), then the δ'-transversality (obviously) implies that the intersection of λX with each simplex and its neighbours in S' in the vicinity of each point $x \in \lambda X \subset \mathbb{R}^M$ has the same combinatorial pattern as the intersection of the tangent space $T_x(\lambda X) \subset \mathbb{R}^M$ with these simplices. Hence, the (cell) partition $\Pi = \Pi_{f'}$ of λX induced from S' can be subdivided into a triangulation of $X = \lambda X$.

Almost all of what we have presented in this section so far was essentially understood by Poincaré, who switched at some point from geometric cycles to triangulations, apparently, in order to prove his duality. (See [42] for pursuing further the first Poincaré approach to homology.)

The language of geometic/generic cycles suggested by Poincaré is well suited for observing and proving the multitude of obvious little things one comes across every moment in topology. (I suspect, geometric, even worse, some algebraic topologists think of cycles while they draw commutative diagrams. Rephrasing J.B.S. Haldane's words: "Geometry is like a mistress to a topologist: he cannot live without her but he's unwilling to be seen with her in public".)

But if you are *far* away from manifolds in the homotopy theory it is easier to work with *cohomology* and use the *cohomology product* rather than intersection product.

The cohomology product is a bilinear pairing, often denoted $H^i \otimes H^j \overset{\smile}{\to} H^{i+j}$, which is the Poincaré dual of the intersection product $H_{n-i} \otimes H_{n-j} \overset{\cap}{\to} H_{n-i-j}$ in closed oriented n-manifolds X.

The $\smile$-product can be defined for all, say triangulated, X as the dual of the intersection product on the relative homology, $H_{M-i}(U;\infty) \otimes H_{M-j}(U;\infty) \to H_{M-i-j}(U;\infty)$, for a small regular neighbourhood $U \supset X$ of X embedded into some $\mathbb{R}^M$. The $\smile$ product, so defined, is invariant under continuous maps $f : X \to Y$:

$$f^*(h_1 \smile h_2) = f^*(h_1) \smile f^*(h_2) \text{ for all } h_1, h_2 \in H^*(Y).$$

It easy to see that the $\smile$-pairing equals the composition of the Künneth homomorphism $H^*(X) \otimes H^*(X) \to H^*(X \times X)$ with the restriction to the diagonal $H^*(X \times X) \to H^*(X_{diag})$.

You can hardly expect to arrive at anything like Serre's finiteness theorem without a linearized (co)homology theory; yet, geometric constructions are of a great help on the way.

Topological and $\mathbb{Q}$-manifolds. The combinatorial proof of the Poincaré duality is the most transparent for open subsets $X \subset \mathbb{R}^n$ where the standard decomposition S of $\mathbb{R}^n$ into cubes is the combinatorial dual of its own translate by a generic vector.

Poincaré duality remains valid for all oriented *topological* manifolds X and also for all *rational homology* or $\mathbb{Q}$-*manifolds*, that are compact triangulated n-spaces where the *link* $L^{n-i-1} \subset X$ of every i-simplex Δ^i in X has the same rational homology as the sphere S^{n-i-1}, where it follows from the (special case of) *Alexander duality.*

The rational homology of the complement to a topologically embedded k-sphere as well as of a rational homology sphere, into S^n (or into a $\mathbb{Q}$-manifold with the rational homology of S^n) equals that of the $(n-k-1)$-sphere.

(The link $L^{n-i-1}(\Delta^i)$ is the union of the simplices $\Delta^{n-i-1} \subset X$ which do not intersect Δ^i and for which there exists an simplex in X which contains Δ^i and Δ^{n-i-1}.)

Alternatively, an n-dimensional space X can be embedded into some $\mathbb{R}^M$ where the duality for X reduces to that for "suitably regular" neighbourhoods $U \subset \mathbb{R}^M$ of X which admit Thom isomorphisms $H_i(X) \leftrightarrow H_{i+M-n}(U_\bullet)$.

If X is a topological manifold, then "locally generic" cycles of complementary dimension intersect at a discrete set which allows one to define their geometric intersection index. Also one can define the intersection of several cycles C_j, $j = 1, ...k$, with $\sum_j dim(C_i) = dim(X)$ as the intersection index of $\times_j C_j \subset X^k$ with $X_{diag} \subset X^k$, but anything more then that can not be done so easily.

Possibly, there is a comprehensive formulation with an obvious invariant proof of the "functorial Poincaré duality" which would make transparent, for example, the multiplicativity of the signature (see below) and the topological nature of rational Pontryagin classes (see section 10) and which would apply to "cycles" of dimensions βN where $N = \infty$ and $0 \le \beta \le 1$ in spaces like these we shall meet in section 11.

Signature. The intersection of (compact) k-cycles in an oriented, possibly non-compact and/or disconnected, $2k$-manifold X defines *a bilinear* form on the homology $H_k(X)$. If k is odd, this form is antisymmetric and if k is even it is symmetric.

The signature of the latter, i.e. the number of positive minus the number of negative squares in the diagonalized form, is called $sig(X)$. This is well defined if $H_k(X)$ has finite rank, e.g. if X is compact, possibly with a boundary.

Geometrically, a diagonalization of the intersection form is achieved with a *maximal* set of *mutually disjoint* k-cycles in X where each of them has a *non-zero* (positive or negative) self-intersection index. (If the cycles are represented by smooth closed oriented *k-submanifolds*, then these indices equal the Euler numbers of the normal bundles of these submanifolds. In fact, such a maximal system of submanifolds always exists as it was derived by Thom from the Serre finiteness theorem.)

Examples. (a) $S^{2k} \times S^{2k}$ has zero signature, since the $2k$-homology is generated by the classes of the two coordinate spheres $[s_1 \times S^{2k}]$ and $[S^{2k} \times s_2]$, which both have zero self-intersections.

(b) The complex projective space $\mathbb{C}P^{2m}$ has signature one, since its middle homology is generated by the class of the complex projective subspace $\mathbb{C}P^m \subset \mathbb{C}P^{2m}$ with the self-intersection = 1.

(c) The tangent bundle $T(S^{2k})$ has signature 1, since $H_k(T(S^{2k}))$ is generated by $[S^{2k}]$ with the self-intersection equal the Euler characteristic $\chi(S^{2k}) = 2$.

It is obvious that $sig(mX) = m \cdot sig(X)$, where mX denotes the disjoint union of m copies of X, and that $sig(-X) = -sig(X)$, where "$-$" signifies reversion of orientation. Furthermore

The oriented boundary X of every compact oriented $(4k+1)$-manifold Y has zero signature.(Rokhlin 1952).

(Oriented boundaries of non-orientable manifolds may have non-zero signatures. For example the double covering $\tilde{X} \to X$ with $sig(\tilde{X}) = 2sig(X)$ non-orientably bounds the corresponding 1-ball bundle Y over X.)

Proof. If k-cycles C_i, $i = 1, 2$, bound relative $(k+1)$-cycles D_i in Y, then the (zero-dimensional) intersection C_1 with C_2 bounds a relative 1-cycle in Y which makes the index of the intersection zero. Hence,

the intersection form vanishes on the kernel $ker_k \subset H_k(X)$ of the inclusion homomorphism $H_k(X) \to H_k(Y)$.

On the other hand, the obvious identity

$$[C \cap D]_Y = [C \cap \partial D]_X$$

and the Poincaré duality in Y show that the orthogonal complement $ker_k^{\perp} \subset H_k(X)$ with respect to the intersection form in X is contained in ker_k. QED

Observe that this argument depends entirely on the Poincaré duality and it equally applies to the topological and $\mathbb{Q}$-manifolds with boundaries.

Also notice that the Künneth formula and the Poincaré duality (trivially) imply the Cartesian multiplicativity of the signature for closed manifolds,

$$sig(X_1 \times X_2) = sig(X_1) \cdot sig(X_2).$$

For example, the products of the complex projective spaces $\times_i \mathbb{C}P^{2k_i}$ have signatures one. (The Künneth formula is obvious here with the cell decompositions of $\times_i \mathbb{C}P^{2k_i}$ into $\times_i (2k_i + 1)$ cells.)

Amazingly, the *multiplicativity of the signature* of *closed* manifolds under covering maps can not be seen with comparable clarity.

Multiplicativity Formula *if $\tilde{X} \to X$ is an l-sheeted covering map, then*

$$sign(\tilde{X}) = l \cdot sign(X).$$

This can be sometimes proved by elementary, means, e.g. if the fundamental group of X is *free*. In this case, there obviously exist closed hypersurfaces $Y \subset X$ and $\tilde{Y} \subset \tilde{X}$ such that $\tilde{X} \setminus \tilde{Y}$ is diffeomorphic to the disjoint union of l copies of $X \setminus Y$. This implies multiplicativity, since *signature is additive*:

removing a closed hypersurface from a manifold does not change the signature.

Therefore,

$$sig(\tilde{X}) = sig(\tilde{X} \setminus \tilde{Y}) = l \cdot sig(X \setminus Y) = l \cdot sig(X).$$

(This "additivity of the signature" easily follows from the Poincaré duality as observed by S. Novikov.)

In general, given a finite covering $\tilde{X} \to X$, there exists an *immersed* hypersurface $Y \subset X$ (with possible self-intersections) such that the covering trivializes over $X \setminus Y$; hence, $\tilde{X}$ can be assembled from the pieces of $X \setminus Y$ where each piece is taken l times. One still has an addition formula for some "stratified signature" but it is rather involved in the general case.

On the other hand, the multiplicativity of the signature can be derived in a couple of lines from the Serre finiteness theorem (see below).

5 The Signature and Bordisms.

Let us prove the multiplicativity of the signature by constructing a compact oriented manifold Y with a boundary, such that the *oriented* boundary $\partial(Y)$ equals $m\tilde{X} - mlX$ for some integer $m \neq 0$.

Embed X into $\mathbb{R}^{n+N}$, $N >> n = 2k = dim(X)$ let $\tilde{X} \subset \mathbb{R}^{n+N}$ be an embedding obtained by a small generic perturbation of the covering map $\tilde{X} \to X \subset \mathbb{R}^{n+N}$ and $\tilde{X}' \subset \mathbb{R}^{n+N}$ be the union of l generically perturbed copies of X.

Let $\tilde{A}_\bullet$ and $\tilde{A}'_\bullet$ be the Atiyah-Thom maps from $S^{n+N} = \mathbb{R}^{n+N}_\bullet$ to the Thom spaces $\tilde{U}_\bullet$ and $U'_\bullet$ of the normal bundles $\tilde{U} \to \tilde{X}$ and $\tilde{U}' \to \tilde{X}'$.

Let $\tilde{P} : \tilde{X} \to X$ and $\tilde{P}' : \tilde{X}' \to X$ be the normal projections. These projections, obviously, induce the normal bundles $\tilde{U}$ and $\tilde{U}'$ of $\tilde{X}$ and $\tilde{X}'$ from the normal bundle $U^\perp \to X$. Let

$$\tilde{P}_\bullet : \tilde{U}_\bullet \to U^\perp_\bullet \text{ and } \tilde{P}'_\bullet : \tilde{U}'_\bullet \to U^\perp_\bullet$$

be the corresponding maps between the Thom spaces and let us look at the two maps f and f' from the sphere $S^{n+N} = \mathbb{R}^{N+n}_\bullet$ to the Thom space $U^\perp_\bullet$,

$$f = \tilde{P}_\bullet \circ \tilde{A}_\bullet : S^{n+N} \to U^\perp_\bullet, \text{ and } f' = \tilde{P}'_\bullet \circ \tilde{A}'_\bullet : S^{n+N} \to U^\perp_\bullet.$$

Clearly

$$[\tilde{\bullet}\tilde{\bullet}'] \qquad f^{-1}(X) = \tilde{X} \text{ and } (f')^{-1}(X) = \tilde{X}'.$$

On the other hand, the homology homomorphisms of the maps f and f' are related to those of $\tilde{P}$ and $\tilde{P}'$ via the Thom suspension homomorphism $S_\bullet : H_n(X) \to H_{n+N}(U^\perp_\bullet)$ as follows

$$f_*[S^{n+N}] = S_\bullet \circ \tilde{P}_*[\tilde{X}] \text{ and } f'_*[S^{n+N}] = S_\bullet \circ \tilde{P}'_*[\tilde{X}'].$$

Since $deg(\tilde{P}) = deg(\tilde{P}') = l$,

$$\tilde{P}_*[\tilde{X}] = \tilde{P}'_*[\tilde{X}'] = l \cdot [X] \text{ and } f'[S^{n+N}] = f[S^{n+N}] = l \cdot S_\bullet[X] \in H_{n+N}(U^\perp_\bullet);$$

hence,

some non-zero m-multiples of the maps $f, f' : S^{n+N} \to U^\perp_\bullet$ can be joined by a (smooth generic) homotopy $F : S^{n+N} \times [0,1] \to U^\perp_\bullet$ by Serre's theorem, since $\pi_i(U^\perp_\bullet) = 0$, $i = 1, ...N-1$.

Then, because of $[\tilde{\bullet}\tilde{\bullet}']$, the pullback $F^{-1}(X) \subset S^{n+N} \times [0,1]$ *establishes a bordism between $m\tilde{X} \subset S^{n+N} \times 0$ and $m\tilde{X}' = mlX \subset S^{n+N} \times 1$.* This implies that

$m \cdot sig(\tilde{X}) = ml \cdot sig(X)$ and since $m \neq 0$ we get $sig(\tilde{X}) = l \cdot sig(X)$. QED.

Bordisms and the Rokhlin-Thom-Hirzebruch Formula. Let us modify our definition of homology of a manifold X by allowing only *non-singular* i-cycles in X, i.e. smooth closed oriented i-submanifolds in X and denote the resulting Abelian group by $\mathcal{B}^o_i(X)$.

If $2i \geq n = dim(X)$ one has a (minor) problem with taking sums of non-singular cycles, since generic i-submanifolds may intersect and their union is unavoidably singular. We assume below that $i < n/2$; otherwise, we replace X by $X \times \mathbb{R}^N$ for $N >> n$, where, observe, $\mathcal{B}^o_i(X \times \mathbb{R}^N)$ does not depend on N for $N >> i$.

Unlike homology, the *bordism groups* $\mathcal{B}^o_i(X)$ may be non-trivial even for a contractible space X, e.g. for $X = \mathbb{R}^{n+N}$. (Every cycle in $\mathbb{R}^n$ equals the boundary of any cone over it but this does not work with manifolds due to the *singularity* at the apex of the cone which is not allowed by the definition of a bordism.) In fact,

if $N >> n$, then the bordism group $\mathcal{B}^o_n = \mathcal{B}^o_n(\mathbb{R}^{n+N})$ is canonically isomorphic to the homotopy group $\pi_{n+N}(V_\bullet)$, where $V_\bullet$ is the Thom space of the tautological oriented $\mathbb{R}^N$-bundle V over the Grassmann manifold $V = Gr^{or}_N(\mathbb{R}^{n+N+1})$ (Thom, 1954).

Proof. Let $X_0 = Gr^{or}_N(\mathbb{R}^{n+N})$ be the Grassmann manifold of *oriented* N-planes and $V \to X_0$ the tautological oriented $\mathbb{R}^N$ bundle over this X_0.

(The space $Gr^{or}_N(\mathbb{R}^{n+N})$ equals the double cover of the space $Gr_N(\mathbb{R}^{n+N})$ of non-oriented N-planes. For example, $Gr^{or}_1(\mathbb{R}^{n+1})$ equals the sphere S^n, while $Gr_1(\mathbb{R}^{n+1})$ is the projective space, that is S^n divided by the $\pm$-involution.)

Let $U^\perp \to X$ be the oriented normal bundle of X with the orientation induced by those of X and of $\mathbb{R}^N \supset X$ and let $G : X \to X_0$ be the oriented Gauss map which assigns to each $x \in X$ the oriented N-plane $G(x) \in X_0$ parallel to the oriented normal space to X at x.

Since G induces $U^\perp$ from V, it defines the Thom map $S^{n+N} = \mathbb{R}^{n+N}_\bullet \to V_\bullet$ and every bordism $Y \subset S^{n+N} \times [0,1]$ delivers a homotopy $S^{n+N} \times [0,1] \to V_\bullet$ between the Thom maps at the two ends $Y \cap S^{n+N} \times 0$ and $Y \cap S^{n+N} \times 1$.

This define a *homomorphism*

$$\tau_{b\pi} : \mathcal{B}^o_n \to \pi_{n+N}(V_\bullet)$$

since the additive structure in $\mathcal{B}^o_n(\mathbb{R}^{i+N})$ agrees with that in $\pi_{i+N}(V^o_\bullet)$. (Instead of checking this, which is trivial, one may appeal to the general principle: "two natural Abelian group structures on the same set must coincide.")

Also note that one needs the extra 1 in $\mathbb{R}^{n+N+1}$, since bordisms Y between manifolds in $\mathbb{R}^{n+N}$ lie in $\mathbb{R}^{n+N+1}$, or, equivalently, in $S^{n+N+1} \times [0,1]$.

On the other hand, the generic pullback construction

$$f \mapsto f^{-1}(X_0) \subset \mathbb{R}^{n+N} \supset \mathbb{R}^{n+N}_\bullet = S^{n+N}$$

defines a homomorphism $\tau_{\pi b} : [f] \to [f^{-1}(X_0)]$ from $\pi_{n+N}(V_\bullet)$ to $\mathcal{B}^o_n$, where, clearly $\tau_{\pi b} \circ \tau_{b\pi}$ and $\tau_{b\pi} \circ \tau_{\pi b}$ are the identity homomorphisms. QED.

Now Serre's $\mathbb{Q}$-sphericity theorem implies the following

Thom Theorem. *The (Abelian) group $\mathcal{B}^o_i$ is finitely generated;*

$\mathcal{B}^o_n \otimes \mathbb{Q}$ *is isomorphic to the rational homology group* $H_i(X_0;\mathbb{Q}) = H_i(X_0) \otimes \mathbb{Q}$ *for* $X_0 = Gr^{or}_N(\mathbb{R}^{i+N+1})$.

Indeed, $\pi_i(V^\bullet) = 0$ for $N >> n$, hence, by Serre,

$$\pi_{n+N}(V_\bullet) \otimes \mathbb{Q} = H_{n+N}(V_\bullet;\mathbb{Q}),$$

while

$$H_{n+N}(V_\bullet;\mathbb{Q}) = H_n(X_0;\mathbb{Q})$$

by the Thom isomorphism.

In order to apply this, one has to compute the homology $H_n(Gr^{or}_N(\mathbb{R}^{N+n+j}));\mathbb{Q})$, which, as it is clear from the above, is independent of $N \geq 2n+2$ and of $j > 1$; thus, we pass to

$$Gr^{or} =_{def} \bigcup_{j,N \to \infty} Gr^{or}_N(\mathbb{R}^{N+j}).$$

Let us state the answer in the language of *cohomology*, with the advantage of the *multiplicative structure* (see section 4) where, recall, the cohomology product $H^i(X) \otimes H^j(X) \xrightarrow{\smile} H^{i+j}(X)$ for closed oriented n-manifold can be defined via the Poincaré duality $H^*(X) \leftrightarrow H_{n-*}(X)$ by the intersection product $H_{n-i}(X) \otimes H_{n-j}(X) \xrightarrow{\cap} H_{n-(i+j)}(X)$.

The cohomology ring $H^*(Gr^{or};\mathbb{Q})$ is the polynomial ring in some distinguished integer classes, called Pontryagin classes $p_k \in H^{4k}(Gr^{or};\mathbb{Z})$, $k = 1,2,3,...$ [51], [21].

(It would be awkward to express this in the homology language when $N = dim(X) \to \infty$, although the cohomology ring $H^*(X)$ is canonically isomorphic to $H_{N-*}(X)$ by Poincaré duality.)

If X is a smooth oriented n-manifold, its Pontryagin classes $p_k(X) \in H^{4k}(X;\mathbb{Z})$ are defined as the classes induced from p_k by the normal Gauss map $G \to Gr^{or}_N(\mathbb{R}^{N+n}) \subset Gr^{or}$ for an embedding $X \to \mathbb{R}^{n+N}$, $N >> n$.

Examples (see [51]). (a) The the complex projective spaces have

$$p_k(\mathbb{C}P^n) = \binom{n+1}{k} h^{2k}$$

for the generator $h \in H^2(\mathbb{C}P^n)$ which is the Poincaré dual to the hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^{n-1}$.

(b) The rational Pontryagin classes of the Cartesian products $X_1 \times X_2$ satisfy

$$p_k(X_1 \times X_2) = \sum_{i+j=k} p_i(X_1) \otimes p_j(X_2).$$

If Q is a unitary (i.e. a product of powers) monomial in p_i of graded degree $n = 4k$, then the value $Q(p_i)[X]$ is called *the (Pontryagin) Q-number.* Equivalently, this is the value of $Q(p_i) \in H^{4i}(Gr^{or};\mathbb{Z})$ on the image of (the fundamental class) of X in Gr^{or} under the Gauss map.

The Thom theorem now can be reformulated as follows.

Two closed oriented n-manifolds are $\mathbb{Q}$-bordant if and only if they have equal Q-numbers for all monomials Q. Thus, $\mathcal{B}^o_n \otimes \mathbb{Q} = 0$, unless n is divisible by 4 and the rank of $\mathcal{B}^o_n \otimes \mathbb{Q}$ for $n = 4k$ equals the number of Q-monomials of graded degree n, that are $\prod_i p_i^{k_i}$ with $\sum_i k_i = k$.

(We shall prove this later in this section, also see [51].)

For example, if $n = 4$, then there is a single such monomial, p_1; if n=8, there two of them: p_2 and p_1^2; if $n = 12$ there three monomials: p_3, p_1p_2 and p_1^3; if $n = 16$ there are five of them.

In general, the number of such monomials, say $\pi(k) = rank(H^{4k}(Gr^{or};\mathbb{Q})) = rank_{\mathbb{Q}}(\mathcal{B}^o_{4k})$ (obviously) equals the number of the conjugacy classes in the permutation group $\Pi(k)$ (which can be seen as a certain subgroup in the Weyl group in $SO(4k)$), where, by the Euler formula, the generating function $E(t) = 1 + \sum_{k=1,2,...} \pi(k)t^k$ satisfies

$$1/E(t) = \prod_{k=1,2,...} (1 - t^k) = \sum_{-\infty<k<\infty} (-1)^k t^{(3k^2-k)/2},$$

Here the first equality is obvious, the second is tricky (Euler himself was not able to prove it) and where one knows now-a-days that

$$\pi(k) \sim \frac{exp(\pi\sqrt{2k/3})}{4k\sqrt{3}} \text{ for } k \to \infty.$$

Since the top Pontryagin classes p_k of the complex projective spaces do not vanish, $p_k(\mathbb{C}P^{2k}) \neq 0$, the products of these spaces constitute a basis in $\mathcal{B}^o_n \otimes \mathbb{Q}$.

Finally, notice that the bordism groups together make a commutative ring under the Cartesian product of manifolds, denoted $\mathcal{B}^o_*$, and the Thom theorem says that

$\mathcal{B}^o_* \otimes \mathbb{Q}$ *is the polynomial ring over $\mathbb{Q}$ in the variables* $[\mathbb{C}P^{2k}]$, $k = 0, 2, 4, ...$

Instead of $\mathbb{C}P^{2k}$, one might take the compact quotients of the *complex hyperbolic spaces* $\mathbb{C}H^{2k}$ for the generators of $\mathcal{B}^o_* \otimes \mathbb{Q}$. The quotient spaces $\mathbb{C}H^{2k}/\Gamma$

have two closely related attractive features: their tangent bundles admit natural *flat connections* and their rational Pontryagin numbers are *homotopy invariant*, see section 10. It would be interesting to find "natural bordisms" between (linear combinations of) Cartesian products of $\mathbb{C}H^{2k}/\Gamma$ and of $\mathbb{C}P^{2k}$, e.g. associated to complex analytic ramified coverings $\mathbb{C}H^{2k}/\Gamma \to \mathbb{C}P^{2k}$.

Since the signature is additive and also multiplicative under this product, it defines a homomorphism $[sig]:\mathcal{B}^o_* \to \mathbb{Z}$ which can be expressed in each degree $4k$ by means of a universal polynomial in the Pontryagin classes, denoted $L_k(p_i)$, by

$$sig(X) = L_k(p_i)[X] \text{ for all closed oriented } 4k\text{-manifolds } X.$$

For example,

$$L_1 = \frac{1}{3}p_1,\ L_2 = \frac{1}{45}(7p_2 - p_1^2),\ L_3 = \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3).$$

Accordingly,

$$sig(X^4) = \frac{1}{3}p_1[X^4], \qquad \text{(Rokhlin 1952)}$$

$$sig(X^8) = \frac{1}{45}(7p_2(X^8) - p_1^2(X^8))[X^8], \qquad \text{(Thom 1954)}$$

and where a concise general formula (see blow) was derived by Hirzebruch who evaluated the coefficients of L_k using the above values of p_i for the products $X = \times_j \mathbb{C}P^{2k_j}$ of the complex projective spaces, which all have $sig(X) = 1$, and by substituting these products $\times_j \mathbb{C}P^{2k_j}$ with $\sum_j 4k_j = n = 4k$, for $X = X^n$ into the formula $sig(X) = L_k[X]$. The outcome of this seemingly trivial computation is unexpectedly beautiful.

Hirzebruch Signature Theorem. Let

$$R(z) = \frac{\sqrt{z}}{\tanh(\sqrt{z})} = 1 + z/3 - z^2/45 + ... = 1 + 2\sum_{l>0}(-1)^{l+1}\frac{\zeta(2l)z^l}{\pi^{2l}} = 1 + \sum_{l>0}\frac{2^{2l}B_{2l}z^l}{(2l)!},$$

where $\zeta(2l) = 1 + \frac{1}{2^{2l}} + \frac{1}{3^{2l}} + \frac{1}{4^{2l}} + ...$ and let

$$B_{2l} = (-1)^l 2l\zeta(1-2l) = (-1)^{l+1}(2l)!\zeta(2l)/2^{2l-1}\pi^{2l}$$

be the *Bernoulli numbers* [48],

$$B_2 = 1/6, B_4 = -1/30, ..., B_{12} = -691/2730, B_{14} = 7/6, ..., B_{30} = 8615841276005/14322, ...$$

Write

$$R(z_1)\cdot ... \cdot R(z_k) = 1 + P_1(z_j) + ... + P_k(z_j) + ...$$

where P_j are homogeneous symmetric polynomials of degree j in $z_1, ..., z_k$ and rewrite

$$P_k(z_j) = L_k(p_i)$$

where $p_i = p_i(z_1, ..., z_k)$ are the elementary symmetric functions in z_j of degree i. The Hirzebruch theorem says that *the above L_k is exactly the polynomial which makes the equality $L_k(p_i)[X] = sig(X)$.*

A significant aspect of this formula is that the Pontryagin numbers and the signature are *integers* while the Hirzebruch polynomials L_k have non-trivial denominators. This yields certain *universal divisibility* properties of the Pontryagin numbers (and sometimes of the signatures) for smooth closed orientable $4k$-manifolds.

But despite a heavy integer load carried by the signature formula, its derivation depends only on the *rational* bordism groups $\mathcal{B}_n^o \otimes \mathbb{Q}$. This point of elementary linear algebra was overlooked by Thom (isn't it incredible?) who derived the signature formula for 8-manifolds from his special and more difficult computation of the true bordism group $\mathcal{B}_8^o$. However, the shape given by Hirzebruch to this formula is something more than just linear algebra.

Question. Is there an implementation of the analysis/arithmetic encoded in the Hirzebruch formula by some infinite dimensional manifolds?

Computation of the Cohomology of the Stable Grassmann Manifold. First, we show that the cohomology $H^*(Gr^{or};\mathbb{Q})$ is multiplicatively generated by some classes $e_i \in H^*(Gr^{or};\mathbb{Q})$ and then we prove that the L_i-classes are multiplicatively independent. (See [51] for computation of the integer cohomology of the Grassmann manifolds.)

Think of the unit tangent bundle $UT(S^n)$ as the space of orthonormal 2-frames in $\mathbb{R}^{n+1}$, and recall that $UT(S^{2k})$ is a rational homology $(4k-1)$-sphere.

Let $W_k = Gr^{or}_{2k+1}(\mathbb{R}^\infty)$ be the Grassmann manifold of oriented $(2k+1)$-planes in $\mathbb{R}^N$, $N \to \infty$, and let W''_k consist of the pairs (w,u) where $w \in W_k$ is an $(2k+1)$-plane $\mathbb{R}^{2k+1}_w \subset \mathbb{R}^\infty$, and u is an orthonormal frame (pair of orthonormal vectors) in $\mathbb{R}^{2k+1}_w$.

The map $p : W''_k \to W_{k-1} = Gr^{or}_{2k-1}(\mathbb{R}^\infty)$ which assigns, to every (w,u), the $(2k-1)$-plane $u^\perp_w \subset \mathbb{R}^{2k+1}_w \subset \mathbb{R}^\infty$ normal to u is a fibration with *contractible fibers* that are spaces of orthonormal 2-frames in $\mathbb{R}^\infty \ominus u^\perp_w = \mathbb{R}^{\infty-(2k-1)}$; hence, *$p$ is a homotopy equivalence.*

A more interesting fibration is $q : W_k \to W''_k$ for $(w,u) \mapsto w$ with the fibers $UT(S^{2k})$. Since $UT(S^{2k})$ is a rational $(4k-1)$-sphere, the kernel of the cohomology homomorphism $q^* : H^*(W''_k;\mathbb{Q}) \to H^*(W_k;\mathbb{Q})$ is generated, as a $\smile$-ideal, by the rational Euler class $e_k \in H^{4k}(W''_k;\mathbb{Q})$.

It follows by induction on k that the rational cohomology algebra of $W_k = Gr^{or}_{2k+1}(\mathbb{R}^\infty)$ is generated by certain $e_i \in H^{4i}(W_k;\mathbb{Q})$, $i = 0,1,...,k$, and since $Gr^{or} = \underset{\longleftarrow k\to\infty}{\lim} Gr^{or}_{2k+1}$, these e_i also generate the cohomology of Gr^{or}.

Direct Computation of the L-Classes for the Complex Projective Spaces. Let $V \to X$ be an oriented vector bundle and, following Rokhlin-Schwartz and Thom, *define* L-classes of V, without any reference to Pontryagin classes, as follows. Assume that X is a manifold with a trivial tangent bundle; otherwise, embed X into some $\mathbb{R}^M$ with large M and take its small regular neighbourhood. By Serre's theorem, there exists, for every homology class $h \in H_{4k}(X) = H_{4k}(V)$, an $m = m(h) \neq 0$ such that the m-multiple of h is representable by a closed $4k$-submanifold $Z = Z_h \subset V$ that equals the pullback of a generic point in the sphere S^{M-4k} under a generic map $V \to S^{M-4k} = \mathbb{R}^{M-4k}_\bullet$ with "compact support", i.e. where all but a compact subset in V goes to $\bullet \in S^{M-4k}$. Observe that such a Z has trivial normal bundle in V.

Define $L(V) = 1 + L_1(V) + L_2(V) + ... \in H^{4*}(V;\mathbb{Q}) = \oplus_k H^{4k}(V;\mathbb{Q})$ by the equality $L(V)(h) = sig(Z_h)/m(h)$ for all $h \in H_{4k}(V) = H_{4k}(X)$.

If the bundle V is induced from $W \to Y$ by an $f : X \to Y$ then $L(V) = f^*(L(W))$, since, for $dim(W) > 2k$ (which we may assume), the generic image of our Z in W has *trivial* normal bundle.

It is also clear that the bundle $V_1 \times V_2 \to X_1 \times X_2$ has $L(V_1 \times V_2) = L(V_1) \otimes L(V_2)$ by the Cartesian multiplicativity of the signature.

Consequently the L-class of the *Whitney sum* $V_1 \oplus V_2 \to X$ of V_1 and V_2 over X, which is defined as the restriction of $V_1 \times V_2 \to X \times X$ to $X_{diag} \subset X \times X$, satisfies

$$L(V_1 \oplus V_2) = L(V_1) \smile L(V_2).$$

Recall that the complex projective space $\mathbb{C}P^k$ – the space of $\mathbb{C}$-lines in $\mathbb{C}^{k+1}$ comes with the canonical $\mathbb{C}$-line bundle represented by these lines and denoted $U \to \mathbb{C}P^k$, while the same bundle with the reversed orientation is denoted U^-. (We always refer to the canonical orientations of $\mathbb{C}$-objects.)

Observe that $U^- = Hom_{\mathbb{C}}(U \to \theta)$ for the trivial $\mathbb{C}$-bundle

$$\theta = \mathbb{C}P^k \times \mathbb{C} \to \mathbb{C}P^k = Hom_{\mathbb{C}}(U \to U)$$

and that the Euler class $e(U^-) = -e(U)$ equals the generator in $H^2(\mathbb{C}P^k)$ that is the Poincaré dual of the hyperplane $\mathbb{C}P^{k-1} \subset \mathbb{C}P^k$, and so e^l is the dual of $\mathbb{C}P^{k-l} \subset \mathbb{C}P^k$.

The Whitney $(k+1)$-multiple bundle of U^-, denoted $(k+1)U^-$, equals the tangent bundle $T_k = T(\mathbb{C}P^k)$ plus θ. Indeed, let $U^\perp \to \mathbb{C}P^k$ be the $\mathbb{C}^k$ bundle of the normals to the lines representing the points in $\mathbb{C}P^k$. It is clear that $U^\perp \oplus U = (k+1)\theta$, i.e. $U^\perp \oplus U$ is the trivial $\mathbb{C}^{k+1}$-bundle, and that, tautologically,

$$T_k = Hom_{\mathbb{C}}(U \to U^\perp).$$

It follows that

$$T_k \oplus \theta = Hom_{\mathbb{C}}(U \to U^\perp \oplus U) = Hom_{\mathbb{C}}(U \to (k+1)\theta) = (k+1)U^-.$$

Recall that

$$sig(\mathbb{C}P^{2k}) = 1; \text{ hence, } L_k((k+1)U^-) = L_k(T_k) = e^{2k}.$$

Now we compute $L(U^-) = 1 + \sum_k L_k = 1 + \sum_k l_{2k}e^{2k}$, by equating e^{2k} and the $2k$-degree term in the $(k+1)$th power of this sum.

$$(1 + \sum_k l_{2k}e^{2k})^{k+1} = 1 + ... + e^{2k} + ...$$

Thus,

$$(1 + l_1e^2)^3 = 1 + 3l_1 + ... = 1 + e^2 + ...,$$

which makes $l_1 = 1/3$ and $L_1(U^-) = e^2/3$.

Then

$$(1 + l_1e^2 + l_2e^4)^5 = 1 + ... + (10l_1 + 5l_2)e^4 + ... = 1 + ... + e^4 + ...$$

which implies that $l_2 = 1/5 - 2l_1 = 1/5 - 2/3$ and $L_2(U^-) = (-7/15)e^4$, etc.

Finally, we compute all L-classes $L_j(T_{2k}) = (L(U^-))^{k+1}$ for $T_{2k} = T(\mathbb{C}P^{2k})$ and thus, all $L(\times_j \mathbb{C}P^{2k_j})$.

For example,

$$(L_1(\mathbb{C}P^8))^2[\mathbb{C}P^8] = 10/3 \text{ while } (L_1(\mathbb{C}P^4 \times \mathbb{C}P^4))^2[\mathbb{C}P^4 \times \mathbb{C}P^4] = 2/3$$

which implies that $\mathbb{C}P^4 \times \mathbb{C}P^4$ and $\mathbb{C}P^8$, which have equal signatures, are *not* rationally bordant, and similarly one sees that the products $\times_j \mathbb{C}P^{2k_j}$ are multiplicatively independent in the bordism ring $\mathcal{B}^o_* \otimes Q$ as we stated earlier.

Combinatorial Pontryagin Classes. Rokhlin-Schwartz and independently Thom applied their definition of L_k, and hence of the *rational* Pontryagin classes, to *triangulated (not necessarily smooth) topological manifolds* X by observing that the pullbacks of *generic* points $s \in S^{n-4k}$ under piece-wise linear map are $\mathbb{Q}$-manifolds and by pointing out that the signatures of $4k$-manifolds are invariant under bordisms by such $(4k+1)$-dimensional $\mathbb{Q}$-manifolds with boundaries (by the Poincaré duality issuing from the Alexander duality, see section 4). Thus, they have shown, in particular, that

rational Pontryagin classes of smooth manifolds are invariant under piece-wise smooth homeomorphisms between smooth manifolds.

The combinatorial pull-back argument breaks down in the topological category since there is no good notion of a generic *continuous* map. Yet, S. Novikov (1966) proved that the L-classes and, hence, the rational Pontryagin classes are invariant under *arbitrary homeomorphisms* (see section 10).

The Thom-Rokhlin-Schwartz argument delivers a definition of rational Pontryagin classes for all $\mathbb{Q}$-manifolds which are by far more general objects than smooth (or combinatorial) manifolds due to possibly enormous (and beautiful) fundamental groups $\pi_1(L^{n-i-1})$ of their links.

Yet, the naturally defined bordism ring $\mathbb{Q}\mathcal{B}^o_n$ of oriented $\mathbb{Q}$-manifolds is only marginally different from $\mathcal{B}^o_*$ in the degrees $n \neq 4$ where the natural homomorphisms $\mathcal{B}^o_n \otimes \mathbb{Q} \to \mathbb{Q}\mathcal{B}^o_n \otimes \mathbb{Q}$ are isomorphisms. This can be easily derived by *surgery* (see section 9) from Serre's theorems. For example, if a $\mathbb{Q}$-manifold X has a single singularity – a cone over $\mathbb{Q}$-sphere Σ then a connected sum of several copies of Σ bounds a smooth $\mathbb{Q}$-ball which implies that a multiple of X is $\mathbb{Q}$-bordant to a smooth manifold.

On the contrary, the group $\mathbb{Q}\mathcal{B}^o_4 \otimes \mathbb{Q}$, is much bigger than $\mathcal{B}^o_4 \otimes \mathbb{Q} = \mathbb{Q}$ as $rank_{\mathbb{Q}}(\mathbb{Q}\mathcal{B}^o_4) = \infty$ (see [45], [22], [23] and references therein).

(It would be interesting to have a notion of "refined bordisms" between $\mathbb{Q}$-manifold that would partially keep track of $\pi_1(L^{n-i-1})$ for $n > 4$ as well.)

The simplest examples of $\mathbb{Q}$-manifolds are one point compactifications $V^{4k}_\bullet$ of the tangent bundles of even dimensional spheres, $V^{4k} = T(S^{2k}) \to S^{2k}$, since the boundaries of the corresponding $2k$-ball bundles are $\mathbb{Q}$-homological $(2k-1)$-spheres – the unit tangent bundles $UT(S^{2k}) \to S^{2k}$.

Observe that the tangent bundles of spheres are *stably trivial* – they become trivial after adding trivial bundles to them, namely the tangent bundle of $S^{2k} \subset \mathbb{R}^{2k+1}$ stabilizes to the trivial bundle upon adding the (trivial) normal bundle of $S^{2k} \subset \mathbb{R}^{2k+1}$ to it. Consequently, the manifolds $V^{4k} = T(S^{2k})$ have all characteristic classes zero, and $V^{4k}_\bullet$ have all $\mathbb{Q}$-classes zero except for dimension $4k$.

On the other hand, $L_k(V_\bullet^{4k}) = sig(V_\bullet^{4k}) = 1$, since the tangent bundle $V^{4k} = T(S^{2k}) \to S^{2k}$ has non-zero Euler number. Hence,

the $\mathbb{Q}$-manifolds $V_\bullet^{4k}$ multiplicatively generate all of $\mathbb{Q}\mathcal{B}_*^o \otimes \mathbb{Q}$ except for $\mathbb{Q}\mathcal{B}_4^o$.

Local Formulae for Combinatorial Pontryagin Numbers. Let X be a closed oriented triangulated (smooth or combinatorial) $4k$-manifold and let $\{S_x^{4k-1}\}_{x\in X_0}$ be the *disjoint* union of the oriented links S_x^{4k-1} of the vertices x in X. Then

there exists, for each monomial Q of the total degree $4k$ in the Pontryagin classes, an assignment of rational numbers $Q[S_x]$ to all S_x^{4k-1}, where $Q[S_x^{4k-1}]$ depend only on the combinatorial types of the triangulations of S_x induced from X, such that the Pontryagin Q-number of X satisfies (Levitt-Rourke 1978),

$$Q(p_i)[X] = \sum_{S_x^{4k-1}\in[[X]]} Q[S_x^{4k-1}].$$

Moreover, there is a canonical assignment of real numbers to S_x^{4k-1} with this property which also applies to all $\mathbb{Q}$-manifolds (Cheeger 1983).

There is no comparable *effective combinatorial* formulae with *a priori* rational numbers $Q[S_x]$ despite many efforts in this direction, see [24] and references therein. (Levitt-Rourke theorem is purely existential and Cheeger's definition depends on the L_2-analysis of differential forms on the non-singular locus of X away from the codimension 2 skeleton of X.)

Questions. Let $\{[S^{4k-1}]_\triangle\}$ be a finite collection of combinatorial isomorphism classes of oriented triangulated $(4k-1)$-spheres let $\mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$ be the $\mathbb{Q}$-vector space of functions $q : \{[S^{4k-1}]_\triangle\} \to \mathbb{Q}$ and let X be a closed oriented triangulated $4k$-manifold homeomorphic to the $4k$-torus (or any parallelizable manifold for this matter) with all its links in $\{[S^{4k-1}]_\triangle\}$.

Denote by $q(X) \in \mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$ the function, such that $q(X)([S^{4k-1}]_\triangle)$ equals the number of copies of $[S^{4k-1}]_\triangle$ in $\{S_x^{4k-1}\}_{x\in X_0}$ and let $\mathcal{L}\{[S]_\triangle\} \subset \mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$ be the linear span of $q(X)$ for all such X.

The above shows that the vectors $q(X)$ of "q-numbers" satisfy, besides $2k+1$ *Euler-Poincaré and Dehn-Somerville equations*, about $\frac{exp(\pi\sqrt{2k/3})}{4k\sqrt{3}}$ linear "Pontryagin relations".

Observe that the Euler-Poincaré and Dehn-Somerville equations do not depend on the ±-orientations of the links but the "Pontryagin relations" are antisymmetric since $Q[-S^{4k-1}]_\triangle = -Q[S^{4k-1}]_\triangle$. Both kind of relations are valid for all $\mathbb{Q}$-manifolds.

What are the codimensions $codim(\mathcal{L}\{[S^{4k-1}]_\triangle\} \subset \mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$, i.e. the numbers of independent relations between the "q-numbers", for "specific" collections $\{[S^{4k-1}]_\triangle\}$?

It is pointed out in [23] that

• the spaces $\mathbb{Q}_\pm^{\mathcal{S}^i}$ of antisymmetric $\mathbb{Q}$-linear combinations of *all* combinatorial spheres make a *chain complex* for the differential $q_\pm^i : \mathbb{Q}_\pm^{\mathcal{S}^i} \to \mathbb{Q}_\pm^{\mathcal{S}^{i-1}}$ defined by the linear extension of the operation of taking the oriented links of all vertices on the triangulated i-spheres $S_\triangle^i \in \mathcal{S}^i$.

• The operation $q_\pm^i$ with values in $\mathbb{Q}_\pm^{\mathcal{S}^{i-1}}$, which is obviously defined on all closed oriented combinatorial i-manifolds X as well as on combinatorial i-spheres, satisfies

$q_{\pm}^{i-1}(q_{\pm}^{i}(X)) = 0$, *i.e. i-manifolds represent i-cycles in this complex.*

Furthermore, it is shown in [23] (as was pointed out to me by Jeff Cheeger) that all such anti-symmetric relations are generated/exhausted by the relations issuing from $q_{\pm}^{n-1} \circ q_{\pm}^{n} = 0$, where this identity can be regarded as an "oriented (Pontryagin in place of Euler-Poincaré) counterpart" to the (Klee)-Dehn-Somerville equations [41].

The exhaustiveness of $q_{\pm}^{n-1} \circ q_{\pm}^{n} = 0$ and its (easy, [25]) Dehn-Somerville counterpart, probably, imply that in most (all?) cases the Euler-Poincaré, Dehn-Somerville, Pontryagin and $q_{\pm}^{n-1} \circ q_{\pm}^{n} = 0$ make the full set of affine (i.e. homogeneous and non-homogeneous linear) relations between the vectors $q(X)$, but it seems hard to effectively (even approximately) evaluate the number of independent relations issuing from for $q_{\pm}^{n-1} \circ q_{\pm}^{n} = 0$ for particular collections $\{[S^{n-1}]_{\triangle}\}$ of allowable links of X^n.

Examples. Let $D_0 = D_0(\Gamma)$ be a *Dirichlet-Voronoi (fundamental polyhedral) domain* of a generic lattice $\Gamma \subset \mathbb{R}^M$ and let $\{[S^{n-1}]_{\triangle}\}$ consist of the (isomorphism classes of naturally triangulated) boundaries of the intersections of D_0 with generic affine n-planes in $\mathbb{R}^M$.

What is $codim(\mathcal{L}\{[S^{n-1}]_{\triangle}\} \subset \mathbb{Q}\{[S^{n-1}]_{\triangle}\})$ in this case?

What are the (affine) relations between the "geometric q-numbers" i.e. the numbers of combinatorial types of intersections σ of λ-scaled submanifolds $X \subset_f \mathbb{R}^M$, $\lambda \to \infty$, (as in the triangulation construction in the previous section) with the Γ-translates of D_0?

Notice, that some of these σ are *not* convex-like, but these are negligible for $\lambda \to \infty$. On the other hand, if λ is sufficiently large all σ can be made convex-like by a small perturbation f' of f by an argument which is similar to but slightly more technical than the one used for the triangulation of manifolds in the previous section.

Is there anything special about the "geometric q-numbers" for "distinguished" X, e.g. for round n-spheres in $\mathbb{R}^N$?

Observe that the *ratios* of the "geometric q-numbers" are asymptotically defined for many *non-compact complete* submanifolds $X \subset \mathbb{R}^M$.

For example, if X is an affine subspace $A = A^n \subset \mathbb{R}^M$, these ratios are (obviously) expressible in terms of the volumes of the connected regions Ω in $D \subset \mathbb{R}^M$ obtained by cutting D along hypersurfaces made of the affine n-subspaces $A' \subset \mathbb{R}^M$ which are parallel to A and which *meet* the $(M-n-1)$-skeleton of D.

What is the number of our kind of relations between these volumes?

There are similar relations/questions for intersection patterns of particular X with other fundamental domains of lattices Γ in Euclidean and some non-Euclidean spaces (where the finer asymptotic distributions of these patterns have a slight arithmetic flavour).

If $f : X^n \to \mathbb{R}^M$ is a generic map *with singularities* (which may happen if $M \leq 2n$) and $D \subset \mathbb{R}^M$ is a small convex polyhedron in $\mathbb{R}^M$ with its faces being δ-transversal to f (e.g. $D = \lambda^{-1}D_0$, $\lambda \to \infty$ as in the triangulations of the previous section), then the pullback $f^{-1}(D) \subset X$ is not necessarily a topological cell. However, some local/additive formulae for certain characteristic numbers may still be available in the corresponding "non-cell decompositions" of X.

For instance, one (obviously) has such a formula for the Euler characteristic for all kind of decompositions of X. Also, one has such a "local formula" for $sig(X)$ and $f : X \to \mathbb{R}$ (i.e for $M = 1$) by Novikov's signature additivity property mentioned at the end of the previous section.

It seems not hard to show that *all* Pontryagin numbers can be thus locally/additively expressed for $M \geq n$, but it is unclear what are precisely the Q-numbers which are combinatorially/locally/additively expressible for given $n = 4k$ and $M < n$.

(For example, if $M = 1$, then the Euler characteristic and the signature are, probably, *the only* "locally/additively expressible" invariants of X.)

Bordisms of Immersions. If the allowed singularities of oriented n-cycles in $\mathbb{R}^{n+k}$ are those of collections of n-planes in general position, then the resulting homologies are the bordism groups of *oriented immersed* manifolds $X^n \subset \mathbb{R}^{n+k}$ (R.Wells, 1966). For example if $k = 1$, this group is isomorphic to the stable homotopy group $\pi_n^{st} = \pi_{n+N}(S^N)$, $N > n+1$, by the Pontryagin pullback construction, since a small generic perturbation of an oriented X^n in $\mathbb{R}^{n+N} \supset \mathbb{R}^{n+1} \supset X^n$ is *embedded* into $\mathbb{R}^{n+N}$ with a *trivial* normal bundle, and where *every* embedding $X^n \to \mathbb{R}^{n+N}$ with the *trivial normal bundle* can be isotoped to such a perturbation of an immersion $X^n \to \mathbb{R}^{n+1} \subset \mathbb{R}^{n+N}$ by the *Smale-Hirsch immersion theorem.* (This is obvious for $n = 0$ and $n = 1$).

Since immersed oriented $X^n \subset \mathbb{R}^{n+1}$ have *trivial* stable normal bundles, they have, for $n = 4k$, *zero* signatures by the Serre finiteness theorem. Conversely, the finiteness of the stable groups $\pi_n^{st} = \pi_{n+N}(S^N)$ can be (essentially) reduced by a (framed) surgery of X^n (see section 9) to the vanishing of these signatures.

The complexity of $\pi_{n+N}(S^N)$ shifts in this picture one dimension down to bordism invariants of the "decorated self-intersections" of immersed $X^n \subset \mathbb{R}^{n+1}$, which are partially reflected in the structure of the l-sheeted coverings of the loci of l-multiple points of X^n.

The Galois group of such a covering may be equal the full permutation group $\Pi(l)$ and the "decorated invariants" live in certain "decorated" bordism groups of the *classifying spaces* of $\Pi(l)$, where the "dimension shift" suggests an inductive computation of these groups that would imply, in particular, Serre's finiteness theorem of the stable homotopy groups of spheres. In fact, this can be implemented in terms of configuration spaces associated to the iterated loop spaces as was pointed out to me by Andras Szùcs, also see [1], [76]. Szùcs, [77].

The simplest bordism invariant of codimension one immersions is the parity of the number of $(n+1)$-multiple points of generically immersed $X^n \subset \mathbb{R}^{n+1}$. For example, the figure $\infty \subset \mathbb{R}^2$ with a single double point represents a *non-zero* element in $\pi_{n=1}^{st} = \pi_{1+N}(S^N)$. The number of $(n+1)$-multiple points also can be *odd* for $n = 3$ (and, trivially, for $n = 0$) but it is always *even* for codimension one immersions of *orientable* n-manifolds with $n \neq 0, 1, 3$, while the non-orientable case is more involved [16], [17].

One knows, (see next section) that every element of the stable homotopy group $\pi_n^{st} = \pi_{n+N}(S^N)$, $N >> n$, can be represented for $n \neq 2, 6, 14, 30, 62, 126$ by an immersion $X^n \to \mathbb{R}^{n+1}$, where X^n is a *homotopy sphere*; if $n = 2, 6, 14, 30, 62, 126$, one can make this with an X^n where $rank(H_*(X_n)) = 4$.

What is the smallest possible size of the topology, e.g. homology, of the *image* $f(X^n) \subset \mathbb{R}^{n+1}$ and/or of the homologies of the (natural coverings of the) subsets of the l-multiple points of $f(X^n)$?

Geometric Questions about Bordisms. Let X be a closed oriented Riemannian n-manifold with locally bounded geometry, which means that every R-ball in X admits a λ-*bi-Lipschitz homeomorphism* onto the Euclidean R-ball.

Suppose X is bordant to zero and consider all compact Riemannian $(n+1)$-manifolds Y extending $X = \partial(Y)$ with its Riemannian tensor and such that the local geometries of Y are bounded by some *constants* $R' << R$ and $l' >> \lambda$ with the obvious precaution near the boundary.

One can show that the infimum of the volumes of these Y is bounded by

$$\inf_{Y} Vol(Y) \leq F(Vol(X)),$$

with the power exponent bound on the function $F = F(V)$. (F also depends on R, λ, R', λ', but this seems non-essential for $R' << R, \lambda' >> \lambda$.)

What is the true asymptotic behaviour of $F(V)$ for $V \to \infty$?

It may be linear for all we know and the above "dimension shift" picture and/or the construction from [23] may be useful here.

Is there a better setting of this question with some curvature integrals and/or spectral invariants rather than volumes?

The real cohomology of the Grassmann manifolds can be analytically represented by invariant differential forms. Is there a compatible analytic/geometric representation of $\mathcal{B}_n^o \otimes \mathbb{R}$? (One may think of a class of *measurable n-foliations*, see section 10, or, maybe, something more sophisticated than that.)

6 Exotic Spheres.

In 1956, to everybody's amazement, Milnor found smooth manifolds Σ^7 which were *not diffeomorphic to* S^N; yet, each of them was decomposable into the union of two 7-balls $B_1^7, B_2^7 \subset \Sigma^7$ intersecting over their boundaries $\partial(B_1^7) = \partial(B_2^7) = S^6 \subset \Sigma^7$ like in the ordinary sphere S^7.

In fact, this decomposition does imply that Σ^7 is "ordinary" in the *topological* category: such a Σ^7 is (obviously) *homeomorphic* to S^7.

The subtlety resides in the "equality" $\partial(B_1^7) = \partial(B_2^7)$; this identification of the boundaries is far from being the identity map from the point of view of either of the two balls – it does not come from any diffeomorphisms $B_1^7 \leftrightarrow B_2^7$.

The equality $\partial(B_1^7) = \partial(B_2^7)$ can be regarded as a self-diffeomorphism f of the round sphere S^6 – the boundary of standard ball B^7, but this f *does not extend* to a diffeomorphism of B^7 in Milnor's example; otherwise, Σ^7 would be diffeomorphic to S^7. (Yet, f radially extends to a *piecewise smooth homeomorphism* of B^7 which yields a piecewise smooth homeomorphism between Σ^7 and S^7.)

It follows, that such an f *can not* be included into a family of diffeomorphisms bringing it to an isometric transformations of S^6. Thus, any geometric "energy minimizing" flow on the diffeomorphism group $diff(S^6)$ either gets stuck or develops singularities. (It seems little, if anything at all, is known about such flows and their singularities.)

Milnor's spheres Σ^7 are rather innocuous spaces – the boundaries of (the total spaces of) 4-ball bundles $\Theta^8 \to S^4$ in some in some $\mathbb{R}^4$-bundles $V \to S^4$, i.e. $\Theta^8 \subset V$ and, thus, our Σ^7 are certain S^3-bundles over S^4.

All 4-ball bundles, or equivalently $\mathbb{R}^4$-bundles, over S^4 are easy to describe: each is determined by two numbers: the *Euler number e*, that is the self-intersection index of $S^4 \subset \Theta^8$, which assumes all integer values, and the

Pontryagin number p_1 (i.e. the value of the Pontryagin class $p_1 \in H^4(S^4)$ on $[S^4] \in H_4(S^4)$) which may be an *arbitrary even integer.*

(Milnor explicitly construct his fibrations with maps of the 3-sphere into the group $SO(4)$ of orientation preserving linear isometries of $\mathbb{R}^4$ as follows. Decompose S^4 into two round 4-balls, say $S^4 = B^4_+ \cup B^4_-$ with the common boundary $S^3_\partial = B^4_+ \cap B^4_-$ and let $f : s_\partial \mapsto O_\partial \in SO(4)$ be a smooth map. Then glue the boundaries of $B^4_+ \times \mathbb{R}^4$ and $B^4_- \times \mathbb{R}^4$ by the diffeomorphism $(s_\partial, s) \mapsto (s_\partial, O_\partial(s))$ and obtain $V^8 = B^4_+ \times \mathbb{R}^4 \cup_f B^4_- \times \mathbb{R}^4$ which makes an $\mathbb{R}^4$-fibration over S^4.

To construct a specific f, identify $\mathbb{R}^4$ with the quaternion line $\mathbb{H}$ and S^3 with the multiplicative group of quaternions of norm 1. Let $f(s) = f_{ij}(s) \in SO(4)$ act by $x \mapsto s^i x s^j$ for $x \in \mathbb{H}$ and the left and right quaternion multiplication. Then Milnor computes: $e = i + j$ and $p_1 = \pm 2(i - j)$.)

Obviously, all Σ^7 are 2-connected, but $H_3(\Sigma^7)$ may be non-zero (e.g. for the trivial bundle). It is not hard to show that Σ^7 has the same homology as S^7, hence, homotopy equivalent to S^7, if and only if $e = \pm 1$ which means that the selfintersection index of the zero section sphere $S^4 \subset \Theta^8$ equals ± 1; we stick to $e = 1$ for our candidates for Σ^7.

The basic example of Σ^7 with $e = \pm 1$ (the sign depends on the choice of the orientation in Θ^8) is the ordinary 7-sphere which comes with the *Hopf fibration* $S^7 \to S^4$, where this S^7 is positioned as the unit sphere in the quaternion plane $\mathbb{H}^2 = \mathbb{R}^8$, where it is freely acted upon by the group $G = S^3$ of the unit quaternions and where S^7/G equals the sphere S^4 representing the quaternion projective line.

If Σ^7 is diffeomorphic to S^7 one can attach the 8-ball to Θ^8 along this S^7-boundary and obtain a *smooth* closed 8-manifold, say Θ^8_+ .

Milnor observes that the signature of Θ^8_+ equals ± 1, since the homology of Θ^8_+ is represented by a single cycle – the sphere $S^4 \subset \Theta^8 \subset \Theta^8_+$ the selfintersection number of which equals the Euler number.

Then Milnor invokes the *Thom signature theorem*

$$45 sig(X) + p_1^2[X] = 7 p_2[X]$$

and concludes that the number $45 + p_1^2$ must be divisible by 7; therefore, the boundaries Σ^7 of those Θ^8 which fail this condition, say for $p_1 = 4$, must be exotic. (You do not have to know the definition of the Pontryagin classes, just remember they are *integer* cohomology classes.)

Finally, using quaternions, Milnor explicitly constructs a Morse function $\Sigma^7 \to \mathbb{R}$ with only two critical points – maximum and minimum on each Σ^7 with $e = 1$; this yields the two ball decomposition. (We shall explain this in section 8.)

(Milnor's topological arguments, which he presents with a meticulous care, became a common knowledge and can be now found in any textbook; his lemmas look apparent to a to-day topology student. The hardest for the modern reader is the final Milnor's lemma claiming that his function $\Sigma^7 \to \mathbb{R}$ is Morse with two critical points. Milnor is laconic at this point: "It is easy to verify" is all what he says.)

The 8-manifolds Θ^8_+ associated with Milnor's exotic Σ^7 can be triangulated with a single non-smooth point in such a triangulation. Yet, they admit no smooth structures compatible with these triangulations since their combinatorial Pontryagin numbers (defined by Rochlin-Schwartz and Thom) fail the divisibility condition issuing from the Thom formula $sig(X^8) = L_2[X^8]$; in fact, they are not *combinatorially bordant* to smooth manifolds.

Moreover, these Θ^8_+ are not even *topologically* bordant, and therefore, they are non-homeomorphic to smooth manifolds by (slightly refined) Novikov's topological Pontryagin classes theorem.

The number of *homotopy spheres*, i.e. of mutually non-diffeomorphic manifolds Σ^n which are homotopy equivalent to S^n is not that large. In fact, it is *finite* for all $n \neq 4$ by the work of Kervaire and Milnor [39], who, eventually, derive this from the Serre finiteness theorem. (One knows now-a-days that every smooth homotopy sphere Σ^n is *homeomorphic* to S^n according to the solution of the Poincaré conjecture by Smale for $n \geq 5$, by Freedman for $n = 4$ and by Perelman for $n = 3$, where "homeomorphic"$\Rightarrow$ "diffeomorphic" for $n = 3$ by *Moise's theorem.*)

Kervaire and Milnor start by showing that for every *homotopy sphere* Σ^n, there exists a smooth map $f : S^{n+N} \to S^N$, $N >> n$, such that the pullback $f^{-1}(s) \subset S^{n+N}$ of a generic point $s \in S^N$ is diffeomorphic to Σ^n. (The existence of such an f with $f^{-1}(s) = \Sigma^n$ is equivalent to the existence of an *immersion* $\Sigma^n \to \mathbb{R}^{n+1}$ by the *Hirsch theorem.*)

Then, by applying *surgery* (see section 9) to the f_0-pullback of a point for a *given* generic map $f_0 : S^{n+N} \to S^N$, they prove that almost all homotopy classes of maps $S^{n+N} \to S^N$ come from homotopy n-spheres. Namely:

• *If $n \neq 4k+2$, then every homotopy class of maps $S^{n+N} \to S^N$, $N >> n$, can be represented by a "Σ^n-map" f, i.e. where the pullback of a generic point is a homotopy sphere.*

If $n = 4k+2$, then the homotopy classes of "Σ^n-maps" constitute a subgroup in the corresponding stable homotopy group, say $K^\perp_n \subset \pi^{st}_n = \pi_{n+N}(S^N)$, $N >> n$, that has index 1 or 2 and which is expressible in terms of the *Kervaire-(Arf) invariant* classifying (similarly to the signature for $n = 4k$) properly defined "self-intersections" of $(k+1)$-cycles mod 2 in $(4k+2)$-manifolds.

One knows today by the work of Pontryagin, Kervaire-Milnor and Barratt-Jones-Mahowald see [9] that

• *If $n = 2, 6, 14, 30, 62$, then the Kervaire invariant can be non-zero*, i.e. $\pi^{st}_n / K^\perp_n = \mathbb{Z}_2$.

Furthermore,

• *The Kervaire invariant vanishes, i.e. $K^\perp_n = \pi^{st}_n$, for $n \neq 2, 6, 14, 30, 62, 126$* (where it remains unknown if $\pi^{st}_{126}/K^\perp_{126}$ equals $\{0\}$ or $\mathbb{Z}_2$).

In other words,

every continuous map $S^{n+N} \to S^N$, $N >> n \neq 2, 6, ..., 126$, is homotopic to a smooth map $f : S^{n+N} \to S^N$, such that the f-pullback of a generic point is a homotopy n-sphere.

The case $n \neq 2^l - 2$ goes back to Browder (1969) and the case $n = 2^l - 2$, $l \geq 8$ is a recent achievement by Hill, Hopkins and Ravenel [37]. (Their proof relies on a generalized homology theory H^{gen}_n where $H^{gen}_{n+256} = H^{gen}_n$.)

If the pullback of a generic point of a smooth map $f : S^{n+N} \to S^N$, is *diffeomorphic* to S^n, the map f may be *non-contractible.* In fact, the set of the homotopy classes of such f makes a cyclic subgroup in the stable homotopy group of spheres, denoted $J_n \subset \pi_n^{st} = \pi_{n+N}(S^N)$, $N >> n$ (and called the *image of the J-homomorphism* $\pi_n(SO(\infty)) \to \pi_n^{st}$). The order of J_n is 1 or 2 for $n \neq 4k-1$; if $n = 4k-1$, then the order of J_n equals the denominator of $|B_{2k}/4k|$, where B_{2k} is the Bernoulli number. The first non-trivial J are

$$J_1 = \mathbb{Z}_2,\ J_3 = \mathbb{Z}_{24},\ J_7 = \mathbb{Z}_{240},\ J_8 = \mathbb{Z}_2,\ J_9 = \mathbb{Z}_2 \text{ and } J_{11} = \mathbb{Z}_{504}.$$

In general, the homotopy classes of maps f such that the f-pullback of a generic point is diffeomorphic to a given homotopy sphere Σ^n, make a J_n-coset in the stable homotopy group π_n^{st}. Thus the correspondence $\Sigma^n \rightsquigarrow f$ defines a map from the set $\{\Sigma^n\}$ of the diffeomorphism classes of homotopy spheres to the factor group π_n^{st}/J_n, say $\mu : \{\Sigma^n\} \to \pi_n^{st}/J_n$.

The map μ (which, by the above, is surjective for $n \neq 2, 6, 14, 30, 62, 126$) is *finite-to-one* for $n \neq 4$, where the proof of this finiteness for $n \geq 5$ depends on *Smale's h-cobordism theorem*, (see section 8). In fact, the homotopy n-spheres make an Abelian *group* ($n \neq 4$) under the *connected sum operation* $\Sigma_1 \# \Sigma_2$ (see next section) and, by applying surgery to manifolds Θ^{n+1} with boundaries Σ^n, where these Θ^{n+1} (unlike the above Milnor's Θ^8) come as pullbacks of generic points under smooth maps from $(n+N+1)$-balls B^{n+N+1} to S^N, Kervaire and Milnor show that

($\bigstar$) *$\mu : \{\Sigma^n\} \to \pi_n^{st}/J_n$ is a homomorphism with a finite ($n \neq 4$) kernel denoted $\mathcal{B}^{n+1} \subset \{\Sigma^n\}$ which is a cyclic group.*

(The homotopy spheres $\Sigma^n \in \mathcal{B}^{n+1}$ bound $(n+1)$-manifolds with trivial tangent bundles.)

Moreover,

($\star$) *The kernel $\mathcal{B}^{n+1}$ of μ is zero for $n = 2m \neq 4$.*

If $n+1 = 4k+2$, then $\mathcal{B}^{n+1}$ is either zero or $\mathbb{Z}_2$, depending on the Kervaire invariant:

($\star$) *If n equals 1, 5, 13, 29, 61 and, possibly, 125, then $\mathcal{B}^{n+1}$ is zero, and $\mathcal{B}^{n+1} = \mathbb{Z}_2$ for the rest of $n = 4k+1$.*

($\star$) *if $n = 4k-1$, then the cardinality (order) of $\mathcal{B}^{n+1}$ equals $2^{2k-2}(2^{2k-1}-1)$ times the numerator of $|4B_{2k}/k|$, where B_{2k} is the Bernoulli number.*

The above and the known results on the stable homotopy groups π_n^{st} imply, for example, that there are no exotic spheres for $n = 5, 6$, there are 28 mutually non-diffeomorphic homotopy 7-spheres, there are 16 homotopy 18-spheres and 523264 mutually non-diffeomorphic homotopy 19-spheres.

By Perelman, there is a single smooth structure on the homotopy 3 sphere and the case $n = 4$ remains open. (Yet, every homotopy 4-sphere is *homeomorphic* to S^4 by Freedman's solution of the $4D$-Poincaré conjecture.)

7 Isotopies and Intersections.

Besides constructing, listing and classifying manifolds X one wants to understand the topology of spaces of maps $X \to Y$.

The space $[X \to Y]_{smth}$ of *all* C^∞ maps carries little geometric load by itself since this space is homotopy equivalent to $[X \to Y]_{cont(inuous)}$.

An analyst may be concerned with completions of $[X \to Y]_{smth}$, e.g. with Sobolev' topologies while a geometer is keen to study geometric structures, e.g. Riemannian metrics on this space.

But from a differential topologist's point of view the most interesting is the space of *smooth embeddings* $F : X \to Y$ which diffeomorphically send X onto a smooth submanifold $X' = f(X) \subset Y$.

If $dim(Y) > 2dim(X)$ then generic f are embeddings, but, in general, you can not produce them at will so easily. However, given such an embedding $f_0 : X \to Y$, there are plenty of smooth homotopies, called *(smooth) isotopies* f_t, $t \in [0,1]$, of it which remain embeddings for every t and which can be obtained with the following

Isotopy Theorem. (Thom, 1954.) Let $Z \subset X$ be a compact smooth submanifold (boundary is allowed) and $f_0 : X \to Y$ is an embedding, where the essential case is where $X \subset Y$ and f_0 is the identity map.

Then every isotopy of $Z \overset{f_0}{\to} Y$ *can be extended to an isotopy of all of* X*. More generally, the restriction map* $R_{|Z} : [X{\to}Y]_{emb} \to [Z{\to}Y]_{emb}$ *is a fibration; in particular, the isotopy extension property holds for an arbitrary family of embeddings* $X \to Y$ *parametrized by a compact space.*

This is similar to the *homotopy extension property* (mentioned in section 1) for spaces of continuous maps $X \to Y$ – the "geometric" cornerstone of the algebraic topology.)

The proof easily reduces with the implicit function theorem to the case, where $X = Y$ and $dim(Z) = dim(W)$.

Since diffeomorphisms are *open* in the space of all smooth maps, one can extend "small" isotopies, those which only slightly move Z, and since diffeomorphisms of Y make a *group*, the required isotopy is obtained as a composition of small diffeomorphisms of Y. (The details are easy.)

Both "open" and "group" are crucial: for example, homotopies by *locally diffeomorphic* maps, say of a disk $B^2 \subset S^2$ to S^2 do not extend to S^2 whenever a map $B^2 \to S^2$ starts overlapping itself. Also it is much harder (yet possible, [12], [40]) to extend *topological* isotopies, since homeomorphisms are, by no means, open in the space of all continuous maps.

For example if $dim(Y) \geq 2dim(Z) + 2.$ then a generic smooth homotopy of Z is an isotopy: Z does not, generically, cross itself as it moves in Y (unlike, for example, a circle moving in the 3-space where self-crossings are stable under small perturbations of homotopies). Hence, every generic homotopy of Z extends to a smooth isotopy of Y.

Mazur Swindle and Hauptvermutung. Let U_1, U_2 be compact n-manifolds with boundaries and $f_{12} : U_1 \to U_2$ and $f_{21} : U_2 \to U_1$ be embeddings which land in the interiors of their respective target manifolds.

Let W_1 and W_2 be the unions (inductive limits) of the infinite increasing sequences of spaces

$$W_1 = U_1 \subset_{f_{12}} U_2 \subset_{f_{21}} U_1 \subset_{f_{12}} U_2 \subset_{f_{12}} ...$$

and

$$W_2 = U_2 \subset_{f_{21}} U_1 \subset_{f_{12}} U_2 \subset_{f_{12}} U_1 \subset_{f_{12}} ...$$

Observe that W_1 and W_2 are open manifolds without boundaries and that they are diffeomorphic since dropping the first term in a sequence $U_1 \subset U_2 \subset U_3 \subset ...$ does not change the union.

Similarly, both manifolds are diffeomorphic to the unions of the sequences

$$W_{11} = U_1 \subset_{f_{11}} U_1 \subset_{f_{11}} ... \text{ and } W_{22} = U_2 \subset_{f_{22}} U_2 \subset_{f_{22...}}$$

for

$$f_{11} = f_{12} \circ f_{21} : U_1 \to U_1 \text{ and } f_{22} = f_{21} \circ f_{12} : U_2 \to U_2.$$

If the self-embedding f_{11} is isotopic to the identity map, then W_{11} is diffeomorphic to the interior of U_1 by the isotopy theorem and the same applies to f_{22} (or any self-embedding for this matter).

Thus we conclude with the above, that, for example,

open normal neighbourhoods U_1^{op} and U_2^{op} of two homotopy equivalent n-manifolds (and triangulated spaces in general) Z_1 and Z_2 in $\mathbb{R}^{n+N}$, $N \geq n+2$, are diffeomorphic (Mazur 1961).

Anybody might have guessed that the "open" condition is a pure technicality and everybody believed so until Milnor's 1961 counterexample to the *Hauptvermutung* – the main conjecture of the combinatorial topology.

Milnor has shown that there are two free isometric actions A_1 and A_2 of the cyclic group $\mathbb{Z}_p$ on the sphere S^3, for every prime $p \geq 7$, such that

the quotient (lens) spaces $Z_1 = S^3/A_1$ and $Z_2 = S^3/A_2$ are homotopy equivalent, but their closed normal neighbourhoods U_1 and U_2 in any $\mathbb{R}^{3+N}$ are not diffeomorphic. (This could not have happened to *simply connected* manifolds Z_i by the h-cobordism theorem.)

Moreover,

the polyhedra P_1 and P_2 obtained by attaching the cones to the boundaries of these manifolds admit no isomorphic simplicial subdivisions.

Yet, the interiors U_i^{op} of these U_i , $i = 1, 2$, are diffeomorphic for $N \geq 5$. In this case,

P_1 and P_2 are homeomorphic as the one point compactifications of two homeomorphic spaces U_1^{op} and U_2^{op}.

It was previously known that these Z_1 and Z_2 are *homotopy equivalent* (J. H. C. Whitehead, 1941); yet, they are *combinatorially non-equivalent* (Reidemeister, 1936) and, hence, by Moise's 1951 positive solution of the Hauptvermutung for 3-manifolds, *non-homeomorphic.*

There are few direct truly geometric constructions of diffeomorphisms, but those available, are extensively used, e.g. *fiberwise linear diffeomorphisms of vector bundles.* Even the sheer existence of the humble homothety of $\mathbb{R}^n$, $x \mapsto tx$, combined with the isotopy theorem, effortlessly yields, for example, the following

$[B \to Y]$-Lemma. *The space of embeddings f of the n-ball (or $\mathbb{R}^n$) into an arbitrary $Y = Y^{n+k}$ is homotopy equivalent to the space of tangent n-frames in Y; in fact the differential $f \mapsto Df|0$ establishes a homotopy equivalence between the respective spaces.*

For example,

the assignment $f \mapsto J(f)|0$ of the Jacobi matrix at $0 \in B^n$ is a homotopy equivalence of the space of embeddings $f : B \to \mathbb{R}^n$ to the linear group $GL(n)$.

Corollary: Ball Gluing Lemma. Let X_1 and X_2 be $(n+1)$-dimensional manifolds with boundaries Y_1 and Y_2, let $B_1 \subset Y_1$ be a smooth submanifold diffeomorphic to the n-ball and let $f : B_1 \to B_2 \subset Y_2 = \partial(A_2)$ be a diffeomorphism.

If the boundaries Y_i of X_i are connected, the diffeomorphism class of the $(n+1)$-manifold $X_3 = X_1 +_f X_2$ obtained by attaching X_1 to X_2 by f and *(obviously canonically) smoothed at the "corner" (or rather the "crease") along the boundary of B_1, does not depend on B_1 and f.*

This X_3 is denoted $X_1 \#_\partial X_2$. For example, this "sum" of balls, $B^{n+1} \#_\partial B^{n+1}$, is again a smooth $(n+1)$-ball.

Connected Sum. The boundary $Y_3 = \partial(X_3)$ can be defined without any reference to $X_i \supset Y_i$, as follows. Glue the manifolds Y_1 an Y_2 by $f : B_1 \to B_2 \subset Y_2$ and then remove the interiors of the balls B_1 and of its f-image B_2.

If the manifolds Y_i (not necessarily anybody's boundaries or even being closed) are connected, then the resulting *connected sum* manifold is denoted $Y_1 \# Y_2$.

Isn't it a waste of glue? You may be wondering why bother glueing the interiors of the balls if you are going to remove them anyway. Wouldn't it be easier *first* to remove these interiors from both manifolds and *then* glue what remains along the spheres $S_i^{n-1} = \partial(B_i)$?

This *is* easier but also it is also a *wrong* thing to do: the result may depend on the diffeomorphism $S_1^{n-1} \leftrightarrow S_2^{n-1}$, as it happens for $Y_1 = Y_2 = S^7$ in Milnor's example; but the connected sum defined with balls is unique by the $[B{\to}Y]$-lemma.

The ball gluing operation may be used many times in succession; thus, for example, one builds "big $(n+1)$-balls" from smaller ones, where this lemma in lower dimension may be used for ensuring the ball property of the gluing sites.

Gluing and Bordisms. Take two closed oriented n-manifold X_1 and X_2 and let

$$X_1 \supset U_1 \underset{f}{\leftrightarrow} U_2 \subset X_2$$

be an *orientation reversing* diffeomorphisms between compact n-dimensional submanifolds $U_i \subset X_i$, $i = 1, 2$ with boundaries. If we glue X_1 and X_2 by f and remove the (glued together) interiors of U_i the resulting manifold, say $X_3 = X_1 +_{-U} X_2$ is naturally oriented and, clearly, it is orientably bordant to the disjoint union of X_1 and X_2. (This is similar to the geometric/algebraic cancellation of cycles mentioned in section 4.)

Conversely, one can give an alternative definition of the oriented bordism group $\mathcal{B}_n^o$ as of the Abelian group generated by oriented n-manifolds with the relations $X_3 = X_1 + X_2$ for all $X_3 = X_1 +_{-U} X_2$. This gives the same $\mathcal{B}_n^o$ even if the only U allowed are those diffeomorphic to $S^i \times B^{n-i}$ as it follows from the handle decompositions induced by Morse functions.

The isotopy theorem is not dimension specific, but the following construction due to Haefliger (1961) generalizing the *Whitney Lemma* of 1944 demonstrates something special about isotopies in high dimensions.

Let Y be a smooth n-manifold and $X', X'' \subset Y$ be smooth closed submanifolds in general position. Denote $\Sigma_0 = X' \cap X'' \subset Y$ and let X be the (abstract) *disjoint union* of X' and X''. (If X' and X'' are connected equividimensional manifolds, one could say that X is a smooth manifold with its two "connected components" X' and X'' being embedded into Y.)

Clearly,

$$dim(\Sigma_0) = n - k' - k'' \text{ for } n = dim(Y),\ n - k' = dim(X') \text{ and } n - k'' = dim(X'').$$

Let $f_t : X \to Y$, $t \in [0,1]$, be a smooth generic homotopy which disengages X' from X'', i.e. $f_1(X')$ does not intersect $f_1(X'')$, and let

$$\tilde{\Sigma} = \{(x', x'', t)\}_{f_t(x')=f_t(x'')} \subset X' \times X'' \times [0,1],$$

i.e. $\tilde{\Sigma}$ consists of the triples (x', x'', t) for which $f_t(x') = f_t(x'')$.

Let $\Sigma \subset X' \cup X''$ be the union $S' \cup S''$, where $S' \subset X'$ equals the projection of $\tilde{\Sigma}$ to the X'-factor of $X' \times X'' \times [0,1]$ and $S'' \subset X''$ is the projection of $\tilde{\Sigma}$ to X''.

Thus, there is a correspondence $x' \leftrightarrow x''$ between the points in $\Sigma = S' \cup S''$, where the two points correspond one to another if $x' \in S'$ meets $x'' \in S''$ at some moment t_* in the course of the homotopy, i.e.

$$f_{t_*}(x') = f_{t_*}(x'') \text{ for some } t_* \in [0,1].$$

Finally, let $W \subset Y$ be the union of the f_t-paths, denoted $[x' *_t x''] \subset Y$, travelled by the points $x' \in S' \subset \Sigma$ and $x'' \in S'' \subset \Sigma$ until they meet at some moment t_*. In other words, $[x' *_t x''] \subset Y$ consists of the union of the points $f_t(x')$ and $f_t(x'')$ over $t \in [0, t_* = t_*(x') = t_*(x'')]$ and

$$W = \bigcup_{x' \in S'} [x' *_t x''] = \bigcup_{x'' \in S''} [x' *_t x''].$$

Clearly,

$$dim(\Sigma) = \dim(\Sigma_0) + 1 = n - k' - k'' + 1 \text{ and } dim(W) = dim(\Sigma) + 1 = n - k' - k'' + 2.$$

To grasp the picture look at X consisting of a round 2-sphere X' (where $k' = 1$) and a round circle X'' (where $k'' = 2$) in the Euclidean 3-space Y, where X and X' intersect at two points x_1, x_2 – our $\Sigma_0 = \{x_1, x_2\}$ in this case.

When X' an X'' move away one from the other by parallel translations in the opposite directions, their intersection points sweep W which equals the intersection of the 3-ball bounded by X' and the flat 2-disc spanned by X''. The boundary Σ of this W consists of two arcs $S' \subset X'$ and $S'' \subset X''$, where S' joins x_1 with x_2 in X' and S'' join x_1 with x_2 in X''.

Back to the general case, we want W to be, generically, a smooth submanifold *without double points* as well as without any other singularities, except for the unavoidable corner in its boundary Σ, where S' meet S'' along Σ_0. We need for this

$$2dim(W) = 2(n - k' - k'' + 2) < n = dim(Y) \text{ i.e. } 2k' + 2k'' > n + 4.$$

Also, we want to avoid an intersection of W with X' and with X'' away from $\Sigma = \partial(W)$. If we agree that $k'' \geq k'$, this, generically, needs

$$dim(W) + dim(X) = (n - k' - k'' + 2) + (n - k') < n \text{ i.e. } 2k' + k'' > n + 2.$$

These inequalities imply that $k' \geq k \geq 3$, and the lowest dimension where they are meaningful is the first Whitney case: $dim(Y) = n = 6$ and $k' = k'' = 3$.

Accordingly, W is called *Whitney's disk*, although it may be non-homeomorphic to B^2 with the present definition of W (due to Haefliger).

Haefliger Lemma (Whitney for $k + k' = n$). Let the dimensions $n - k' = dim(X')$ and $n - k'' = dim(X'')$, where $k'' \geq k'$, of two submanifolds X' and X'' in the ambient n-manifold Y satisfy $2k' + k'' > n + 2$.

Then every homotopy f_t of (the disjoint union of) X' and X'' in Y which disengages X' from X'', can be replaced by a disengaging homotopy f_t^{new} which is an isotopy, on both manifolds, i.e. $f_t^{new}(X')$ and $f^{new}(X'')$ reman smooth without self intersection points in Y for all $t \in [0, 1]$ and $f_1^{new}(X')$ does not intersect $f_1^{new}(X'')$.

Proof. Assume f_t is smooth generic and take a small neighbourhood $U_{3\varepsilon} \subset Y$ of W. By genericity, this f_t is an isotopy of X' as well as of X'' *within* $U_{3\varepsilon} \subset Y$: the intersections of $f_t(X')$ and $f_t(X'')$ with $U_{3\varepsilon}$, call them $X'_{3\varepsilon}(t)$ and $X''_{3\varepsilon}(t)$ are smooth submanifolds in $U_{3\varepsilon}$ for all t, which, moreover, do not intersect away from $W \subset U_{3\varepsilon}$.

Hence, by the Thom isotopy theorem, there exists an isotopy F_t of $Y \setminus U_\varepsilon$ which equals f_t on $U_{2\varepsilon} \setminus U_\varepsilon$ and which is constant in t on $Y \setminus U_{3\varepsilon}$.

Since f_t and F_t within $U_{3\varepsilon}$ are equal on the overlap $U_{2\varepsilon} \setminus U_\varepsilon$ of their definition domains, they make together a homotopy of X' and X'' which, obviously, satisfies our requirements.

There are several immediate generalizations/applications of this theorem.

(1) One may allow self-intersections Σ_0 *within* connected components of X, where the necessary homotopy condition for removing Σ_0 (which was expressed with the disengaging f_t in the present case) is formulated in terms of maps $\tilde{f} : X \times X \to Y \times Y$ commuting with the involutions $(x_1, x_2) \leftrightarrow (x_2, x_1)$ in $X \times X$ and $(y_1, y_2) \leftrightarrow (y_2, y_1)$ in $Y \times Y$ and having the pullbacks $\tilde{f}^{-1}(Y_{diag})$ of the diagonal $Y_{diag} \subset Y \times Y$ equal $X_{diag} \subset X \times X$, [33].

(2) One can apply all of the above to p parametric families of maps $X \to Y$, by paying the price of the extra p in the excess of $dim(Y)$ over $dim(X)$, [33].

If $p = 1$, this yield an isotopy classification of embeddings $X \to Y$ for $3k > n+3$ by homotopies of the above symmetric maps $X \times X \to Y \times Y$, which shows, for example, that there are *no knots* for these dimensions (Haefliger, 1961).

if $3k > n + 3$, then every smooth embedding $S^{n-k} \to \mathbb{R}^n$ is smoothly isotopic to the standard $S^{n-k} \subset \mathbb{R}^n$.

But if $3k = n + 3$ and $k = 2l + 1$ is odd then there are

infinitely many isotopy of classes of embeddings $S^{4l-1} \to \mathbb{R}^{6l}$, (Haefliger 1962).

Non-triviality of such a knot $S^{4l-1} \to \mathbb{R}^{6l}$ is detected by showing that a map $f_0 : B^{4l} \to \mathbb{R}^{6l} \times \mathbb{R}_+$ extending $S^{4l-1} = \partial(B^{4l})$ can not be turned into an embedding, keeping it transversal to $\mathbb{R}^{6l} = \mathbb{R}^{6l} \times 0$ and with its boundary equal our knot $S^{4l-1} \subset \mathbb{R}^{6l}$.

The Whitney-Haefliger W for f_0 has dimension $6l + 1 - 2(2l + 1) + 2 = 2l + 1$ and, generically, it transversally intersects B^{4l} at several points.

The resulting (properly defined) intersection index of W with B is non-zero (otherwise one could eliminate these points by Whitney) and it does not depend on f_0. In fact, it equals the linking invariant of Haefliger. (This is reminiscent of the "higher linking products" described by Sullivan's minimal models, see section 9.)

(3) In view of the above, one must be careful if one wants to relax the dimen-

sion constrain by an inductive application of the Whitney-Haefliger disengaging procedure, since obstructions/invariants for removal "higher" intersections which come on the way may be not so apparent. (The structure of "higher self-intersections" of this kind for Euclidean hypersurfaces carries a significant information on the stable homotopy groups of spheres.)

But this is possible, at least on the $\mathbb{Q}$-level, where one has a comprehensive algebraic control of self-intersections of all multiplicities for maps of codimension $k \geq 3$. Also, even without tensoring with $\mathbb{Q}$, the higher intersection obstructions tend to vanish in the combinatorial category.

For example,

there is no combinatorial knots of codimension $k \geq 3$ (Zeeman, 1963).

The essential mechanism of knotting $X = X^n \subset Y = Y^{n+2}$ depends on the fundamental group Γ of the complement $U = Y \subset X$. The group Γ may look a nuisance when you want to untangle a knot, especially a surface X^2 in a 4-manifold, but these $\Gamma = \Gamma(X)$ for various $X \subset Y$ form beautifully intricate patterns which are poorly understood.

For example, the groups $\Gamma = \pi_1(U)$ capture *the étale cohomology of algebraic manifolds* and the *Novikov-Pontryagin classes of topological manifolds* (see section 10). Possibly, the groups $\Gamma(X^2)$ for surfaces $X^2 \subset Y^4$ have much to tell us about the smooth topology of 4-manifolds.

There are few systematic ways of constructing "simple" $X \subset Y$, e.g. immersed submanifolds, with "interesting" (e.g. far from being free) fundamental groups of their complements.

Offhand suggestions are pullbacks of (special singular) divisors X_0 in complex algebraic manifolds Y_0 under generic maps $Y \to Y_0$ and immersed subvarieties X^n in cubically subdivided Y^{n+2}, where X^n are made of n-sub-cubes $\square^n$ *inside* the cubes $\square^{n+2} \subset Y^{n+2}$ and where these interior $\square^n \subset \square^{n+2}$ are parallel to the n-faces of $\square^{n+2}$.

It remains equally unclear what is the possible topology of self-intersections of immersions $X^n \to Y^{n+2}$, say for $S^3 \to S^5$, where the self-intersection makes a link in S^3, and for $S^4 \to S^6$ where this is an immersed surface in S^4.

(4) One can control the position of the image of $f^{new}(X) \subset Y$, e.g. by making it to land in a given open subset $W_0 \subset W$, if there is no homotopy obstruction to this.

The above generalizes and simplifies in the combinatorial or "piecewise smooth" category, e.g. for "unknotting spheres", where the basic construction is as follows

Engulfing. Let X be a piecewise smooth polyhedron in a smooth manifold Y.

If $n - k = dim(X) \leq dim(Y) - 3$ *and if* $\pi_i(Y) = 0$ *for* $i = 1, ...dim(Y)$, *then there exists a smooth isotopy* F_t *of* Y *which eventually (for* $t = 1$*) moves* X *to a given (small) neighbourhood* $B_\circ$ *of a point in* Y.

Sketch of the Proof. Start with a generic f_t. This f_t does the job away from a certain W which has $dim(W) \leq n - 2k + 2$. This is $< dim(X)$ under the above assumption and the proof proceeds by induction on $dim(X)$.

This is called "engulfing" since $B_\circ$, when moved by the time reversed isotopy, engulfs X; engulfing was invented by Stallings in his approach to the *Poincaré Conjecture* in the combinatorial category, which goes, roughly, as follows.

Let Y be a smooth n-manifold. Then, with a simple use of two mutually dual smooth triangulations of Y, one can decompose Y, for each i, into the union of *regular neighbourhoods* U_1 and U_2 of smooth subpolyhedra X_1 and X_2 in Y of dimensions i and $n-i-1$ (similarly to the *handle body decomposition* of a 3-manifold into the union of two thickened graphs in it), where, recall, a neighbourhood U of an $X \subset Y$ is regular if there exists an isotopy $f_t : U \to U$ which brings all of U arbitrarily close to X.

Now let Y be a homotopy sphere of dimension $n \geq 7$, say $n = 7$, and let $i = 3$ Then X_1 and X_2, and hence U_1 and U_2, can be engulfed by (diffeomorphic images of) balls, say by $B_1 \supset U_1$ and $B_2 \supset U_2$ with their centers denoted $0_1 \in B_1$ and $0_2 \in B_2$.

By moving the 6-sphere $\partial(B_1) \subset B_2$ by the radial isotopy in B_2 toward 0_2, one represents $Y \smallsetminus 0_2$ by the union of an increasing sequence of isotopic copies of the ball B_1. This implies (with the isotopy theorem) that $Y \smallsetminus 0_2$ is diffeomorphic to $\mathbb{R}^7$, hence, Y is homeomorphic to S^7.

(A refined generalization of this argument delivers the Poincaré conjecture in the combinatorial and topological categories for $n \geq 5$. See [68] for an account of techniques for proving various "Poincaré conjectures" and for references to the source papers.)

8 Handles and h-Cobordisms.

The original approach of Smale to the Poincaré conjecture depends on *handle decompositions* of manifolds – counterparts to cell decompositions in the homotopy theory.

Such decompositions are more flexible, and by far more abundant than triangulations and they are better suited for a match with algebraic objects such as homology. For example, one can sometimes realize a basis in homology by suitably chosen cells or handles which is not even possible to formulate properly for triangulations.

Recall that an *i-handle of dimension n* is the ball B^n decomposed into the product $B^n = B^i \times B^{n-i}(\varepsilon)$ where one think of such a handle as an ε-thickening of the unit i-ball and where

$$A(\varepsilon) = S^i \times B^{n-1}(\varepsilon) \subset S^{n-1} = \partial B^n$$

is seen as an ε-neighbourhood of its *axial $(i-1)$-sphere* $S^{i-1} \times 0$ – *an equatorial i-sphere in S^{n-1}.*

If X is an n-manifold with boundary Y and $f : A(\varepsilon) \to Y$ a smooth embedding, one can attach B^n to X by f and the resulting manifold (with the "corner" along $\partial A(\varepsilon)$ made smooth) is denoted $X +_f B^n$ or $X +_{S^{i-1}} B^n$, where the latter subscript refers to the f-image of the axial sphere in Y.

The effect of this on the boundary, i.e. modification

$$\partial(X) = Y \rightsquigarrow_f Y' = \partial(X +_{S^{i-1}} B^n)$$

does not depend on X but only on Y and f. It is called an *i-surgery of Y at the sphere* $f(S^{i-1} \times 0) \subset Y$.

The manifold $X = Y \times [0,1] +_{S^{i-1}} B^n$, where B^n is attached to $Y \times 1$, makes a bordism between $Y = Y \times 0$ and Y' which equals the surgically modified $Y \times 1$-component of the boundary of X. If the manifold Y is oriented, so is X, unless

$i = 1$ and the two ends of the 1-handle $B^1 \times B^{n-1}(\varepsilon)$ are attached to the same connected component of Y with opposite orientations.

When we attach an i-handle to an X along a *zero-homologous* sphere $S^{i-1} \subset Y$, we create a new i-cycle in $X +_{S^{i-1}} B^n$; when we attach an $(i+1)$-handle along an i-sphere in X which is *non-homologous to zero*, we "kill" an i-cycle.

These creations/annihilations of homology may cancel each other and a handle decomposition of an X may have by far more handles (balls) than the number of independent homology classes in $H_*(X)$.

Smale's argument proceeds in two steps.

(1) The overall algebraic cancellation is decomposed into "elementary steps" by "reshuffling" handles (in the spirit of J.H.C. Whitehead's theory of the *simple homotopy type*);

(2) each elementary step is implemented geometrically as in the example below (which does not elucidate the case $n = 6$).

Cancelling a 3*-handle by a* 4*-handle.* Let $X = S^3 \times B^4(\varepsilon_0)$ and let us attach the 4-handle $B^7 = B^4 \times B^3(\varepsilon)$, $\varepsilon << \varepsilon_0$, to the (normal) ε-neighbourhood $A_\sim$ of some sphere

$$S^3_\sim \subset Y = \partial(X) = S^3 \times S^3(\varepsilon_0) \text{ for } S^3(\varepsilon_0) = \partial B^4(\varepsilon_0).$$

by some diffeomorphism of $A(\varepsilon) \subset \partial(B^7)$ onto $A_\sim$.

If $S^3_\sim = S^3 \times b_0$, $b_0 \in S^3(\varepsilon_0)$, is the standard sphere, then the resulting $X_\sim = X +_{S^3_\sim} B^7$ is obviously diffeomorphic to B^7: adding $S^3 \times B^4(\varepsilon_0)$ to B^7 amounts to "bulging" the ball B^7 over the ε-neighbourhood $A(\varepsilon)$ of the axial 3-sphere on its boundary.

Another way to see it is by observing that this addition of $S^3 \times B^4(\varepsilon_0)$ to B^7 can be decomposed into gluing two balls in succession to B^7 as follows.

Take a ball $B^3(\delta) \subset S^3$ around some point $s_0 \in S^3$ and decompose $X = S^3 \times B^4(\varepsilon_0)$ into the union of two balls that are

$$B^7_\delta = B^3(\delta) \times B^4(\varepsilon_0)$$

and

$$B^7_{1-\delta} = B^3(1-\delta) \times B^4(\varepsilon_0) \text{ for } B^3(1-\delta) =_{def} S^3 \setminus B^3(\delta).$$

Clearly, the attachment loci of $B^7_{1-\delta}$ to X and of B^7_δ to $X + B^7_{1-\delta}$ are diffeomorphic (after smoothing the corners) to the 6-ball.

Let us modify the sphere $S^3 \times b_0 \subset S^3 \times B^4(\varepsilon_0) = \partial(X)$ by replacing the original standard embedding of the 3-ball

$$B^3(1-\delta) \to B^7_{1-\delta} = B^3(1-\delta) \times S^3(\varepsilon_0) \subset \partial(X)$$

by another one, say,

$$f_\sim : B^3(1-\delta) \to B^7_{1-\delta} = B^3(1-\delta) \times S^3(\varepsilon_0) = \partial(X),$$

such that $f_\sim$ equals the original embedding near the boundary of $\partial(B^3(1-\delta)) = \partial(B^3(\delta)) = S^2(\delta)$.

Then the same "ball after ball" argument applies, since the first gluing site where $B^7_{1-\delta}$ is being attached to X, albeit "wiggled", remains diffeomorphic to B^6 by the isotopy theorem, while the second one does not change at all. So we conclude:

whenever $S^3_\sim \subset S^3 \times S^3(\varepsilon_0)$ transversally intersect $s_0 \times S^3(\varepsilon_0)$, $s_0 \in S^3$, at a single point, the manifold $X_\sim = X +_{S^3_\sim} B^7$ is diffeomorphic to B^7.

Finally, by Whitney's lemma, every embedding $S^3 \to S^3 \times S^3(\varepsilon_0) \subset S^3 \times B^4(\varepsilon_0)$ which is *homologous in* $S^3 \times B^4(\varepsilon_0)$ to the standard $S^3 \times b_0 \subset S^3 \times B^4(\varepsilon_0)$, can be isotoped to another one which meets $s_0 \times S^3(\varepsilon_0)$ transversally at a single point. Hence,

the handles do cancel one another: if a sphere

$$S^3_\sim \subset S^3 \times S^3(\varepsilon_0) = \partial(X) \subset X = S^3 \times B^4(\varepsilon_0),$$

is homologous in X to

$$S^3 \times b_0 \subset X = S^3 \times B^4(\varepsilon_0),\ b_0 \in B^4(\varepsilon),$$

then the manifold $X +_{S^3_\sim} B^7$ is diffeomorphic to the 7-ball.

Let us show in this picture that Milnor's sphere Σ^7 minus a small ball is diffeomorphic to B^7. Recall that Σ^7 is fibered over S^4, say by $p : \Sigma^7 \to S^4$, with S^3-fibers and with the Euler number $e = \pm 1$.

Decompose S^4 into two round balls with the common S^3-boundary, $S^4 = B^4_+ \cup B^4_-$. Then Σ^7 decomposes into $X_+ = p^{-1}(B^4_+) = B^4_+ \times S^3$ and $X_- = p^{-1}(B^4_-) = B^4_- \times S^3$, where the gluing diffeomorphism between the boundaries $\partial(X_+) = S^3_+ \times S^3$ and $\partial(X_-) = S^3_- \times S^3$ for $S^3_\pm = \partial B^4_\pm$, is homologically the same as for the Hopf fibration $S^7 \to S^4$ for $e = \pm 1$.

Therefore, if we decompose the S^3-factor of $B^4_- \times S^3$ into two round balls, say $S^3 = B^3_1 \cup B^3_2$, then either $B^4_- \times B^3_1$ or $B^4_- \times B^3_2$ makes a 4-handle attached to X_+ to which the handle cancellation applies and shows that $X_+ \cup (B^4_- \times B^3_1)$ is a smooth 7-ball. (All what is needed of the Whitney's lemma is obvious here: the zero section $X \subset V$ in an oriented $\mathbb{R}^{2k}$-bundle $V \to X = X^{2k}$ with $e(V) = \pm 1$ can be perturbed to $X' \subset V$ which *transversally* intersect X at a *single* point.)

The handles shaffling/cancellation techniques do not solve the existence problem for diffeomorphisms $Y \leftrightarrow Y'$ but rather reduce it to the existence of *h-cobordisms* between manifolds, where a compact manifold X with two boundary components Y and Y' is called an h-cobordism (between Y and Y') if the inclusion $Y \subset X$ is a homotopy equivalence.

Smale h-Cobordism Theorem. *If an h-cobordism has $dim(X) \geq 6$ and $\pi_1(X) = 1$ then X is diffeomorphic to $Y \times [0,1]$, by a diffeomorphism keeping $Y = Y \times 0 \subset X$ fixed. In particular, h-cobordant simply connected manifolds of dimensions ≥ 5 are diffeomorphic.*

Notice that the Poincaré conjecture for the homotopy spheres Σ^n, $n \geq 6$, follows by applying this to Σ^n minus two small open balls, while the case $m = 1$ is solved by Smale with a construction of an h-cobordism between Σ^5 and S^5.

Also Smale's handle techniques deliver the following geometric version of the Poincaré connectedness/contractibilty correspondence (see section 4).

Let X be a closed n-manifold, $n \geq 5$, with $\pi_i(X) = 0$, $i = 1, ..., k$. Then X contains a $(n-k-1)$-dimensional smooth sub-polyhedron $P \subset X$, such that the complement of the open (regular) neighbourhood $U_\varepsilon(P) \subset X$ of P is diffeomorphic to the n-ball, (where the boundary $\partial(U_\varepsilon)$ is the $(n-1)$-sphere "ε-collapsed" onto $P = P^{n-k-1}$).

If $n = 5$ and if the *normal bundle of X embedded into some $\mathbb{R}^{5+N}$ is trivial*, i.e. if the normal Gauss map of X to the Grassmannian $Gr(\mathbb{R}^{5+N})$ is contractible, then Smale proves, assuming $\pi_1(X) = 1$, that

one can choose $P = P^3 \subset X = X^5$ that equals the union of a smooth topological segment $s = [0,1] \subset X$ and several spheres S_i^2 and S_i^3, where each S_i^3 meets s at one point, and also transversally intersects S_i^2 at a single point and where there are no other intersections between s, S_i^2 and S_i^3.

In other words,

(Smale 1965) *X is diffeomorphic to the connected sum of several copies of $S^2 \times S^3$.*

The triviality of the bundle in this theorem is needed to ensure that all embedded 2-spheres in X have trivial normal bundles, i.e. their normal neighbourhoods split into $S^2 \times \mathbb{R}^3$ which comes handy when you play with handles.

If one drops this triviality condition, one has

Classification of Simply Connected 5-Manifolds. (Barden 1966) *There is a finite list of explicitly constructed 5-manifolds X_i, such that every closed simply connected manifold X is diffeomorphic to the connected sum of X_i.*

This is possible, in view of the above Smale theorem, since all simply connected 5-manifolds X have "almost trivial" normal bundles e.g. their only possible Pontryagin class $p_1 \in H^4(X)$ is zero. Indeed $\pi_1(X) = 1$ implies that $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)] = 0$ and then $H^4(X) = H_1(X) = 0$ by the Poincaré duality.

When you encounter bordisms, the generecity sling launches you to the stratosphere of algebraic topology so fast that you barely discern the geometric string attached to it.

Smale's cells and handles, on the contrary, feel like slippery amebas which merge and disengage as they reptate in the swamp of unruly geometry, where n-dimensional cells continuously collapse to lower dimensional ones and keep squeezing through paper-thin crevices. Yet, their motion is governed, for all we know, by the rules dictated by some *algebraic K-theory* (theories?)

This motion hardly can be controlled by any traditional geometric flow. First of all, the "simply connected" condition can not be encoded in geometry ([53], [28] [54] and also breaking the symmetry by dividing a manifold into handles along with "genericity" poorly fare in geometry.

Yet, some generalized "Ricci flow with partial collapse and surgeries" in the "space of (generic, random?) amebas" might split away whatever it fails to untangle and bring fresh geometry into the picture.

For example, take a compact locally symmetric space $X_0 = S/\Gamma$, where S is a non-compat irreducible symmetric space of rank $\geq$ and make a 2-surgery along some non-contractible circle $S^1 \subset X_0$. The resulting manifold X has finite fundamental group by Margulis' theorem and so a finite covering $\tilde{X} \to X$ is simply connected. What can a geometric flow do to these X_0 and $\tilde{X}$? Would it bring X back to X_0?

9 Manifolds under Surgery.

The Atiyah-Thom construction and Serre's theory allows one to produce "arbitrarily large" manifolds X for the *m-domination* $X_1 \succ_m X_2$, $m > 0$, meaning that there is a map $f : X_1 \to X_2$ of degree m.

Every such f between closed connected oriented manifolds induces a *surjective* homomorphisms $f_{*i} : H_i(X_1;\mathbb{Q}) \to H_i(X_1;\mathbb{Q})$ for all $i = 0, 1, ..., n$, (as we know from section 4), or equivalently, an injective cohomology homomorphism $f^{*i} : H^i(X_2;\mathbb{Q}) \to H^i(X_2;\mathbb{Q})$.

Indeed, by the Poincaré $\mathbb{Q}$-duality, the cup-product (this the common name for the product on cohomology) pairing $H^i(X_2;\mathbb{Q}) \otimes H^{n-i}(X_2;\mathbb{Q}) \to \mathbb{Q} = H^n(X_2;\mathbb{Q})$ is faithful; therefore, if f^{*i} vanishes, then so does f^{*n}. But the latter amounts to multiplication by $m = deg(f)$,

$$H^n(X_2;\mathbb{Q}) = \mathbb{Q} \to_{\cdot d} \mathbb{Q} = H^n(X_1;\mathbb{Q}).$$

(The main advantage of the cohomology product over the intersection product on homology is that the former is preserved by all continuous maps,

$f^{*i+j}(c_1 \cdot c_2) = f^{*i}(c_1) \cdot f^{*j}(c_2)$ for all $f : X \to Y$ and all $c_1 \in H^i(Y)$, $c_2 \in H^j(Y)$.)

If $m = 1$, then (by the full cohomological Poincaré duality) the above remains true for all coefficient fields $\mathbb{F}$; moreover, the induced homomorphism $\pi_i(X_1) \to \pi_i(X_2)$ is surjective as it is seen by looking at the lift of $f : X_1 \to X_2$ to the induced map from the covering $\tilde{X}_1 \to X_1$ induced by the universal covering $\tilde{X}_2 \to X_2$ to $\tilde{X}_2$. (A map of degree $m > 1$ sends $\pi_1(X_1)$ to a subgroup in $\pi_1(X_2)$ of a finite index dividing m.)

Let us construct manifolds starting from *pseudo-manifolds*, where a compact oriented n-dimensional pseudo-manifold is a triangulated n-space X_0, such that

• every simplex of dimension $< n$ in X_0 lies in the boundary of an n-simplex,

• The complement to the union of the $(n-2)$-simplices in X_0 is an oriented manifold.

Pseudo-manifolds are infinitely easier to construct and to recognize than manifolds: essentially, these are simplicial complexes with exactly two n-simplices adjacent to every $(n-1)$-simplex.

There is no comparably simple characterization of triangulated n-manifolds X where the links $L^{n-i-1} = L_{\Delta^i} \subset X$ of the i-simplices must be topological $(n-i-1)$-spheres. But even deciding if $\pi_1(L^{n-i-1}) = 1$ is *an unsolvable problem* except for a couple of low dimensions.

Accordingly, it is very hard to produce manifolds by combinatorial constructions; yet, one can "dominate" any pseudo-manifold by a manifold, where, observe, the notion of degree perfectly applies to oriented pseudo-manifolds.

Let X_0 be a connected oriented n-pseudomanifold. Then there exists a smooth closed connected oriented manifold X and a continuous map $f : X \to X_0$ of degree $m > 0$.

Moreover, given an oriented $\mathbb{R}^N$-bundle $V_0 \to X_0$, $N \geq 1$, one can find an m-dominating X, which also admits a smooth embedding $X \subset \mathbb{R}^{n+N}$, such that our $f : X \to X_0$ of degree $m > 0$ induces the normal bundle of X from V_0.

Proof. Since that the first $N-1$ homotopy groups of the Thom space of $V_\bullet$ of V_0 vanish (see section 5), Serre's m-sphericity theorem delivers a map $f_\bullet : S^{n+N} \to V_\bullet$ a non-zero degree m, provided $N > n$. Then the "generic pullback" X of $X_0 \subset V_0$ (see section 3) does the job as it was done in section 5 for Thom's bordisms.

In general, if $1 \leq N \leq n$, the m-sphericity of the fundamental class $[V_\bullet] \in H_{n+N}(V_\bullet)$ is proven with the *Sullivan's minimal models*, see theorem 24.5 in [19]

The minimal model, of a space X is a *free (skew)commutative differential algebra* which, in a way, extends the cohomology algebra of X and which faithfully encodes *all* homotopy $\mathbb{Q}$-invariants of X. If X is a smooth N-manifold it can be seen in terms of "higher linking" in X.

For example, if two cycles $C_1, C_2 \subset X$ of codimensions i_1, i_2, satisfy $C_1 \sim 0$ and $C_1 \cap C_2 = 0$, then the (first order) *linking class* between them is an element in the quotient group $H_{N-i_1-i_2-1}(X)/(H_{N-i_1-1}(X) \cap [C_2])$ which is defined with a plaque $D_1 \in \partial^{-1}(C_1)$, i.e. such that $\partial(D_1) = C_1$, as the image of $[D_1 \cap C_2]$ under the quotient map

$$H_{N-i_1-i_2-1}(X) \ni [D_1 \cap C_2] \mapsto H_{N-i_1-i_2-1}(X)/(H_{N-i_1-1}(X) \cap [C_2]).$$

Surgery and the Browder-Novikov Theorem (1962 [8],[55]). Let X_0 be a smooth closed simply connected oriented n-manifold, $n \geq 5$, and $V_0 \to X_0$ be a *stable* vector bundle where "stable" means that $N = rank(V) >> n$. We want to modify the smooth structure of X_0 keeping its homotopy type unchanged but with its original normal bundle in $\mathbb{R}^{n+N}$ replaced by V_0.

There is an obvious algebraic-topological obstruction to this highlighted by Atiyah in [2] which we call $[V_\bullet]$-*sphericity* and which means that there exists a *degree one*, map $f_\bullet$ of S^{n+N} to the Thom space $V_\bullet$ of V_0, i.e. $f_\bullet$ sends the generator $[S^{n+N}] \in H_{n+N}(S^{n+N}) = \mathbb{Z}$ (for some orientation of the sphere S^{n+N}) to the fundamental class of the Thom space, $[V_\bullet \in H_{n+N}(V_\bullet) = \mathbb{Z}$, which is distinguished by the orientation in X. (One has to be pedantic with orientations to keep track of possible/impossible algebraic cancellations.)

However, this obstruction is "$\mathbb{Q}$-nonessential", [2] : the set of the vector bundles admitting such an $f_\bullet$ constitutes a coset of a subgroup of *finite index* in Atiyah's (reduced) K-*group* by Serre's finiteness theorem.

Recall that $K(X)$ is the Abelian group formally generated by the isomorphism classes of vector bundles V over X, where $[V_1] + [V_2] =_{def} 0$ whenever the *Whitney sum* $V_1 \oplus V_2$ is isomorphic to a trivial bundle.

The *Whitney sum* of an $\mathbb{R}^{n_1}$-bundle $V_1 \to X$ with an $\mathbb{R}^{n_2}$-bundle $V_2 \to X$, is the $\mathbb{R}^{n_1+n_2}$-bundle over X. which equals the fiber-wise Cartesian product of the two bundles.

For example the Whitney sum of the tangent bundle of a smooth submanifold $X^n \subset W^{n+N}$ and of its normal bundle in W equals the tangent bundle of W restricted to X. Thus, it is *trivial* for $W = \mathbb{R}^{n+N}$, i.e. it isomorphic to $\mathbb{R}^{n+N} \times X \to X$, since the tangent bundle of $\mathbb{R}^{n+N}$ is, obviously, trivial.

Granted an $f_\bullet : S^{n+N} \to V_\bullet$ of degree 1, we take the "generic pullback" X of X_0,

$$X \subset \mathbb{R}^{n+N} \subset \mathbb{R}_\bullet^{n+N} = S^{n+N},$$

and denote by $f : X \to X_0$ the restriction of $f_\bullet$ to X, where, recall, f induces the normal bundle of X from V_0. .

The map $f : X_1 \to X_0$, which is clearly *onto*, is far from being injective – it may have uncontrollably complicated folds. In fact, it is not even a homotopy equivalence – the homology homomorphism induced by f

$$f_{*i} : H_i(X_1) \to H_i(X_0),$$

is, as we know, surjective and it may (and usually does) have non-trivial kernels $ker_i \subset H_i(X_1)$. However, these kernels can be "killed" by a "surgical implementation" of the obstruction theory (generalizing the case where $X_0 = S^n$ due to Kervaire-Milnor) as follows.

Assume $ker_i = 0$ for $i = 0, 1, ..., k-1$, invoke Hurewicz' theorem and realize the cycles in ker_k by k-spheres mapped to X_1, where, observe, the f-images of these spheres are contractible in X_0 by a relative version of the (elementary) Hurewicz theorem.

Furthermore, if $k < n/2$, then these spheres $S^k \subset X_1$ are generically *embedded* (no self-intersections) and have *trivial* normal bundles in X_1, since, essentially, they come from $V \to X_1$ via contractible maps. Thus, small neighbourhoods (ε-annuli) $A = A_\varepsilon$ of these spheres in X_1 split: $A = S^k \times B^{n-k}_\varepsilon \subset X_1$.

It follows, that the corresponding spherical cycles can be killed by $(k+1)$-surgery (where X_1 now plays the role of Y in the definition of the surgery); moreover, it is not hard to arrange a map of the resulting manifold to X_0 with the same properties as f.

If $n = dim(X_0)$ is odd, this works up to $k = (n-1)/2$ and makes *all* ker_i, including $i > k$, equal zero by the Poincaré duality.

Since

a continuous map between *simply connected* spaces which induces an *isomorphism on homology* is a *homotopy equivalence* by the (elementary) *Whitehead theorem*,

the resulting manifold X is a *homotopy equivalent* to X_0 via our surgically modified map f, call it $f_{srg} : X \to X_0$.

Besides, by the construction of f_{srg}, this map induces the normal bundle of X from $V \to X_0$. Thus we conclude,

the Atiyah $[V_\bullet]$*-sphericity is the only condition for realizing a stable vector bundle* $V_0 \to X_0$ *by the normal bundle of a smooth manifold* X *in the homotopy class of a given odd dimensional simply connected manifold* X_0.

If n is even, we need to kill k-spheres for $k = n/2$, where an extra obstruction arises. For example, if k is even, the surgery does not change the signature; therefore, the Pontryagin classes of the bundle V *must satisfy* the Rokhlin-Thom-Hirzebruch formula to start with.

(There is an additional constrain for the tangent bundle $T(X)$ – the equality between the Euler characteristic $\chi(X) = \sum_{i=0,...,n} (-1)^i rank_{\mathbb{Q}}(H_i(X))$ and the Euler number $e(T(X))$ that is the self-intersection index of $X \subset T(X)$.)

On the other hand the equality $L(V)[X_0] = sig(X_0)$ (obviously) implies that $sig(X) = sig(X_0)$. It follows that

the intersection form on $ker_k \subset H_k(X)$ *has zero signature*,

since all $h \in ker_k$ have zero intersection indices with the pullbacks of k-cycles from X_0.

Then, assuming $ker_i = 0$ for $i < k$ and $n \neq 4$, one can use Whitney's lemma and realize a basis in $ker_k \subset H_k(X_1)$ by $2m$ *embedded* spheres $S^k_{2j-1}, S^k_{2j} \subset X_1$, $i = 1, ...m$, which have zero self-intersection indices, one point crossings between S^k_{2j-1} and S^k_{2j} and no other intersections between these spheres.

Since the spheres $S^k \subset X$ with $[S^k] \in ker_k$ have trivial *stable* normal bundles $U^\perp$ (i.e. their Whitney sums with trivial 1-bundles, $U^\perp \oplus \mathbb{R}$, are trivial), the normal bundle $U^\perp = U^\perp(S^k)$ of such a sphere S^k is trivial if and only if the Euler number $e(U^\perp)$ vanishes.

Indeed any oriented k-bundle $V \to B$, such that $V \times \mathbb{R} = B \times \mathbb{R}^{k+1}$, is induced from the tautological bundle V_0 over the oriented Grassmannian $Gr_k^{or}(\mathbb{R}^{k+1})$, where $Gr_k^{or}(\mathbb{R}^{k+1}) = S^k$ and V_0 is the tangent bundle $T(S^k)$. Thus, the Euler class of V is induced from that of $T(S^k)$ by the classifying map, $G : B \to S^k$. If $B = S^k$ then the Euler number of $e(V)$ equals $2deg(G)$ and if $e(V) = 0$ the map G is contractible which makes $V = S^k \times \mathbb{R}^k$.

Now, observe, $e(U^\perp(S^k))$ is conveniently equal to the self-intersection index of S^k in X. ($e(U^\perp(S^k))$ equals, by definition, the self-intersection of $S^k \subset U^\perp(S^k)$ which is the same as the self-intersection of this sphere in X.)

Then it easy to see that the $(k+1)$-surgeries applied to the spheres S^k_{2j}, $j = 1, ..., m$, kill all of ker_k and make $X \to X_0$ a homotopy equivalence.

There are several points to check (and to correct) in the above argument, but everything fits amazingly well in the lap of the linear algebra (The case of odd k is more subtle due to the Kervaire-Arf invariant.)

Notice, that our starting X_0 does not need to be a manifold, but rather a *Poincaré (Browder) n-space*, i.e. a finite cell complex satisfying the Poincaré duality: $H_i(X_0, \mathbb{F}) = H^{n-i}(X_0, \mathbb{F})$ for all coefficient fields (and rings) $\mathbb{F}$, where these "equalities" must be coherent in an obvious sense for different $\mathbb{F}$.

Also, besides the *existence* of smooth n-manifolds X, the above *surgery argument* applied to a bordism Y between homotopy equivalent manifolds X_1 and X_2. Under suitable conditions on the normal bundle of Y, such a bordism can be surgically modified to an h-cobordism. Together with the h-cobordism theorem, this leads to an algebraic *classification* of smooth structures on simply connected manifolds of dimension $n \geq 5$. (see [55]).

Then the Serre finiteness theorem implies that

there are at most finitely many smooth closed simply connected n-manifolds X in a given a homotopy class and with given Pontryagin classes $p_k \in H^{4k}(X)$.

Summing up, the question "What are manifolds?" has the following

1962 Answer. *Smooth closed simply connected n-manifolds for* $n \geq 5$*, up to a "finite correction term", are "just" simply connected Poincaré n-spaces X with distinguished cohomology classes* $p_i \in H^{4i}(X)$*, such that* $L_k(p_i)[X] = sig(X)$ *if* $n = 4k$.

This is a fantastic answer to the "manifold problem" undreamed of 10 years earlier. Yet,

• Poincaré spaces are not classifiable. Even the candidates for the cohomology rings are not classifiable over $\mathbb{Q}$.

Are there special "interesting" classes of manifolds and/or coarser than $diff$ classifications? (Something mediating between bordisms and h-cobordisms maybe?)

• The $\pi_1 = 1$ is very restrictive. The surgery theory extends to manifolds with an arbitrary fundamental group Γ and, modulo *the Novikov conjecture* – a non-simply connected counterpart to the relation $L_k(p_i)[X] = sig(X)$ (see next section) – delivers a comparably exhaustive answer in terms of the "Poincaré complexes over (the group ring of) Γ" (see [81]).

But this does not tells you much about "topologically interesting" Γ, e.g. fundamental groups of n-manifold X with the universal covering $\mathbb{R}^n$ (see [13] [14] about it).

10 Elliptic Wings and Parabolic Flows.

The geometric texture in the topology we have seen so far was all on the side of the "entropy"; topologists were finding gentle routes in the rugged landscape of all possibilities, you do not have to sweat climbing up steep energy gradients on these routs. And there was no essential new analysis in this texture for about 50 years since Poincaré.

Analysis came back with a bang in 1963 when Atiyah and Singer discovered the index theorem.

The underlying idea is simple: the "difference" between dimensions of two spaces, say Φ and Ψ, can be defined and be *finite* even if the spaces themselves are *infinite dimensional*, provided the spaces come with a linear (sometimes non-linear) *Fredholm* operator $D : \Phi \to \Psi$. This means, there exists an operator $E : \Psi \to \Phi$ such that $(1 - D \circ E) : \Psi \to \Psi$ and $(1 - E \circ D) : \Phi \to \Phi$ are compact operators. (In the non-linear case, the definition(s) is local and more elaborate.)

If D is Fredholm, then the spaces $ker(D)$ and $coker(D) = \Psi/D(\Phi)$ are finite dimensional and the *index* $ind(D) = dim(ker(D)) - dim(coker(D))$ is (by a simple argument) is a *homotopy invariant* of D in the space of *Fredholm* operators.

If, and this is a "big IF", you can associate such a D to a geometric or topological object X, this index will serve as an invariant of X.

It was known since long that *elliptic* differential operators, e.g. the ordinary Laplace operator, are Fredholm under suitable (boundary) conditions but most of these "natural" operators are self-adjoint and always have zero indices: they are of no use in topology.

"Interesting" elliptic differential operators D are scares: the ellipticity condition is a tricky inequality (or, rather, non-equality) between the coefficients of D. In fact, all such (linear) operators currently in use descend from a single one: *the Atiyah-Singer-Dirac operator on spinors.*

Atiyah and Singer have computed the indices of their geometric operators in terms of traditional topological invariants, and thus discovered new properties of the latter.

For example, they expressed the signature of a closed smooth Riemannian manifold X as an index of such an operator D_{sig} acting on differential forms on X. Since the *parametrix* operator E for an elliptic operator D can be obtained by piecing together local parametrices, the very existence of D_{sig} implies the multiplicativity of the signature.

The elliptic theory of Atiyah and Singer and their many followers, unlike the classical theory of PDE, is functorial in nature as it deals with many interconnected operators at the same time in coherent manner.

Thus smooth structures on potential manifolds (Poincaré complexes) define a functor from the homotopy category to the category of "Fredholm diagrams" (e.g. operators – one arrow diagrams); one is tempted to forget manifolds and study such functors per se. For example, a closed smooth manifold represents a homology class in Atiyah's K-theory – the index of D_{sig}, twisted with vector bundles over X with connections in them.

Interestingly enough, one of the first topological applications of the index theory, which equally applies to all dimensions be they big or small, was the solution (Massey, 1969) of the *Whitney 4D-conjecture* of 1941 which, in a simplified form, says the following.

The number $N(Y)$ of possible normal bundles of a closed connected non-orientable surface Y embedded into the Euclidean space $\mathbb{R}^4$ equals $|\chi(Y)-1|+1$, where χ denotes the Euler characteristic.Equivalently, there are $|\chi(Y)-1| = 1$ possible homeomorphisms types of small normal neighbourhoods of Y in $\mathbb{R}^4$.

If Y is an orientable surface then $N(Y) = 1$, since a small neighbourhood of such a $Y \subset \mathbb{R}^4$ is homeomorphic to $Y \times \mathbb{R}^2$ by an elementary argument.

If Y is non-orientable, Whitney has shown that $N(Y) \geq |\chi(Y) - 1| + 1$ by constructing $N = |\chi(Y)-1|+1$ embeddings of each Y to $\mathbb{R}^4$ with different normal bundles and then conjectured that one could not do better.

Outline of Massey's Proof. Take the (unique in this case) ramified double covering X of $S^4 \supset \mathbb{R}^4 \supset Y$ branched at Y with the natural involution $I : X \to X$. Express the *signature of* I, that is the quadratic form on $H_2(X)$ defined by the intersection of cycles C and $I(C)$ in X, in terms of the *Euler number* $e^\perp$ of the normal bundle of $Y \subset \mathbb{R}^4$ as $sig = e^\perp/2$ (with suitable orientation and sign conventions) by applying the *Atiyah-Singer equivariant signature theorem.* Show that $rank(H_2(X)) = 2 - \chi(Y)$ and thus establish the bound $|e^\perp/2| \leq 2 - \chi(Y)$ in agreement with Whitney's conjecture.

(The experience of the high dimensional topology would suggest that $N(Y) = \infty$. Now-a-days, multiple constrains on topology of embeddings of surfaces into 4-manifolds are derived with Donaldson's theory.)

Non-simply Connected Analytic Geometry. The Browder-Novikov theory implies that, besides the Euler-Poincaré formula, there is a *single* "$\mathbb{Q}$-essential (i.e. non-torsion) homotopy constraint" on tangent bundles of closed *simply connected* $4k$-manifolds– the Rokhlin-Thom-Hirzebruch signature relation.

But in 1966, Sergey Novikov, in the course of his proof of the *topological* invariance of the of the *rational Pontryagin classes*, i.e of the homology homomorphism $H_*(X^n;\mathbb{Q}) \to H_*(Gr_N(\mathbb{R}^{n+N});\mathbb{Q})$ induced by the normal Gauss map, found the following new relation for *non-simply connected* manifolds X.

Let $f : X^n \to Y^{n-4k}$ be a smooth map. Then the signature of the $4k$-dimensional pullback manifold $Z = f^{-1}(y)$ of a generic point, $sig[f] = sig(Z)$, does not depend on the point and/or on f within a given homotopy class $[f]$ by the generic pull-back theorem and the cobordism invariance of the signature, but it may change under a homotopy equivalence $h : X_1 \to X_2$.

By an elaborate (and, at first sight, circular) *surgery + algebraic K-theory* argument, Novikov proves that

if Y is a k-torus, then $sig[f \circ h] = sig[f]$,

where the simplest case of the projection $X \times \mathbb{T}^{n-4k} \to \mathbb{T}^{n-4k}$ is (almost all) what is needed for the topological invariance of the Pontryagin classes. (See [27] for a simplified version of Novikov's proof and [63] for a different approach to the topological Pontryagin classes.)

Novikov conjectured (among other things) that a similar result holds for an arbitrary closed manifold Y with *contractible* universal covering. (This would imply, in particular, that if an oriented manifold Y' is orientably homotopy equivalent to such a Y, then it is bordant to Y.) Mishchenko (1974) proved this for manifolds Y admitting metrics of *non-positive curvature* with a use of an index theorem for operators on infinite dimensional bundles, thus linking the Novikov conjecture to geometry.

(Hyperbolic groups also enter Sullivan's existence/uniqueness theorem of *Lipschitz structures* on topological manifolds of dimensions ≥ 5.

A bi-Lipschitz homeomorphism may look very nasty. Take, for instance, infinitely many disjoint round balls $B_1, B_2, ...$ in $\mathbb{R}^n$ of radii $\to 0$, take a diffeomorphism f of B_1 fixing the boundary $\partial(B_1)$ an take the scaled copy of f in each B_i. The resulting homeomorphism, fixed away from these balls, becomes quite complicated whenever the balls accumulate at some closed subset, e.g. a hypersurface in $\mathbb{R}^n$. Yet, one can extend the signature index theorem and some of the Donaldson theory to this unfriendly bi-Lipschitz, and even to quasi-conformal, environment.)

The Novikov conjecture remains unsolved. It can be reformulated in purely group theoretic terms, but the most significant progress which has been achieved so far depends on geometry and on the index theory.

In a somewhat similar vein, Atiyah (1974) introduced square integrable (also called L_2) cohomology on non-compact manifolds $\tilde{X}$ with cocompact discrete group actions and proved the *L_2-index theorem.* For example, he has shown that

if a compact Riemannian $4k$-manifolds has non-zero signature, then the universal covering $\tilde{X}$ admits a non-zero square summable harmonic $2k$-form.

This L_2-index theorem was extended to measurable *foliated spaces* (where "measurable" means the presence of *transversal measures*) by Alain Connes, where the two basic manifolds' attributes– the smooth structure and the measure – are separated: the *smooth structures in the leaves* allow differential operators while the *transversal measures* underly integration and where the two cooperate in the "non-commutative world" of Alain Connes.

If X is a compact measurably and smoothly n-foliated (i.e. almost all leaves are smooth n-manifolds) leaf-wise oriented space then one naturally defines Pontryagin's numbers which are real numbers in this case.

(Every closed manifold X can be regarded as a measurable foliation with the "transversal Dirac δ-measure" supported on X. Also complete Riemannian manifolds of *finite volume* can be regarded as such foliations, provided the universal coverings of these have locally bounded geometries [11].)

There is a natural notion of bordisms between measurable foliated spaces, where the Pontryagin numbers are obviously, bordism invariant.

Also, the L_2-signature, (which is also defined for leaves being $\mathbb{Q}$-manifolds) is bordism invariant by Poincaré duality.

The corresponding L_k-number, $k = n/4$, satisfies here the Hirzebruch formula with the L_2-signature (sorry for the mix-up in notation: $L_2 \neq L_{k=2}$): $L_k(X) = sig(X)$ by the Atiyah-Connes L_2-index theorem [11].

It seems not hard to generalize this to measurable foliated spaces where leaves are topological (or even topological $\mathbb{Q}$) manifolds.

Questions. Let X be a measurable leaf-wise oriented n-foliated space with zero Pontryagin numbers, e.g. $n \neq 4k$. Is X orientably bordant to zero, provided every leaf in X has measure zero.

What is the counterpart to the Browder-Novikov theory for measurable foliations?

Measurable foliations can be seen as transversal measures on some *universal topological foliation*, such as the *Hausdorff moduli space* X of the isometry classes of pointed complete Riemannian manifolds L with uniformly *locally bounded geometries* (or locally bounded covering geometries [11]), which is tautologically foliated by these L. Alternatively, one may take the space of pointed triangulated manifolds with a uniform bound on the numbers of simplices adjacent to the points in L.

The simplest transversal measures on such an X are weak limits of convex combinations of Dirac's δ-measures supported on closed leaves, but most (all?) known interesting examples descent from group actions, e.g. as follows.

Let L be a Riemannian symmetric space (e.g. the complex hyperbolic space $\mathbb{C}H^n$ as in section 5), let the isometry group G of L be embedded into a locally compact group H and let $O \subset H$ be a compact subgroup such that the intersection $O \cap G$ equals the (isotropy) subgroup $O_0 \subset G$ which fixes a point $l_0 \in L$. For example, H may be the special linear group $SL_N(\mathbb{R})$ with $O = SO(N)$ or H may be an adelic group.

Then the quotient space $\tilde{X} = H/O$ is naturally foliated by the H-translate copies of $L = G/O_0$.

This foliation becomes truly interesting if we pass from $\tilde{X}$ to $X = \tilde{X}/\Gamma$ for a discrete subgroup $\Gamma \subset H$, where H/Γ has finite volume. (If we want to make sure that all leaves of the resulting foliation in X are manifolds, we take Γ without torsion, but singular *orbifold* foliations are equally interesting and amenable to the general index theory.)

The full vector of the Pontryagin numbers of such an X depends, up to rescaling, only on L but it is unclear if there are "natural (or any) bordisms" between different X with the same L.

Linear operators are difficult to delinearize keeping them topologically interesting. The two exceptions are the *Cauchy-Riemann operator* and the signature operator in dimension 4. The former is used by Thurston (starting from late 70s) in his $3D$-geometrization theory and the latter, in the form of the *Yang-Mills equations*, begot Donaldson's $4D$-theory in 1983 and a decade later the Seiberg-Witten theory ([15] [84]).

The logic of Donaldson's approach resembles that of the index theorem. Yet, his operator $D : \Phi \to \Psi$ is *non-linear* Fredholm and instead of the index he studies the bordism-like invariants of (finite dimensional!) pullbacks $D^{-1}(\psi) \subset \Phi$ of suitably generic ψ.

These invariants for the Yang-Mills and Seiberg-Witten equations unravel an incredible richness of the smooth $4D$-topological structures which remain invisible from the perspectives of pure topology" and/or of linear analysis.

The non-linear Ricci flow equation of Richard Hamilton, the parabolic relative of Einstein, does not have any built-in topological intricacy; it is similar to the plain heat equation associated to the ordinary Laplace operator. Its potential role is not in exhibiting new structures but, on the contrary, in showing that these do not exist by ironing out bumps and ripples of Riemannian metrics. This potential was realized in dimension 3 by Perelman in 2003:

The Ricci flow on Riemannian 3*-manifolds, when manually redirected at its singularities, eventually brings every closed Riemannian* 3*-manifold to a canonical geometric form predicted by Thurston.*

(Possibly, there is a non-linear analysis on foliated spaces, where solutions of, e.g. parabolic Hamilton-Ricci for $3D$ and of elliptic Yang-Mills/Seiberg-Witten for $4D$, equations fast, e.g. L_2, decay on each leaf and where "decay" for non-linear objects may refer to a decay of distances between pairs of objects.)

There is hardly anything in common between the proofs of Smale and Perelman of the Poincaré conjecture. Why the statements look so similar? Is it the same "Poincaré conjecture" they have proved? Probably, the answer is "no" which raises another question: what is the high dimensional counterpart of the Hamilton-Perelman $3D$-structure?

To get a perspective let us look at another, seemingly remote, fragment of mathematics – the theory of algebraic equations, where the numbers $2, 3$ and 4 also play an exceptional role.

If topology followed a contorted path 2→5...→4→3, algebra was going straight 1→2→3→4→5... and it certainly did not stop at this point.

Thus, by comparison, the Smale-Browder-Novikov theorems correspond to non-solvability of equations of degree ≥ 5 while the present day $3D$- and $4D$-theories are brethren of the magnificent formulas solving the equations of degree 3 and 4.

What does, in topology, correspond to the Galois theory, class field theory, the modularity theorem... ?

Is there, in truth, anything in common between this algebra/arthmetic and geometry?

It seems so, at least on the surface of things, since the reason for the particularity of the numbers 2, 3, 4 in both cases arises from the same formula:

$$4 =_3 2 + 2 :$$

a 4 element set has exactly 3 partitions into two 2-element subsets and where, observe $3 < 4$. No number $n \geq 5$ admits a similar class of decompositions.

In algebra, the formula $4 =_3 2 + 2$ implies that the alternating group $A(4)$ admits an epimorphism onto $A(3)$, while the higher groups $A(n)$ are simple non-Abelian.

In geometry, this transforms into the splitting of the Lie algebra $so(4)$ into $so(3) \oplus so(3)$. This leads to the splitting of the space of the 2-forms into self-dual and anti-self-dual ones which underlies the Yang-Mills and Seiberg-Witten equations in dimension 4.

In dimension 2, the group $SO(2)$ "unfolds" into the geometry of Riemann surfaces and then, when extended to $homeo(S^1)$, brings to light the conformal field theory.

In dimension 3, Perelman's proof is grounded in the infinitesimal $O(3)$-symmetry of Riemannian metrics on 3-manifolds (which is broken in Thurston's theory and even more so in the high dimensional topology based on surgery) and depends on the irreducibility of the space of traceless curvature tensors.

It seems, the geometric topology has a long way to go in conquering high dimensions with all their symmetries.

11 Crystals, Liposomes and Drosophila.

Many geometric ides were nurtured in the cradle of manifolds; we want to follow these ideas in a larger and yet unexplored world of more general "spaces".

Several exciting new routes were recently opened to us by the high energy and statistical physics, e.g. coming from around the string theory and non-commutative geometry – somebody else may comment on these, not myself. But there are a few other directions where geometric spaces may be going.

Infinite Cartesian Products and Related Spaces. A crystal is a collection of identical molecules $mol_\gamma = mol_0$ positioned at certain sites γ which are the elements of a discrete (crystallographic) group Γ.

If the space of states of each molecule is depicted by some "manifold" M, and the molecules do not interact, then the *space X of states* of our "crystal" equals the Cartesian power $M^\Gamma = \times_{\gamma \in \Gamma} M_\gamma$.

If there are inter-molecular constrains, X will be a subspace of M^Γ; furthermore, X may be a quotient space of such a subspace under some equivalence relation, where, e.g. two states are regarded equivalent if they are indistinguishable by a certain class of "measurements".

We look for mathematical counterparts to the following physical problem. Which properties of an individual molecule can be determined by a given class of measurement of the whole crystal?

Abstractly speaking, we start with some category $\mathcal{M}$ of "spaces" M with Cartesian (direct) products, e.g. a category of finite sets, of smooth manifolds or of algebraic manifolds over some field. Given a countable group Γ, we enlarge this category as follows.

Γ-*Power Category* $\Gamma^{\mathcal{M}}$. The objects $X \in \Gamma^{\mathcal{M}}$ are projective limits of finite Cartesian powers M^Δ for $M \in \mathcal{M}$ and finite subsets $\Delta \subset \Gamma$. Every such X is naturally acted upon by Γ and the admissible morphisms in our Γ-category are Γ-*equivariant* projective limits of morphisms in $\mathcal{M}$.

Thus each morphism, $F : X = M^\Gamma \to Y = N^\Gamma$ is defined by a single morphism in $\mathcal{M}$, say by $f : M^\Delta \to N = N$ where $\Delta \subset \Gamma$ is a *finite* (sub)set.

Namely, if we think of $x \in X$ and $y \in Y$ as M- and N-valued functions $x(\gamma)$ and $y(\gamma)$ on Γ then the value $y(\gamma) = F(x)(\gamma) \in N$ is evaluated as follows:

translate $\Delta \subset \Gamma$ to $\gamma\Delta \subset \Gamma$ by γ, restrict $x(\gamma)$ to $\gamma\Delta$ and apply f to this restriction $x|\gamma\Delta \in M^{\gamma\Delta} = M^\Delta$.

In particular, every morphism $f : M \to N$ in $\mathcal{M}$ tautologically defines a morphism in $\mathcal{M}^\Gamma$, denoted $f^\Gamma : M^\Gamma \to N^\Gamma$, but $\mathcal{M}^\Gamma$ has many other morphisms in it.

Which concepts, constructions, properties of morphisms and objects, etc. from $\mathcal{M}$ "survive" in $\Gamma^{\mathcal{M}}$ for a given group Γ? In particular, what happens to topological invariants which are multiplicative under Cartesian products, such as the Euler characteristic and the signature?

For instance, let M and N be manifolds. Suppose M admits *no topological embedding* into N (e.g. $M = S^1$, $N = [0,1]$ or $M = \mathbb{R}P^2$, $N = S^3$). When does M^Γ admit an injective morphism to N^Γ in the category $\mathcal{M}^\Gamma$?

(One may meaningfully reiterate these questions for *continuous* Γ-equivariant maps between Γ-Cartesian products, since *not all* continuous Γ-equivariant maps lie in $\mathcal{M}^\Gamma$.)

Conversely, let $M \to N$ be a map of non-zero degree. When is the corresponding map $f^\Gamma : M^\Gamma \to N^\Gamma$ equivariantly homotopic to a *non-surjective* map?

Let Σ be a hypersurface of bi-degree (p,q) in $\mathbb{C}P^n \times \mathbb{C}P^n$ and $\Gamma = \mathbb{Z}$. Let $P_k(s)$ denote the Poincaré polynomial of $\Sigma_{gen}(G/k\mathbb{Z})$, $k = 1, 2, ...$ and let

$$P(s,t) = \sum_{k=1}^{\infty} t^k P(s) = \sum_{k,i} t^k s^i rank(H_i(\Sigma_{gen}(G/k\mathbb{Z})).$$

Observe that the function $P(s,t)$ depends only on n, and (p,q).

Is $P(s,t)$ meromorphic in the two complex variables s and t? Does it satisfy some "nice" functional equation?

Similarly, if $\mathbb{F} = \mathbb{F}_p$, we ask the same question for the generating function in two variables counting the $\mathbb{F}_{p^l}$-points of $\Sigma(G/k\mathbb{Z})$.

Γ-*Quotients.* These are defined with equivalence relations $R \subset X \times X$ where R are subobjects in our category.

The transitivity of (an *equivalence* relation) R, and it is being a *finitary* defined sub-object are hard to satisfy simultaneously. Yet, *hyperbolic dynamical systems* provide encouraging examples at least for the category $\mathcal{M}$ of *finite sets.*

If $\mathcal{M}$ is the category of finite sets then subobjects in $\mathcal{M}^\Gamma$, defined with subsets $\Sigma \subset M \times M$ are called *Markov* Γ*-shifts.* These are studied, mainly for $\Gamma = \mathbb{Z}$, in the context of *symbolic dynamics* [44], [7].

Γ-*Markov quotients* Z of Markov shifts are defined with equivalence relations $R = R(\Sigma') \subset Y \times Y$ which are Markov subshifts. (These are called *hyperbolic* and/or *finitely presented* dynamical systems [20], [26].)

If $\Gamma = \mathbb{Z}$, then the counterpart of the above $P(s,t)$, now a function only in t, is, essentially, what is called the ζ*-function* of the dynamical system which counts the number of periodic orbits. It is shown in [20] with a use of *(Sinai-Bowen) Markov partitions* that this function is rational in t for all $\mathbb{Z}$-Markov quotient systems.

The local topology of Markov quotient (unlike that of shift spaces which are Cantor sets) may be quite intricate, but some are *topological manifolds.*

For instance, classical *Anosov systems* on *infra-nilmanifolds* V and/or *expanding endomorphisms* of V are representable as a $\mathbb{Z}$- Markov quotient via Markov partitions [35].

Another example is where Γ is the fundamental group of a closed n-manifold V of negative curvature. The *ideal boundary* $Z = \partial_\infty(\Gamma)$ is a topological $(n-1)$-sphere with a Γ-action which admits a Γ-Markov quotient presentation [26].

Since the topological S^{n-1}-bundle $S \to V$ associated to the universal covering, regarded as the principle Γ bundle, is, obviously, isomorphic to the unit tangent bundle $UT(V) \to V$, the Markov presentation of $Z = S^{n-1}$ defines the topological Pontryagin classes p_i of V in terms of Γ.

Using this, one can reduce the homotopy invariance of the Pontryagin classes p_i of V to the ε*-topological* invariance.

Recall that an ε*-homeomorphism* is given by a pair of maps $f_{12} : V_1 \to V_2$ and $f_{21} : V_2 \to V_1$, such that the composed maps $f_{11} : V_1 \to V_1$ and $f_{22} : V_2 \to V_2$ are ε-close to the respective identity maps for some metrics in V_1, V_2 and a small $\varepsilon > 0$ depending on these metrics.

Most known proofs, starting from Novikov's, of invariance of p_i under homeomorphisms equally apply to ε-homeomorphisms.

This, in turn, implies the *homotopy* invariance of p_i if the homotopy can be "rescaled" to an ε-homotopy.

For example, if V is a *nil-manifold* $\tilde{V}/\Gamma$, (where $\tilde{V}$ is a nilpotent Lie group homeomorphic to $\mathbb{R}^n$) with an *expanding endomorphism* $E : V \to V$ (such a V is a $\mathbb{Z}$-Markov quotient of a shift), then a large negative power $\tilde{E}^{-N} : \tilde{V} \to \tilde{V}$ of the lift $\tilde{E} : \tilde{V} \to \tilde{V}$ brings any homotopy close to identity. Then the ε-topological invariance of p_i implies the homotopy invariance for these V. (The case of $V = \mathbb{R}^n/\mathbb{Z}^n$ and $\tilde{E} : \tilde{v} \to 2\tilde{v}$ is used by Kirby in his topological *torus trick*.)

A similar reasoning yields the homotopy invariance of p_i for many (manifolds with fundamental) groups Γ, e.g. for *hyperbolic groups*.

Questions. Can one effectively describe the local and global topology of Γ-Markov quotients Z in combinatorial terms? Can one, for a given (e.g. hyperbolic) group Γ, "classify" those Γ-Markov quotients Z which are *topological manifolds* or, more generally, *locally contractible* spaces?

For example, can one describe the classical Anosov systems Z in terms of the combinatorics of their $\mathbb{Z}$-Markov quotient representations? How restrictive is the assumption that Z is a topological manifold? How much the topology of the local dynamics at the periodic points in Z restrict the topology of Z (E.g. we want to incorporate pseudo-Anosov automorphisms of surfaces into the general picture.)

It seems, as in the case of the hyperbolic groups, (irreducible) $\mathbb{Z}$-Markov quotients becomes more scarce/rigid/symmetric as the topological dimension and/or the local topological connectivity increases.

Are there interesting Γ-Markov quotients over categories $\mathcal{M}$ besides finite sets? For example, can one have such an object over the category of algebraic varieties over $\mathbb{Z}$ with non-trivial (e.g. positive dimensional) topology in the spaces of its $\mathbb{F}_{p^i}$-points?

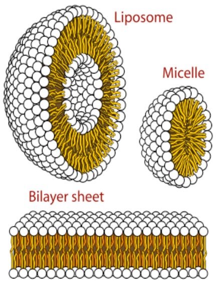
Liposomes and *Micelles* are surfaces of *membranes* surrounded by water which are assembled of rod-like (phospholipid) molecules oriented normally to the surface of the membrane with *hydrophilic* "heads" facing the exterior and the interior of a cell while the *hydrophobic* "tails" are buried inside the membrane.

These surfaces satisfy certain partial differential equations of rather general nature (see [30]). If we heat the water, membranes dissolve: their constituent molecules become (almost) randomly distributed in the water; yet, if we cool the solution, the surfaces and the equations they satisfy re-emerge.

Question. Is there a (quasi)-canonical way of associating statistical ensembles $\mathcal{S}$ to geometric system S of PDE, such that the equations emerge at low temperatures T and also can be read from the properties of high temperature states of $\mathcal{S}$ by some "analytic continuation" in T?

The architectures of liposomes and micelles in an ambient space, say W, which are composed of "somethings" normal to their surfaces $X \subset W$, are reminiscent of Thom-Atiyah representation of submanifolds with their normal bundles by generic maps $f_\bullet : W \to V_\bullet$, where $V_\bullet$ is the Thom space of a vector bundle V_0 over some space X_0 and where manifolds $X = f_\bullet^{-1}(X_0) \subset W$ come with their normal bundles induced from the bundle V_0.

The space of these "generic maps" $f_\bullet$ looks as an intermediate between an individual "deterministic" liposome X and its high temperature randomization.



Can one make this precise?

Poincaré-Sturtevant Functors. All what the brain knows about the geometry of the space is a flow S_{in} of electric impulses delivered to it by our sensory organs. All what an alien browsing through our mathematical manuscripts would directly perceive, is a flow of symbols on the paper, say G_{out}.

Is there a natural functorial-like transformation $\mathcal{P}$ from sensory inputs to mathematical outputs, a map between "spaces of flows" $\mathcal{P} : \mathcal{S} \to \mathcal{G}$ such that $\mathcal{P}(S_{in})$"="G_{out}?

It is not even easy to properly state this problem as we neither know what our "spaces of flows" are, nor what the meaning of the equality "=" is.

Yet, it is an essentially mathematical problem a solution of which (in a weaker form) is indicated by Poincaré in [60]. Besides, we all witness the solution of this problem by our brains.

An easier problem of this kind presents itself in the classical genetics.

What can be concluded about the geometry of a genome of an organism by observing the phenotypes of various representatives of the same species (with no molecular biology available)?

This problem was solved in 1913, long before the advent of the molecular biology and discovery of DNA, by 19 year old Alfred Sturtevant (then a student in T. H. Morgan's lab) who reconstructed *the linear structure* on the set of genes on a chromosome of Drosophila melanogaster from *samples of a probability measure* on the space of gene linkages.

Here mathematics is more apparent: the geometry of a space X is represented by something like a measure on the set of subsets in X; yet, I do not know how to formulate clear-cut mathematical questions in either case (compare [29], [31]).

Who knows where manifolds are going?

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13 Bibliography.

References

[1] P. Akhmet'ev, Geometric approach to stable homotopy groups of spheres. The Adams-Hopf invariants, Fundam. Prikl. Mat., 13:8, pp. 3-15, (2007).

[2] M. Atiyah, Thom complexes. Proc. London Math. Soc. (3) 11. 291-310. (1961).

[3] M. Atiyah, I. Singer, The Index of Elliptic Operators on Compact Manifolds, Bull. Amer. Math. Soc. 69: 322-433, (1963).

[4] D. Barden, Simply connected five manifolds. Ann. Math., 82, 365-385 (1965).

[5] M. Bertelson, Topological invariant for discrete group actions. Lett. Math. Phys. 62, 147-156 (2004).

[6] S. Buoncristiano, C. P. Rourke, B. J. Sanderson, A geometric approach to homology theory. London Mathematical Society Lecture Note Series, no. 18, Cambridge Univ. Press, (1976)

[7] M. Boyle, Open problems in symbolic dynamics. url http://www-users.math.umd.edu/ mmb/open/, (2008).

[8] W. Browder, Homotopy type of differentiable manifolds, Colloq. Algebraic Topology, pp. 42-46, Aarhus University, (1962). reprinted in "Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)", 97-100, LMS Lecture Notes 226 (1995)

[9] E. Brown Jr., The Kervaire invariant and surgery theory. www.math.rochester.edu/u/faculty/doug/otherpapers/brown2.pdf

[10] J. Cheeger, Spectral geometry of singular Riemannian spaces, J. Differential Geom. V. 18. 4. pp. 575-657, (1983).

[11] J. Cheeger and M. Gromov, On the characteristic numbers of complete manifolds of bounded curvature and finite volume, Differential geometry and complex analysis, H. E. Rauch Memorial Volume, Springer-Verlag, Berlin, (1985).

[12] A. Chernavskii, Local contractibility of the group of homeomorphisms of a manifold, Math. USSR Sb. , 8 : 3 pp. 287-333, (1969),

[13] M. Davis, Poincaré duality groups, from: "Surveys on surgery theory, Vol. 1", Ann. of Math. Stud. 145, Princeton Univ. Press, Princeton, NJ, 167193, (2000).

[14] M. Davis, The Geometry and Topology of Coxeter Groups,. London Mathematical Society Monographs, Princeton University Press, (2008).

[15] S. Donaldson, An Application of Gauge Theory to Four Dimensional Topology, Journal of Differential Geometry 18 (2): 279-315, (1983).

[16] P. Eccles, Multiple points of codimension one immersions of oriented manifolds. Math. Proc. Cambridge Philos. Soc. 87, 213-220 (1980).

[17] P. Eccles, Codimension one immersions and the Kervaire invariant one problem, Math. Proc. Camb. Phil. Soc. 90, 483-493 (1981).

[18] S. Eilenberg, Cohomology and continuous mappings, Ann. of Math., 41, 231-251, (1940).

[19] Y. Felix, S. Halperin, J.-C. Thomas, Rational homotopy theory, Springer, (2001).

[20] D. Fried, Finitely presented dynamical systems. Ergodic Theory and Dynamical Systems, 7, pp 489-507, (1987).

[21] D. Fuks, Classical Manifolds. In Topology-1. Sovr. Probl. Math. Fund. Napr. (Itogi. Nauki- Tech.), M., VINITI, (1986).

[22] M. Furuta, Homology cobordism group of homology 3-spheres, Invent. Math. 100, 339-355, (1990).

[23] A. Gaifullin, The construction of combinatorial manifolds with prescribed sets of links of vertices, Izvestiya RAN: Ser. Mat. 72:5, 3-62, (2008).

[24] A. Gaifullin, Configuration spaces, bistellar moves, and combinatorial formulae for the first Pontryagin class, Tr. Mat. Inst. Steklova, Volume 268, pp. 76-93, (2010).

[25] M. Gromov, On the number of simplices of subdivisions of finite complexes, Mat. Zametki, 3:5, pp. 511-522, (1968).

[26] M. Gromov, Hyperbolic manifolds, groups and actions, Ann. Math. Studies 97, 183-215, Princeton University Press, Princeton, (1981).

[27] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, Functional analysis on the eve of the 21st century, Vol. II , 1-213, Progr. Math., 132, Birkhäuser, (1996).

[28] M. Gromov. Spaces and questions, Geom. Funct. Anal., Special Volume, Part I:118-161, (2000).

[29] M. Gromov, Mendelian Dynamics and Sturtevant's Paradigm, in "Geometric and Probabilistic Structures", Contemporary Mathematics Series v. 469, (ed, Keith Burns,, Dmitry Dolgopyat and Yakov Pesin), pp 227-242, American Mathematical Society, Providence RI (2007).

[30] M. Gromov, Crystals, Proteins, Stability and Isoperimetry. Submitted to Bulletin of AMS, (2010). http://www.ihes.fr/~gromov/PDF/pansu-crystals-isoper.pdf

[31] M. Gromov, Structures, Learning and Ergosystems. www.ihes.fr/~gromov/PDF/ergobrain.pdf

[32] R. Hamilton Three-manifolds with positive Ricci curvature. J. Differential Geom. Volume 17, Number 2, 255-306, (1982).

[33] A. Haefliger, Plongements diffrentiables de variétés dans variétés, Comment. Math. Helv. 36, 47-82, (1961).

[34] A. Haefliger, Knotted (4k-1)-spheres in 6k-space. Ann. of Math. (2) 75, 452-466, (1962).

[35] B. Hasselblatt, Hyperbolic dynamical systems. Handbook of Dynamical Systems 1A, 239-319, Elsevier North Holland, (2002).

[36] H. Hopf, Abbildungsklassen n-dimensionaler Mannigfaltigkeiten. Math. Annalen 96, pp. 225-250, (1926).

[37] M.Hill, M. Hopkins, D.Ravenel. On the non-existence of elements of Kervaire invariant one.
www.math.rochester.edu/u/faculty/doug/kervaire_082609.pdf

[38] W. Hurewicz, Beiträge zur Topologie der Deformationen I-II. Proc. Ned. Akad. Weten. Ser. A , 38, pp. 112-119; 521-528, (1935).

[39] M. Kervaire, J. Milnor, Groups of homotopy spheres. I Ann. of Math., 77:3, pp. 504-537 (1963).

[40] R. Kirby, L. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings and Triangulations, Ann. of Math. Studies 88 (1977).

[41] V. Klee, A combinatorial analogue of Poincaré's duality theorem Canad. J. Math. 16, pp, 517-531 (1964).

[42] M. Kreck, Differential Algebraic Topology, Graduate Studies in Mathematics, Volume 110, (2010).

[43] N. Levitt, C. Rourke, The existence of combinatorial formulae for characteristic classes, Trans. Amer. Math. Soc. 239, 391-397, (1978).

[44] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, (1995).

[45] N. Martin, On the difference between homology and piecewise-linear bundles, J. London Math. Soc. (2), 6:2, 197-204, (1973).

[46] W. Massey. Proof of a conjecture of Whitney. Pacific J. Math. Volume 31, Number 1 (1969), 143-156.

[47] B. Mazur, Stable equivalence of differentiable manifolds, Bull. Amer. Math. Soc. 67, 377-384, (1961).

[48] B. Mazur, Bernoulli numbers and the unity of mathematics.
www.math.wisc.edu/~boston/Bernoulli.pdf

[49] J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math. (2) 64 (1956) 399-405, (1956).

[50] J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Annals of Mathematics 74 (2): 575-590 (1961).

[51] J. Milnor and J. Stasheff, Characteristic classes,. Princeton. Univ. Press (1974).

[52] A. Mishchenko, Infinite-dimensional representations of discrete groups, and higher signatures, Math. USSR-Izv., 8:1, 85-111, (1974).

[53] A. Nabutovsky, Combinatorics of the space of Riemannian structures and logic phenomena of Euclidean Quantum Gravity, in "Perspectives in Riemannian Geometry", ed. by V. Apostolov et al., CRM Proceedings and Lecture Notes, vol. 40, 223-248, AMS, Providence, RI, (2006).

[54] A. Nabutovsky, S. Weinberger, The fractal geometry of Riem/Diff, Geometriae Dedicata 101, 1-54, (2003).

[55] S. Novikov, Homotopy equivalent smooth manifolds I. (In Russian). Izv. Akad. Nauk S.S.S.R. ser. mat. 28 (2) (1964),365-474.Also, Translations Amer. Math. Soc. 48, 271-396, (1965).

[56] S. Novikov, On manifolds with free Abelian fundamental group and their application, Izv. Akad. Nauk SSSR, v. 30, N 1, 207-246, (1966).

[57] S. Novikov, Topology. 1 : General survey, Berlin etc. : Springer, 1996. - 319 c. (Encyclopaedia of mathematical sciences). Per. izd. : Itogi nauki i tekniki, Sovremennye problemy matematiki, Fundamental'nye napravleniya..., Topologiya... - Moscow.

[58] G. Perelman, Ricci flow with surgery on three-manifolds. arXiv:math.DG/0303109, (2003).

[59] S. Podkorytov, An alternative proof of a weak form of Serre's theorem, J. of Math. Sci. vol 110, No.4 2875-2881, (2002).

[60] H. Poincaré, Science and hypothesis, London and Newcastle-on-Tyne: The Walter Scott Publishing Co., (1905).

[61] L. Pontryagin, Classification of continuous transformations of a complex into a sphere, Dokl. Akad. Nauk SSSR , 19, pp. 361-363, (In Russian) (1938).

[62] L. Pontryagin, Homotopy classification of the mappings of an $(n+2)$-dimensional sphere on an n-dimensional one, Dokl. Akad. Nauk SSSR (N.S.) 70, 957-959, (Russian) (1950).

[63] A. Ranicki, M. Weiss, On the construction and topological invariance of the Pontryagin classes arXiv:0901.0819, (2009).

[64] V. Rokhlin, Summary of results in homotopy theory of continuous transformations of a sphere into a sphere, Uspekhi Mat. Nauk, 5:6(40), 88-101, (1950).

[65] V. Rokhlin, New results in the theory of 4-dimensional manifolds, Dokl. Akad. Nauk. SSSR 84, 221-224, (Russian) (1952).

[66] V. Rokhlin, On Pontrjagin characteristic classes, Dokl. Akad. Nauk SSSR 113, 276-279, (1957).

[67] V. Rokhlin, A. Schwarz, The combinatorial invariance of Pontryagin classes, Dokl. Akad Nauk SSSR, 114, 490-493, (1957).

[68] C. Rourke, Essay on the Poincaré conjecture, http://msp.warwick.ac.uk/˜cpr/.

[69] A. Russon, Orangutans: Wizards of the Rain Forest, Key Porter books, (2004).

[70] J-P. Serre, Homologie singulière des espaces fibrés. Applications", Annals of Mathematics, 54 (3): 425-505, (1951).

[71] J. Simons, Minimal varieties in riemannian manifolds, Ann. Math. 88, 62-105, (1968).

[72] S. Smale, Generalized Poincaré's conjecture in dimensions greater than four. Ann. of Math. (2) 74,1961 391-406, (1961).

[73] S. Smale: On the structure of 5-manifolds. Ann. Math., 75, 38-46, (1962).

[74] J. Stallings, Polyhedral homotopy-spheres, Bull. Amer. Math. Soc. Volume 66, Number 6, 485-488, (1960).

[75] D. Sullivan, Hyperbolic geometry and homeomorphisms, Geometric Topology., Proc. Georgia Topology Conf., Athens, Ga., Academic Press, New York, pp. 543-555, (1977).

[76] A. Szùcs, Cobordism of singular maps, Geometry and Topology 12, 2379-2452 (2008).

[77] A. Szùcs, A geometric proof of a theorem of Serre on p-components of $\pi_{n+k}(S^k)$, Periodica Matematica Hungarica, 29: pp. 9-13, (1977).

[78] R. Thom, Quelques propriétés globales des variétés différentiables, Commentarii Mathematici Helvetici 28: 17-86, (1954).

[79] R. Thom, Les classes caractéristiques de Pontrjagin des variétés triangulées,. Topologia Algebraica, Mexico, 54-67, (1958).

[80] W. Thurston, The geometry and topology of 3-manifolds, Princeton lecture notes (1978-1981).

[81] C. T. C. Wall, Surgery on compact manifolds, Mathematical Surveys and Monographs, 69 (2nd ed.), Providence, R.I.: American Mathematical Society, (1999).

[82] H. Whitney, On the topology of differentiable manifolds, from: "Lectures in Topology", University of Michigan Press,101-141, (1941).

[83] H. Whitney, The self-intersection of a smooth n-manifold in $2n$-space, Ann. of Math. 45, 220-246, (1944).

[84] E. Witten, Monopoles and four-manifolds, Math. Res. Lett. 1 (6): 769-796,(1994).

[85] E. Zeeman, Unknotting combinatorial balls. Ann. of Math. (2) 78, 501–526, (1963).