Riemann Curvature, Ricci, and Scalar Curvature in Geodesic Coordinates

Geodesic (Normal) Coordinates

Let $(x^1, \ldots, x^n)$ be normal coordinates centered at a point p in a Riemannian manifold (M, g) with Levi–Civita connection ∇. By construction,

$$g_{ij}(p) = \delta_{ij}, \qquad \partial_k g_{ij}(p) = 0, \qquad \Gamma^i_{jk}(p) = 0.$$

Christoffel Symbols

The Christoffel symbols (connection coefficients) of the Levi–Civita connection are defined by

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{i\ell}(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}).$$

They determine the covariant derivative:

$$(\nabla_j V)^i = \partial_j V^i + \Gamma^i_{jk} V^k, \qquad (\nabla_j \omega)_i = \partial_j \omega_i - \Gamma^k_{ji}\, \omega_k.$$

At the origin p of normal coordinates,

$$\Gamma^i_{jk}(p) = 0, \qquad \partial_\ell \Gamma^i_{jk}(p) = -\tfrac{1}{3}\big(R^i{}_{jk\ell} + R^i{}_{kj\ell}\big).$$

Deriving Γ^i_{jk} from Metric-Compatibility and Torsion-Free Conditions

The Levi–Civita connection is the unique connection ∇ that is (i) torsion-free and (ii) metric-compatible. These two axioms determine the Christoffel symbols.

Axioms

- Torsion-free: $\Gamma^i_{jk} = \Gamma^i_{kj}$.
- Metric-compatibility: $\nabla_k g_{ij} = 0$, i.e.

$$\partial_k g_{ij} = g_{mj}\, \Gamma^m_{ik} + g_{im}\, \Gamma^m_{jk}.$$

Algebraic Derivation

Write the metric-compatibility equation for the three index permutations (i,j,k), (j,k,i), and (i,k,j):

$$
\begin{aligned}
&(1) \quad \partial_k g_{ij} = g_{mj}\,\Gamma^m_{ik} + g_{im}\,\Gamma^m_{jk},\\
&(2) \quad \partial_i g_{jk} = g_{mk}\,\Gamma^m_{ji} + g_{jm}\,\Gamma^m_{ki},\\
&(3) \quad \partial_j g_{ik} = g_{mk}\,\Gamma^m_{ij} + g_{im}\,\Gamma^m_{kj}.
\end{aligned}
$$

Using torsion-free symmetry ($\Gamma^m_{ki} = \Gamma^m_{ik}$ and $\Gamma^m_{kj} = \Gamma^m_{jk}$), form

$$(1) + (2) - (3): \quad \partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik} = 2\, g_{mj}\,\Gamma^m_{ik}.$$

Contract with $g^{\ell j}$ to solve for Γ:

$$2\,\Gamma^\ell_{ik} = g^{\ell j}\big(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}\big),$$

which yields the standard formula

$$\Gamma^\ell_{ik} = \tfrac{1}{2}\, g^{\ell j}\big(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}\big).$$

Koszul Formula (Coordinate-Free)

For vector fields X, Y, Z,

$$2\, g(\nabla_X Y, Z) = X\, g(Y,Z) + Y\, g(Z,X) - Z\, g(X,Y) + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y).$$

In a coordinate frame $\{\partial_i\}$ where $[\partial_i, \partial_j] = 0$, this reduces to the coordinate expression above.

Riemann Curvature Tensor

Using the convention

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk},$$

we obtain at the origin p:

$$R_{ijkl}(p) = \tfrac{1}{2}\big(\partial_i\partial_k g_{jl} + \partial_j\partial_l g_{ik} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il}\big)\Big|_p.$$

Ricci and Scalar Curvature

The Ricci tensor and scalar curvature at p follow by contraction:

$$\mathrm{Ric}_{jl}(p) = R^i{}_{jil}(p) = g^{ik}(p) R_{ijkl}(p) = \delta^{ik} R_{ijkl}(p),$$

$$S(p) = g^{jl}(p)\, \mathrm{Ric}_{jl}(p) = \delta^{jl}\, \mathrm{Ric}_{jl}(p).$$

Metric Expansions Near p

The Taylor expansions of the metric and its inverse are

$$g_{ij}(x) = \delta_{ij} - \tfrac{1}{3}R_{ikj\ell}(p)x^k x^\ell - \tfrac{1}{6}R_{ikj\ell;m}(p)x^k x^\ell x^m + O(|x|^4),$$
$$g^{ij}(x) = \delta^{ij} + \tfrac{1}{3}R^i{}_k{}^j{}_\ell(p)x^k x^\ell + \tfrac{1}{6}R^i{}_k{}^j{}_{\ell;m}(p)x^k x^\ell x^m + O(|x|^4).$$

Christoffel Symbols Expansion

$$\Gamma^i_{jk}(x) = -\tfrac{1}{3}(R^i{}_{jk\ell} + R^i{}_{kj\ell})(p)x^\ell - \tfrac{1}{6}(R^i{}_{jk\ell;m} + R^i{}_{kj\ell;m})(p)x^\ell x^m + O(|x|^3).$$

Volume Density

$$\sqrt{\det g(x)} = 1 - \tfrac{1}{6}\operatorname{Ric}_{k\ell}(p)x^k x^\ell - \tfrac{1}{12}\operatorname{Ric}_{k\ell;m}(p)x^k x^\ell x^m + O(|x|^4).$$

Laplacian on Scalars

$$\Delta f = \frac{1}{\sqrt{\det g}}\partial_i\Big(\sqrt{\det g}\, g^{ij}\partial_j f\Big) = \delta^{ij}\partial_i\partial_j f - \tfrac{1}{3}\operatorname{Ric}_{k\ell}(p)x^k\partial_\ell f + O(|x|^2\partial f, |x|\,\partial^2 f).$$

Curvature Symmetries

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}, \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

Einstein Summation Convention

Definition

In tensor calculus, the **Einstein summation convention** (or *summation over repeated indices*) states that whenever an index variable appears once as an upper (contravariant) index and once as a lower (covariant) index in a single term, it is implicitly summed over all its possible values.

$$A^i B_i = \sum_{i=1}^{n} A^i B_i \tag{1}$$

Key Rules

- **Free indices:** appear only once in an expression and remain as indices of the resulting tensor.
- **Dummy (summed) indices:** appear twice (once upper, once lower) and are summed over; they disappear from the result.

- Summation occurs only when an index appears once up and once down.
- If an index appears twice both up or both down, it is not summed unless explicitly indicated.

Examples

1. Dot Product

$$v^i w_i = \sum_{i=1}^{n} v^i w_i \tag{2}$$

2. Matrix-Vector Multiplication

$$y^i = A^i{}_j x^j = \sum_{j=1}^{n} A^i{}_j x^j \tag{3}$$

3. Metric Contraction

$$g_{ij} v^i w^j = \sum_{i,j=1}^{n} g_{ij} v^i w^j \tag{4}$$

Remarks

The position of an index (up or down) reflects whether the component is covariant or contravariant. The Einstein convention greatly simplifies tensor notation by omitting explicit summation signs.

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Consider a 4-form Φ on $\mathbb{R}^n$,

$$\Phi = \Phi(x_1, x_2, x_3, x_4), \qquad x_i \in \mathbb{R}^n,\ i = 1, \ldots, 4,$$

which is symmetric under the following permutations of the entries

$$x_1 \leftrightarrow x_2,\ x_3 \leftrightarrow x_4 \quad \text{and} \quad (x_1, x_2) \leftrightarrow (x_3, x_4)$$

This means Φ is a symmetric bilinear form on the symmetric square $(\mathbb{R}^n)^2$, and so the dimension of the space of all such forms Φ equals $\frac{n(n+1)}{4}\left(1 + \frac{n(n+1)}{2}\right)$. There is a canonical splitting of Φ into the sum $\Phi = \Phi^+ + \Phi^-$, where Φ^+ is the symmetric 4-form on $\mathbb{R}^n$ obtained by the complete symmetrization of Φ, and where $\Phi^- = \Phi - \Phi^+$ satisfies

$$\Phi^-(x_1, x_2, x_3, x_4) = \Phi(x_1, x_2, x_3, x_4) - \Phi(x_1, x_4, x_3, x_2)$$

The form Φ^- has the symmetry type of curvature tensors. These constitute a space of dimension

$$\frac{n(n+1)}{4}\left(1 + \frac{n(n+1)}{2}\right) - \frac{n(n+1)(n+2)(n+3)}{24} = \frac{n^2(n^2-1)}{12}.$$

Now, let $f\colon V \to W$ be an isometric C^2-immersion between Riemannian C^∞-manifolds $V = (V, g)$ and (W, h) of dimension n and q respectively. The map f then induces the form Φ of the above type on every tangent space $T_v(V)$, $v \in V$. Namely, take local coordinates u_i in V which are geodesic at $v \in V$ and define $\Phi = \Phi_f$ by

$$\Phi(\partial_i, \partial_j, \partial_k, \partial_l) = \langle \nabla_{ij} f, \nabla_{kl} f \rangle,$$

for $\partial_i = \partial f(v)/\partial u_i$ and for the covariant derivatives ∇_{ij} and their scalar products in W. The part Φ^- of $\Phi = \Phi_f$ depends only on the curvature tensor R of the induced metric $g = f^*(h)$ by the Gauss theorema egregium

$$R(g) = \Phi_f^-.$$

The remarkable feature of this formula is the absence of third derivatives of f in the expression for $R(f^*(h))$, where $h \mapsto f^*(h)$ is a first order differential operator and $g \mapsto R(g)$ is a second order operator. However, the composition of the two is a second (not third!) order operator.

The symmetric part Φ_f^+ also has a simple geometric interpretation. Let γ be

a geodesic in V in the direction of some unit vector $\partial \in T_v(V)$. Then the value $\Phi_f^+(\partial, \partial, \partial, \partial)$ equals the (curvature)^2 of the curve $f(\gamma) \subset W$ at $w = f(v) \in W$.

Observe that the form Φ_f can be identified with the quadratic form on the symmetric square $(T(V))^2$ which is induced from h by the *second differential* D_f^2. This D_f^2 maps $(T(V))^2$ to the normal bundle $N_f \to V$ by sending $\partial_i \otimes \partial_j$ to $P_N(\nabla_{ij} f)$, for all bivectors $\partial_i \otimes \partial_j \in (T(V))^2$ and for the normal projection $P_N\colon T(W)|V \to N_f$.

C^∞-Immersions with Given Curvature. The isometric immersion relation for maps $f\colon (V,g) \to \mathbb{R}^q$ prescribes the scalar products

$$\langle \partial_1 f, \partial_2 f \rangle = g(\partial_1, \partial_2) \tag{1}$$

for all vectorfields ∂_1 and ∂_2 in V. This implies

$$\langle \partial_1 \partial_2 f, \partial_2 f \rangle = \tfrac{1}{2}\partial_1 g(\partial_2, \partial_2)$$

and so

$$\langle \partial_2^2 f, \partial_1 f \rangle = \partial_2 g(\partial_1, \partial_2) - \tfrac{1}{2}\partial_1 g(\partial_2, \partial_2). \tag{1'}$$

Since commuting fields satisfy

$$\partial_2 \partial_3 = \tfrac{1}{4}[(\partial_2 + \partial_3)^2 - (\partial_2 - \partial_3)^2], \tag{$*$}$$

the Eqs. (1) express via (1′) the scalar products

$$\langle \partial_2 \partial_3 f, \partial_1 f \rangle \tag{2}$$

by combinations of derivatives of g. Next, for commuting fields ∂_i, $i = 1, \ldots, 4$,

$$\partial_4 \langle \partial_2 \partial_3 f, \partial_1 f \rangle - \partial_2 \langle \partial_3 \partial_4 f, \partial_1 f \rangle = \langle \partial_2 \partial_3 f, \partial_1 \partial_4 f \rangle - \langle \partial_1 \partial_2 f, \partial_3 \partial_4 \mathrm{f} \rangle, \tag{3}$$

and so we express (3) by combinations of second derivatives of g. This amounts to Gauss' formula $\Phi_f^- = R(g)$.