Lectures on Curvature

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Euclidean, Spherical, Hyperbolic

In geodesic polar coordinates for the three model spaces of constant curvature:

Space	Metric	Radial factor $f(r)$
Euclidean $\mathbb{R}^n$	$dr^2 + r^2\, d\Omega^2_{n-1}$	r
Sphere S^n (radius 1)	$dr^2 + \sin^2 r\, d\Omega^2_{n-1}$	$\sin r$
Hyperbolic $\mathbb{H}^n$	$dr^2 + \sinh^2 r\, d\Omega^2_{n-1}$	$\sinh r$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) **Riemannian Variation Formula.** Let h_t, $t \in [0,\varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0,\varepsilon]$.

Notice that an arbitrary Riemannian metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0,\varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A^*_t = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A^*_t,$$

or, for brevity,

$$\partial_\nu h = 2A^*,$$

where

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1,...,n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y,y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

SIGN CONVENTION. The first derivative of h changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ of a hypersurface $Y \subset X$ depends on the *coorientation* of Y in X, where our convention is such that

the boundaries of *convex* domains have *positive (semi)definite* second fundamental forms A^*, also denoted II_Y, hence, *positive* mean curvatures, with respect to *the outward* normal vector fields.[1]

(2.1.B) First Variation Formula. This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t{}^2$$

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for C^1-smooth functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t{}^3$$

Now comes the first formula with the Riemannian curvature in it.

0.1 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannian manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y,\tau) = \kappa(X,\tau) + \wedge^2 A^*(\tau),$$

[1] At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.

[2] This come with the *minus* sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal(2019].

[3] This remains true for Lipschitz functions but if ψ is (badly) non-differentiable, e.g. it is equal to the characteristic function of a domain $U \subset Y$, then the derivative $\partial_{\psi\nu} vol_{n-1}(Y_t)$ may become (much) larger than this integral.

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y,y) = Sc(X,y) + \sum_{i\neq j} \alpha_i^*(y)\alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

• $\alpha_i^*(y)$, $i = 1, ..., n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;

• α^*-sum is taken over all ordered pairs (i,j) with $j \neq i$;

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν, τ_i) for ν being a unit (defined up to ±-sign) normal vector to Y;

• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu,\nu).$$

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i\neq j} \alpha_i\alpha_j = \left(\sum_i \alpha_i\right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

(Sc ≥ −2Ric) $$Sc(Y) \geq -2Ricci(\nu,\nu).$$

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n\max_i(c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape* that is the symmetric on $T(Y)$ associated with A^* via the Riemannian scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau,\tau) = \langle A(\tau), \tau\rangle_g \text{ for all } \tau \in T(Y).$$

0.2 Variation of the Curvature of Equidistant Hypersurfaces and Weyl's Tube Formula

(2.3.A) Second Main Formula of Riemannian Geometry.[4] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape operators $A_t = A(Y_t)$ satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where B_t is the symmetric associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $\left(\tau, \nu = \frac{d}{dt}\right)$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry. [5]

Tracing this formula yields

(2.3.B) Hermann Weyl's Tube Formula.

$$trace\left(\frac{dA_t}{dt}\right) = -||A^*||^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -||A^*||^2 - Ricci(\nu, \nu),$$

where

$$||A^*||^2 = ||A||^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y, y) = Sc(X, y) + \sum_{i\neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu, \nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i\neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

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[4]The first main formula is *Gauss' Theorema Egregium.*

[5]Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darmos(Gravitation einsteinienne) 1927] with the reference to Darboux' textbook of 1897.

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $\|A^*\|^2 = \sum_i (\alpha_i^*)^2$.

0.3 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called *umbilic* if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $Y = (Y, h)$ be a smooth Riemannian (n-1)-manifold and $\varphi = \varphi(t) > 0$, $t \in [0, \varepsilon]$ be a smooth positive function. Let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0, \varepsilon]$.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'(t)}{\varphi(t)}$, $i = 1, ..., n-1$ for

$$A_t^* = \frac{\varphi'(t)}{\varphi(t)} h_t \text{ for } \varphi' = \frac{d\varphi(t)}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}.$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2 h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

(c) What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[6]

Cylindrical Extension Exercise. Let Y be a smooth manifold, $X = Y \times \mathbb{R}_+$, let g_0 be a Riemannian metric in a neighbourhood of the boundary $Y = Y \times \{0\} = \partial X$, let h denote the Riemannian metric in Y induced from g_0 and let Y has *constant mean curvature* in X with respect to g_0.

Let X' be a (convex if you wish) ball in the standard (i.e complete simply connected) space with constant sectional curvature and of the same dimension n as X, let $Y' = \partial X'$ be its boundary sphere, let, let $Sc(h) > 0$ and let the mean and the scalar curvatures of Y and Y' are related by the following (comparison) inequality.

[<] $$\frac{|mean.curv_{g_0}(Y)|^2}{Sc(h,y)} < \frac{|mean.curv(Y')|^2}{Sc(Y')} \text{ for all } y \in Y.$$

Show that

if Y is compact, there exists a smooth positive function $\varphi(t)$, $0 \le t < \infty$, which is constant at infinity and such that the the warped product metric $g = \varphi^2 h + dt^2$ has

the same Bartnik data *as g_0, i.e.*

$$g|Y = h_0 \text{ and } mean.curv_g(Y) = mean.curv_{g_0}(Y),$$

Then show that

one *can't make* $Sc(g) \ge Sc(X')$ in general, if [<] is relaxed to the corresponding *non-strict* inequality, where an example is provided by the Bartnik data of $Y' \in X'$ itself.[7]

Vague Question. What are "simple natural" Riemannian metrics g on $X = Y \times \mathbb{R}_+$ with given Bartnik data $(Sc(Y), mean, curv(Y))$, where $Y \subset X$ is allowed *variable* mean curvature, and what are possibilities for lower bound on the scalar curvatures of such g granted $|mean.curv(Y,y)|^2/Sc(Y,y) < C$, e..g. for $C = |mean.curv(Y')|^2/Sc(Y')$ for Y' being a sphere in a space of constant curvature.

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[6] See §12 in [GL(complete) 1983].

[7] It follows from [Brendle-Marques(balls in S^n)N 2011] that the the cylinder $S^{n-1} \times \mathbb{R}_+$ admits a complete Riemannian metric g cylindrical at infinity which has $Sc(g) > n(n-1)$, and which has the same Bartnik data as the boundary sphere X'_0 in the hemisphere X' in the unit n-sphere. But the non-deformation result from [Brendle-Marques(balls in S^n) 2011], suggests that this might be impossible for the Bartnik data of *small* balls in the round sphere.

0.3.1 Higher Warped Products

Let Y and S be Riemannian manifolds with the metrics denoted dy^2 (which now play the role of the above dt^2) and ds^2 (instead of h), let $\varphi > 0$ be a smooth function on Y, and let

$$g = \varphi^2(y)ds^2 + dy^2$$

be the corresponding warped metric on $Y \times S$,

Then

$(\star\star)$

$$Sc(g)(y,s) = Sc(Y)(y) + \frac{1}{\varphi(y)^2}Sc(S)(s) - \frac{m(m-1)}{\varphi^2(y)}||\nabla\varphi(y)||^2 - \frac{2m}{\varphi(y)}\Delta\varphi(y),$$

where $m = dim(S)$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace on Y.

To prove this, apply the above c $(\star)$ to $l \times S$ for naturally parametrised geodesics $l \subset Y$ passing trough y and then average over the space of these l, that is the unit tangent sphere of Y at y.

The most relevant example here is where S is the real line $\mathbb{R}$ or the circle S^1 also denoted $\mathbb{T}^1$ and where $(\star)$ reduces to

$$(\star\star)_1 \qquad Sc(g)(y,s) = Sc(Y)(y) - \frac{2}{\varphi}\Delta\varphi(y).$$[8]

For instance, if the $L = -\Delta + \frac{1}{2}Sc$ on Y is strictly positive, that is the lowest eigenvalue λ is strictly positive and if φ equals to the corresponding eigenfunction of L, then

$$-\Delta\varphi = \lambda\cdot\varphi - \frac{1}{2}Sc\cdot\varphi$$

and

$$Sc(g) = 2\lambda > 0,$$

The basic feature of the metrics $\varphi^2(y)ds^2 + dy^2$ on $Y \times \mathbb{R}$ is that they are $\mathbb{R}$-*invariant*, where the quotients $(Y\times\mathbb{R})/\mathbb{Z} = Y \times \mathbb{T}^1$ carry the corresponding $\mathbb{T}^1$-*invariant* metrics, while *the* $\mathbb{R}$-*quotients are isometric to* Y.

Besides $\mathbb{R}$-invariance, a characteristic feature of warped product metrics is *integrability* of the tangent hyperplane field normal to the $\mathbb{R}$-orbits, where $Y \times \{0\} \subset Y \times \mathbb{R}$, being normal to these orbits, serves as an integral variety for this field.

Also notice that $Y = Y \times \{0\} \subset Y \times \mathbb{R}$ is totally geodesic with respect to the metric $\varphi^2(y)ds^2 + dy^2$, while the ($\mathbb{R}$-invariant) *curvature* (vector field) *of the* $\mathbb{R}$-*orbits* is equal to the *gradient field* $\nabla\varphi$ extended from Y to $Y\times\mathbb{R}$. coordinates

In what follows, we emphasize $\mathbb{R}$-invariance and interchangeably speak of $\mathbb{R}$-invariant metrics on $Y \times \mathbb{R}$ and metrics warped with factors φ^2 over Y.

Gauss-Bonnet $g^{\rtimes}$-*Exercise.* Let the above S be the Euclidean space $\mathbb{R}^N$ (make it $\mathbb{T}^n$ if you wish to keep compactness) with coordinates $t_1, ..., t_N$, let

$$\Phi(y) = (\varphi_1(y), ..., \varphi_i(y), ..., \varphi_N(y))$$

be an N-tuple of smooth positive function on a Riemannian mnanifold $Y = (Y, g)$ and define the (iterated t warped product) metric $g^{\rtimes} = g^{\rtimes}_{\Phi}$ on $Y \times S$ as follows:

$$g^{\rtimes} = g(y) + \varphi_1^2(y)dt_1^2 + \varphi_2^2(y)dt_2^2 + ... + \varphi_N^2(y)dt_N^2$$

[8] The roles of Y and $S = \mathbb{R}$ and notationally reversed here with respect to those in $(\star)$

Show that the scalar curvature of this metric, which, being $\mathbb{R}^N$-invariant, is regarded as a function on Y, satisfies:

$$Sc(g^{\rtimes}, y) = Sc(g) - 2\sum_{i=1}^{N} \Delta_g \log \varphi_i - \sum_{i=1}^{N} (\nabla_g \log \varphi_i)^2 - \left(\sum_{i=1}^{N} \nabla_g \log \varphi_i\right)^2,$$

thus

$$\int_Y Sc(g^{\rtimes}, y)dy \le \int_Y Sc(g, y)dy,$$

and, following [Zhu(rigidity) 2019], obtain the following

"Warped" Gauss-Bonnet Inequality for Closed Surfaces Y:

$$\int_Y Sc(g^{\rtimes}, y)dy \le 4\pi\chi(Y)$$

for the (iterated) warped product metrics $g^{\rtimes} = g^{\rtimes}_{\phi}$ for all positive N-tuples of Φ of positive functions on Y. [9]

0.4 Second Variation Formula

The Weyl formula also yields the following formula for the *second derivative* of the $(n-1)$-volume of a cooriented hypersurface $Y \subset X$ under a normal deformation of Y in X, where the scalar curvature of X plays an essential role.

The deformations we have in mind are by vector fields directed by geodesic normal to Y, where in the simplest case the norm of his field equals one.

In this case we have an equidistant motion $Y \mapsto Y_t$ as earlier and the second derivative of $vol_{n-1}(Y_t)$, denoted here $Vol = Vol_t$, is expressed in terms of of the shape $A_t = A(Y_t)$ of Y_t and the Ricci curvature of X, where, recall $trace(A_t) = mean.curv(Y_t)$ and

$$\partial_\nu Vol = \int_Y mean.curv(Y)dy$$

by the first variation formula.

Then, by Leibniz' rule,

$$\partial^2_\nu Vol = \partial_\nu \int_Y trace(A(y))dy = \int_Y trace^2(A(y))dy + \int_Y trace(\partial_\nu A(y))dy,$$

and where, by Weyl's formula,

$$trace(\partial_\nu A) = -trace(A^2) - Ricci(\nu, \nu)$$

for the normal unit field ν.

Thus,

$$\partial^2_\nu Vol = \int_Y (mean.curv)^2 - trace(A^2) - Ricci(\nu, \nu),$$

which, combining this with the above expression

$$Ricci(\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2\right),$$

[9] See [Zhu() 2019] and sections **??**, **??** for applications and generalizations.

shows that

$$\partial_\nu^2 Vol = \int \frac{1}{2}\left(Sc(Y) - Sc(X) + mean.curv^2 - \|A^*\|^2\right).$$

In particular, if $Sc(X) \geq 0$ and Y is minimal, then,

$$(\textstyle\int \mathrm{Sc} \geq 2\partial^2 \mathrm{Vol}) \qquad \int_Y Sc(Y,y)dy \geq 2\partial_\nu^2 Vol$$

(compare with the $(Sc \geq -2Ric)$ in 2.2).

Warning. Unless Y is minimal and despite the notation ∂_ν^2, this derivative depends on how the normal filed on $Y \subset X$ is extended to a vector filed on (a neighbourhood of Y in) X.

Illuminative Exercise. Check up this formula for concentric spheres of radii t in the spaces with constant sectional curvatures that are S^n, $\mathbb{R}^n$ and $\mathbf{H}^n$.

Now, let us allow a non-constant geodesic field normal to Y, call it $\psi\nu$, where $\psi(y)$ is a smooth function on Y and write down the full second variation formula as follows:

$$\partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y \|d\psi(y)\|^2 dy + R(y)\psi^2(y)dy$$

for

$$[\circ\circ] \qquad R(y) = \frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - \|A^*(Y)\|^2\right),$$

where $M(y)$ stands for the mean curvature of Y at $y \in Y$ and $\|A^*(Y)\|^2 = \sum_i (\alpha^*)^2$, $i = 1, ..., n-1$.

Notice, that the "new" term $\int_Y \|d\psi(y)\|^2 dy$ depends only on the normal field itself, while the R-term depends on the extension of $\psi\nu$ to X, unless

Y is minimal, where $[\circ\circ]$ reduces to

$$[**] \qquad \partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y \|d\psi\|^2 + \frac{1}{2}\left(Sc(Y) - Sc(X) - \|A^*\|^2\right)\psi^2.$$

Furthermore, if Y is volume minimizing in its neighbourhood, then $\partial_{\psi\nu}^2 vol_{n-1}(Y) \geq 0$; therefore,

$$[\star\star] \qquad \int_Y (\|d\psi\|^2 + \frac{1}{2}(Sc(Y))\psi^2 \geq \frac{1}{2}\int_Y (Sc(X,y) + \|A^*(Y)\|^2)\psi^2 dy$$

for all non-zero functions $\psi = \psi(y)$.

Then, if we recall that

$$\int_Y \|d\psi\|^2 dy = \int_Y \langle -\Delta\psi, \psi\rangle dy,$$

we will see that $[\star\star]$ says that

the $\psi \mapsto -\Delta\psi + \frac{1}{2}Sc(Y)\psi$ *is greater than*[10] $\psi \mapsto \frac{1}{2}(Sc(X,y) + \|A^*(Y)\|^2)\psi$.

Consequently,

if $Sc(X) > 0$, then the $-\Delta + \frac{1}{2}Sc(Y)$ on Y is positive.

[10] $A \geq B$ for selfadjoint operators signifies that $A - B$ is positive semidefinite.

Justification of the $||d\psi||^2$ *Term.* Let $X = Y \times \mathbb{R}$ with the product metric and let $Y = Y_0 = Y \times \{0\}$ and $Y_{\varepsilon\psi} \subset X$ be the graph of the function $\varepsilon\psi$ on Y. Then

$$vol_{n-1}(Y_{\varepsilon\psi}) = \int_Y \sqrt{1 + \varepsilon^2||d\psi||^2} dy = vol_{n-1}(Y) + \frac{1}{2}\int_Y \varepsilon^2||d\psi||^2 + o(\varepsilon^2)$$

by the Pythagorean theorem

and

$$\frac{d^2 vol_{n-1}(Y_{\varepsilon\psi})}{d^2\varepsilon} = ||d\psi||^2 + o(1).$$

by the binomial formula.

This proves [∘∘] for product manifolds and the general case follows by *linearity/naturality/functoriality* of the formula [∘∘].

Naturality Problem. All "true formulas" in the Riemannian geometry should be derived with minimal, if any, amount of calculation – only on the basis of their "naturality" and/or of their validity in simple examples, where these formulas are obvious.

Unfortunately, this "naturality principle" is absent from the textbooks on differential geometry, but, I guess, it may be found in some algebraic articles (books?).

Exercise. Derive the second main formula 2.3.A by pure thought from its manifestations in the examples in the above *illuminative exercise.*[11]

0.5 Conformal Laplacian and the Scalar Curvature of Conformally and non-Conformally Scaled Riemannian Metrics

Let (X_0, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and let $\varphi = \varphi(x)$ be a smooth positive function on X.

Then, by a straightforward calculation,[12]

◐ $$Sc(\varphi^2 g_0) = \gamma_n^{-1}\varphi^{-\frac{n+2}{2}} L(\varphi^{\frac{n-2}{2}}),$$

where L is the *conformal Laplace* on (X_0, g_0)

$$L(f(x)) = -\Delta f(x) + \gamma_n Sc(g_0, x) f(x)$$

for the ordinary Laplace (Beltrami) $\Delta f = \Delta_{g_0} f = \sum_i \partial_{ii} f$ and $\gamma_n = \frac{n-2}{4(n-1)}$.

Thus, we conclude to the following.

Kazdan-Warner Conformal Change Theorem. [13] Let $X = (X, g_0)$ be a closed Riemannian manifold, such the the conformal Laplace L is positive.

Then X admits a Riemannian metric g (conformal to g_0) for which $Sc(g) > 0$.

Proof. Since L is positive, its first eigenfunction, say $f(x)$ is positive[14] and since $L(f) = \lambda f$, $\lambda > 0$,

[11] I haven't myself solved this exercise.
[12] There must be a better argument.
[13] [Kazdan-Warner(conformal) 1975]: *Scalar curvature and conformal deformation of Riemannian structure.*
[14] We explain this in section ??.

$$Sc\left(f^{\frac{4}{n-2}}g_0\right) = \gamma_n^{-1}L(f)f^{-\frac{n+2}{n-2}} = \gamma_n^{-1}f^{\frac{2n}{n-2}} > 0.$$

Example: Schwarzschild metric. If (X_0, g_0) is the Euclidean 3-space, and $f = f(x)$ is positive function, then

the sign of $Sc(f^4 g_0)$ is equal to that of $-\Delta f$.

In particular, since the function $\frac{1}{r} = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}$, is harmonic, *the Schwarzschild metric $g_{Sw} = \left(1 + \frac{m}{2r}\right)^4 g_0$ has zero scalar curvature.*

If $m > 0$, then this metric is defined for all $r > 0$ and it is invariant under the involution $r \mapsto \frac{m^2}{r}$.

If $m = 0$, this the flat Euclidian metric.

If $m < 0$, then this metric is defined only for $r > m$ with a singularity ar $r = m$.

$$S[g] = \frac{1}{16\pi G}\int_M R\sqrt{-g}\,d^4x + \int_M \mathcal{L}_{\text{matter}}\sqrt{-g}\,d^4x$$

$$\delta S = 0$$

$$\delta S_g = \frac{1}{16\pi G}\int_M \left(R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

$$\delta S_{\text{matter}} = -\tfrac{1}{2}\int_M T_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

$$R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} = 8\pi G\,T_{\mu\nu}$$

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Einstein–Hilbert action principle.

$$S[g] = \frac{1}{16\pi G}\int_M R\sqrt{-g}\,d^4x + \int_M \mathcal{L}_{\text{matter}}\sqrt{-g}\,d^4x$$

$$\delta S = 0$$

The variation is taken with respect to the metric $g_{\mu\nu}$.
Result of the Variation:

$$\delta S_g = \frac{1}{16\pi G}\int_M \left(R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

and from the matter part,

$$\delta S_{\text{matter}} = -\tfrac{1}{2}\int_M T_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

Setting the total variation $\delta S = 0$ for arbitrary $\delta g^{\mu\nu}$ gives

$$R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} = 8\pi G\,T_{\mu\nu}$$

Notes:

- R: scalar curvature - $R_{\mu\nu}$: Ricci tensor - $g_{\mu\nu}$: spacetime metric - $T_{\mu\nu}$: stress–energy tensor - $g = \det(g_{\mu\nu})$

If the manifold has a boundary, one adds the Gibbons–Hawking–York term to make the variation well-defined:

$$S_{\mathrm{GHY}} = \frac{1}{8\pi G} \int_{\partial M} K \sqrt{|h|}\, d^3x$$

where K is the trace of the extrinsic curvature of the boundary and h is the determinant of the induced metric on ∂M.