Lectures on Immersions with Controlled Curvatures

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Historical Preamble: Heinz Hopf: Selected Topics in Geometry, New York University 1946, Notes by Peter Lax.[1]

Heinz Hopf (19 November 1894 – 3 June 1971)

In 1925, he proved that any simply connected complete Riemannian 3-manifold of constant sectional curvature is globally isometric to Euclidean, spherical, or hyperbolic space.

In 1931, Hopf discovered the Hopf invariant of maps $S^3 \to S^2$ ("element of the architecture of our world" in the words of Penrose) and proved that the Hopf fibration has invariant 1. This:

(1) disproved the then standing intuitive conjecture that the continuous maps between spheres $S^N \to S^n$, $N > n$, are contractible;

(2) Opened the door to the world of vector bundles and the topology of spinors, where the curvature of the Hopf bundle is 1/2 curvature of the 2-sphere.

(Hopf bundle and Dirac Monopole `https://personal.math.ubc.ca/~mihmar/HopfDirac.pdf`,`https://www.sciencedirect.com/science/article/abs/pii/S0393044002001213` `https://ncatlab.org/nlab/show/Hopf%20fibration`)

Peter David Lax (1 May 1926 – 16 May 2025)

After the war ended, Lax remained with the Army at Los Alamos for another year and eventually returned to NYU for the 1946–1947 academic year.

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1 Definitions, Problems and First Examples

Notation: $\overrightarrow{\mathbf{curv}}_\tau$, $\mathbf{curv}^\perp_\mathbf{x}$ and $\mathbf{curv}^\perp(\mathbf{X})$. Let $f : X \to \mathbb{R}^N$ be a smooth immersion, let $\tau = \tau_x \in T_x(X)$ be a tangent vector and let $\gamma_\tau \subset X$ be a geodesic

[1] `https://link.springer.com/book/10.1007/3-540-39482-6` Among many other things, there is a proof of Legendre-Cauchy-A. Schur "Arms-Bow-Lemma" on pp 31-32 in these lecture (attributed by Hopf to E. Schmidt), which has been reproduced in all further publication concerning this theorem. e.g. in `https://www.scribd.com/document/759520702/Chern-Curves-and-surfaces-in-Euclidean-spaces`

in X issuing from x with the speed τ. Then the *normal curvature vector*

$$\overrightarrow{curv}_\tau(X) \in \mathbb{T}_{f(x)}(X) \in \mathbb{R}^N) = \mathbb{R}^N$$

is equal the acceleration (the second derivative) at $f(x)$ of a point moving along the curve $f(\gamma)$ in $\mathbb{R}^N$.

Granted this, define

$$curv_x^\perp(X) = \sup_{\|\tau_x\|=1} curv_{\tau_x}(X) \text{ and } curv^\perp(X) = \sup_{x\in X} curv_x(X).$$

If $dim(X) = 1$, say $X = [0,1]$ and the curve $X \overset{f}{\hookrightarrow} \mathbb{R}^N$ is parametrized by arc length, that is

$$\left\|\frac{df(x)}{dx}\right\| = 1,$$

then this is the usual curvature of a curve,

$$\overrightarrow{curv}(X,x) = \frac{d^2f(x)}{dx^2} \text{and } curv^\perp(X) = \sup_{x\in X}\left\|\frac{d^2f(x)}{dx^2}\right\|.$$

Thus,

the normal curvature $curv^\perp(X \overset{f}{\hookrightarrow} \mathbb{R}^N)$ *is equal to the supremum of the normal curvatures of the* f*-images in* $\mathbb{R}^N$ *of the geodesics from* X.

Locality of the Curvature and Curvature of Submanifolds. Since curvature of an immersion at a point $x \in X$ is a local invariant and since immersions locally are embeddings, the definition and many properties of curvatures of immersions formally follow from those for submanifolds $X \subset \mathbb{X}^N$. In this in mind, we may often (but not always) speak of curvatures of "immersed submanifolds", and accordingly simplify our notation.

Curvatures of Spheres. Spheres $S^n(R)$ of radius R of all dimensions n in the N-space $\mathbb{R}^N$, $N > n$, satisfy

$$curv^\perp(S^n(R)) = \|\overrightarrow{curv}_\tau^\perp(S^n(R))\| = 1/R \text{ for all unit tangent vectors } \tau \in T(S^n(R)).$$

❂ *The unit* n*-spheres* $S^n(R=1) \subset B^N(1)$, *are the only closed connected immersed* n*-sub-manifolds with curvatures* ≤ 1 *in the Euclidean space* $\mathbb{R}^N$, *which are contained in the unit* N*-ball* $B^N(1)$, *except for* $n = 1$, *where multiple covering of the unit circle are also such manifolds.*

This follows by the *maximum principle* applied to the distance function from X to the boundary $\partial B^N(1))$ or equivalently to the squared distance to the centre of the ball B^N denoted $r^2(x)$.

In fact, since $curv^\perp(X) \leq 1$, the second derivatives of r^2 along geodesics parametrized by the arc length satisfy:

$$\|r''r\| \leq 1 \text{ and } (r^2)'' = 2(r''r + \|r'\|^2) \geq 0, \text{ since } \|r'\|^2 = 1.$$

This says that r^2 is *a convex*, hence constant=1 function on X. Thus, X is contained in the unit sphere $S^{N-1}(1) = \partial B(1)$, where it has zero normal curvature by Pythagorean formula (1.1.C); hence, totally geodesic, (compare with 3.B).

Compact, Closed, Complete. Curvature has a limited effect on immersion of *open manifolds*, i.e. those which contain*no compact connected components without boundaries*, called *closed manifolds.*

For instance, according to the generalized *Smale-Hirsch h-principle*[2], an arbitrary immersion f of open manifold X to an open subset $U \subset \mathbb{R}^N$ admits a homotopy (even a regular homotopy[3] to an immersion f_ε, such that

$$curv^\perp(X \overset{f_\varepsilon}{\hookrightarrow} U) \leq \varepsilon \text{ for a given } \varepsilon > 0.$$

In what follows, we focus on immersions of closed manifold X, where much of what we do equally applies to *complete immersed* manifolds $X \hookrightarrow \mathbb{R}^N$, i.e. where the induced Riemannian metrics in X, sometimes called *inner metrics*, are *geodesically complete*: geodesics starting at all point $x \in X$ extend infinitely in all directions $\tau_x \in T_x(X)$.

Exercise. Generalize ❂ to complete immersed $X \hookrightarrow B^N(1)$.

Extremal Immersions. We are much interested in *$curv^\perp$-extremal immersions* between Riemannian manifolds, $f : X \hookrightarrow Y$, especially for $Y = \mathbb{R}^N$, which minimize some geometric size invariant of the image $f(X) \subset Y$, such as $diam_Y(f(X))$, among all immersions with $curv^\perp \leq c$[4] or among all such immersion *regularly homotopic* to a given one. Beside the diameter, it may be, some kind of *width*, the *radius of the minimal ball* which contained $f(X)$, etc.

If we don't ¯specify any invariant, we call an immersion $f_0 : X \hookrightarrow Y$ *simple extremal* if it admits *no regular homotopy* $f_t : X \hookrightarrow Y$, such that $curv^\perp(f_1) < curv^\perp(f_0)$, where the local version of this says that all regular homotopies f_t, satisfy $curv^\perp(f_t) \geq curv^\perp(f_0)$ for $t > 0$.

If $Y = \mathbb{R}^N$, then this may be applied to the convex hull $Y_0 = conv(f(X)) \supset f(X)$ and then an immersion $f_0 : X \hookrightarrow \mathbb{R}^N$ is called *conv-$curv^\perp$-extremal* if one can't decrease the normal curvature of f_0 by a regular homotopy of immersions $f_t : X \hookrightarrow conv(f_0(X))$.

1.A. *Basic Spherical Example.* By ❂, spheres $S^n(1/c) \subset \mathbb{R}^N$ are extremal with respect to all above criteria.

1.B. *Piecewise C^2 Circular Example.* Some naturally arising submanifolds with bounded normal curvatures, e.g. many extremal ones are C^1-smooth and only *piecewise* C^2.[5]

For instance, immersed closed curves, which go around several circles (possibly going around each circle many times) in the figure below, have $curv^\perp$ equal to the curvature of the smallest circle.

These curves are C^1-smooth but they are not C^2: their curvatures jump as they switch the tracks from one circle to another at the contact points between circles.

1.C. ∞*-Subexample.* Let $f : S^1 \hookrightarrow B^2(1) \subset \mathbb{R}^2$ be a C^1 immersion with

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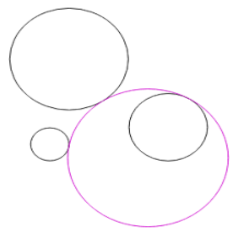
[2]See [C-E-M] and references therein.

[3]A regular homotopy is a path in the space of C^1 immersions with the usual C^1-topology, that is $f_t : X \to Y$, $t \in [0, 1]$, where the differential df_t of f_t in x-variables, $x \in X$, is continuous in t.

[4]Our definitions of $curv^\perp$ naturally generalize to all Riemannian manifold Y receiving immersions from X.

[5]This is a well know phenomenon in the *optimal control theory*, where one is predominantly concerned with $n = 1$, [Feld 1965].



curvature

$$curv^{\perp}(S^1 \overset{f}{\hookrightarrow} \mathbb{R}^2) \le 2.$$

If the corresponding oriented Gauss map to the unit circle

$$\overrightarrow{G}_f = \frac{df}{\|f\|} : S^1 \to S^1 \subset \mathbb{R}^2$$

has degree zero (hence contractible), then the image of f is equal to the union of two circles of radii 1/2, which meet at the center of the disc $B^2(1)$, where they are tangent one to another. Thus the figure ∞ immersion is "radially extremal": it *minimizes the radius of the 2-ball.* around it. (We shall explain why this is so in section 10).

Bi-invariants $\mathbf{curv}^{\perp}_{\mathbf{min}}(\mathbf{X},\mathbf{Y})$ ***and*** $\mathbf{Imm}_{\perp\le\mathbf{c}}(\mathbf{X},\mathbf{Y})$. Let X be a smooth closed manifold and and Y a Riemannian manifold and let $curv^{\perp}_{min}(X,Y)$ be the infimum of normal curvatures of smooth immersions $X \hookrightarrow Y$.

Now, if we choose and fix a particular Y, e.g. the unit ball in $\mathbb{R}^N$, the number $curv^{\perp}_{min}(X,Y)$ becomes a *topological invariant* of X, the value of which is unknown for most n-manifolds and $N > n$.

Dually, given a topological n-manifold, e.g. (homeomorphic to) the product of spheres, the minimal $curv_{\min}(X,Y)$ of immersions $X \hookrightarrow Y$ appears as a *metric invariant* of Y, which is unknown in most cases, for instance, for the N-balls and cubes $Y \subset \mathbb{R}^N$.

The number $\mathbf{curv}^{\perp}_{\mathbf{min}}(\mathbf{X},\mathbf{Y})$ carries only a small part of the information about immersions $f : X \hookrightarrow Y$ with curvatures $curv^{\perp}(f) \le c$.

A more comprehensive information is contained in the *homotopy types* of the spaces of immersions with $curv^{\perp}(f) \le c$, denoted $Imm_{\perp\le c}(X,Y)$ and the *homotopy classes* of the inclusion maps

$$Imm_{\perp\le c_1}(X,Y) \subset Imm_{\perp\le c_2}(X,Y),\ c_1 \le c_2,$$

where much of this information is encoded by the diagram of the natural (co)homology homomorphisms between these spaces.

1.1 Alternative Definitions of Normal Curvature

1.1.A. The full *second order infinitesimal information* of a smooth submanifold X in a Riemannian manifold Y, e.g. in the Euclidean N-space, at a point $x \in X$ is algebraically represented by the *second fundamental form* that is a *symmetric bilinear* form on X with values in the normal vector space $T^{\perp}(X) \subset T(Y)$, denoted

$$\mathrm{II}(X,x) = \mathrm{II}(X,x,\tau_1,\tau_2) = \mathrm{II}_x(\tau_1,\tau_2),$$

where $\tau_1, \tau_2 \in T_x(X)$ are tangent vectors to X and where the value $\mathrm{II}(\tau_1, \tau_2)$ is a vector in $T_x(Y)$ normal to the tangent (sub)space $T_x(X) \subset T_x(Y)$. This form in the case $Y = \mathbb{R}^N$ is defined as the *second differential* of a vector function, say $\Phi : T_x(X) \to T_x^\perp(X)$, such that the graph of Φ in a neighbourhood of $x \in \mathbb{R}^N \supset X$ is equal to $X \subset \mathbb{R}^N = T_x(\mathbb{R}^N) = T_x(X) \oplus T_x^\perp(X)$,

$$\mathrm{II}_x(\tau_1, \tau_2) = \partial_{\tau_1}\partial_{\tau_2}\Phi(x)$$

In the general case, this definition applies by equating $T_x(Y))$ with a small neighbourhood in Y via the exponential map $\exp_x : T_x(Y) \to Y$.

Exercises. **1.1.B.** Show that $\mathrm{II}(\tau, \tau)$ is equal to the second (covariant) derivative in Y of the geodesic in X issuing from x with the velocity τ, and that

$$[\tau_1\tau_2]_\leq \qquad \|\mathrm{II}_x(\tau_1, \tau_2)\| \leq curv_x^\perp(X)$$

for all $x \in X$ and all unit tangent vectors $\tau_1, \tau_2 \in T_X(X)$.

1.1.C. Pythagorean Curvature Composition Let $X \hookrightarrow Y \hookrightarrow Z$ be isometric embeddings (or immersions) between Riemannian manifolds, i.e the Riemannian metrics in Y and in X are induced from a Riemannian metric in Z. Show that

$$curv_\tau^\perp(X \hookrightarrow Z) = \sqrt{(curv_\tau^\perp(X \hookrightarrow Y))^2 + (curv_\tau^\perp(Y \hookrightarrow Z))^2}$$

for all tangent vectors $\tau \in T(X) \hookrightarrow T(Y) \hookrightarrow T(Z)$.

For instance, if $X \hookrightarrow Y = S^{N-1}(1) \hookrightarrow Z = \mathbb{R}^N$ then

$$curv_\tau^\perp(X \hookrightarrow \mathbb{R}^N) = \sqrt{(curv_\tau^\perp(X \hookrightarrow S^{N-1}))^2 + 1}.$$

1.1.D. Geodesic free definition of $curv^\perp$. Show that the normal curvature $curv_\tau^\perp(X \hookrightarrow Y) = \|\overrightarrow{curv}_\tau\|$, $\|\tau\| = 1$, is equal to the infimum of the Y-curvatures $curv_Y^\perp$-curvatures of the curves in X tangent to τ.

This description of $curv^\perp$, which doesn't refer to geodesics, has an advantage of being applicable to mechanical systems with *non-holonomic* constrains that are submanifolds in the tangent bundle of Y, say $\mathcal{X} \subset T(Y)$ rather $X \subset$.

1.1.E. Metric Definition of II. Let $Y = (Y, g)$ be a Riemannian manifold, e.g. $Y = (\mathbb{R}^N, g = \sum_{j=1}^N dy_j^2)$, let $X \subset Y$ be a smooth submanifold, let $\nu \in T_x^\perp(X)$ be a normal vector to X at x and $\tilde{\nu}$ be a smooth vector field on Y, which extend ν_x.

Let $g_{|X}$ be the restriction of the Riemannian quadratic form g to X and let $\tilde{g}'_{|X}$ be the restriction of (Lie) *derivative of g by the field $\tilde{\nu}$ to X.*

Show that the value $\tilde{g}'_{|X}(\tau_1, \tau_2)$ for $\tau_1, \tau_2 \in T_x(X)$ depends only on ν but not on the extension $\tilde{\nu}$ of ν.

Moreoever, show that

$$\tilde{g}'_{|X}(\tau_1, \tau_2) = \langle \nu, \mathrm{II}_x(\tau_1, \tau_2)\rangle_g,$$

and that the second fundamental form II is *uniquely determined* by this identity.

(The definition of the second fundamental form II as the derivative $\tilde{g}'_{|X}$ of the induced Riemannian form uses no covariant derivatives or geodesics either in X or in Y.)

1.1.F. Normal Curvature Defined via the Gauss Map. Let $\mathcal{H} = Gr_n(N)$ be the space of n-dimensional linear subspaces $H \subset \mathbb{R}^N$ and natuarally identify the tangent space $T_H(\mathcal{H}$ with the space of linear maps from H to the normal space $H^\perp \subset \mathbb{R}^N$,

$$T_H(\mathcal{H}) = hom(H, H^\perp).$$

Let $f : X \hookrightarrow \mathbb{R}^N$ be a smooth immersed submanifold and $\overleftrightarrow{G} : X \to Gr_n(N)$, $n = dim(X)$, be the (non-oriented) Gauss map where $\overleftrightarrow{G}(x)$ is the linear subspace parallel to tangent subspace of X in R^N (regarded as an affine subspace) at x.

Let $D_x\overleftrightarrow{G} : T_x(X) \to T_x(X)^\perp$ be the differential of the map $\overleftrightarrow{G}$ at $x \in X$ regarded as a linear operator $T_x(X) \to T_x^\perp(X)^\perp$.

Show that

the normal curvature of X at x is equal to the norm of the operator $D_x\overleftrightarrow{G}$,

$[D\overleftrightarrow{G}]^\perp$ $$curv_x^\perp(X) = \sup_{\tau \in T_x(X), \|\tau\} = 1} \|D_x\overleftrightarrow{G}(\tau)\|$$

and derive from this the following corollary.

1.1.G. Angular Arc Inequality. If the (inner) distance between two points $x_1, x_1 \in X$ satisfies

$$dist_X(x_1, \underline{x}) \le \alpha(curv^\perp(X))^{-1}, \ \alpha \le \pi/2,$$

then the angles between vectors $\tau \in T_{x_1}(X)$ and their images $\bar{\tau}$ under the normal projection $T_{x_1}(X) \to T_{x_1}(X)$ satisfy

$$\angle(\tau, \bar{\tau}) \le \alpha,$$

where the equality holds if and only if there exists a

planar α-arc of radius $\frac{1}{curv^\perp(X)}$, which is contained in X, which join x_1 with $\underline{x}$ and such that τ is tangent to this arc at its x_1-end.

Conversely, the inequality $\angle(\tau, \bar{\tau}) \le \epsilon/c + o(\epsilon)$, $c \ge 0$, for all pairs of ϵ-infinitesimally closed points implies that $curv^\perp(X) \le c$.

no non-zero tangent vector $\tau_1 \in T_{x_1}(X)$ is normal to $T_{\underline{x}}(X)$.

Moreover the same non-normality conclusion holds if

$$dist_X(x_1, \underline{x}) \le \frac{\pi}{2}(curv^\perp(X))^{-1},$$

unless there exists a

planar semicircle of radius $\frac{1}{curv^\perp(X)}$ contained in X and joining x_1 with $\underline{x}$.

1.1.H. Polygonal Curves. Given a spacial polygonal curve P with vertices p_i let $c_i = c_i(P) = c(P, p_i)$ denote the "external" angles of P,

$$c_i = c(P, p_i) = \pi - \angle(P, p_i)$$

where $0 \le \angle(P, p_i) \le \pi$ is the angle between two segments of the curve adjacent to p_i that are $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$.[6]

[6] According to our present convention even non-convex planar polygons have all there angles measured between zero and π.

◯ Let $P \subset \mathbb{R}^N$ be a *closed. connected* spacial polygonal curve with k vertices p_i (where $p_1 = p_k$ by the cyclic convention) and let us decompose P to triangles $\triangle_j$, e.g. by drawing $k-3$ segments $[p_1, p_i] \subset \mathbb{R}^N$. for all $i \neq 1, 2, k-1$.[7]

Observe that the angles of triangles $\triangle_{j_i}$ adjacent to p_i satisfy the following "trianagle kind inequaity"

$$\sum_{j_i} \angle(\triangle_{j_i}, p_i) \geq \angle(P, p_i) \text{ for all } i,$$

where the equality implies that

- the triangles $\triangle_{j_i}$ lie in the same 2-plane (which depends on i)
- the triangles $\triangle_{j_i}$ do not overlap
- the union of these trianles is convex.

It follows that the sum of the "external angles" of P satisfies

$$\sum_{i=1}^{k} c_i \geq 2\pi,$$

where the equality $\sum_{i=1}^{k} c_i \geq 2\pi$ holds if and only if P is a *planar convex* curve.

Exercise. Let $P \hookrightarrow \mathbb{R}^2$ be a planar connected, immersed *locally convex* polygonal curve.

Show that $\sum_i c_i(P) = 2\pi d$, where d is a positive integer and that two such curves P_0 and P_1 can be joint by a homotopy P_t, $t \in [0,1]$ of immersed locally convex curves if and only if $\sum c_i(P_0) = \sum c_i(P_1)$.

1.1.I. Polygonal Approximation Exercises. Let a spacial curve X be represented by a continuous map $x : [0, l] \to \mathbb{R}^N$ and let $s_i = \varepsilon i \in [0, l]$, for $i = 1, 2, ..., k$ and $\varepsilon = l/k$ and let $P_\varepsilon \hookrightarrow \mathbb{R}^N$ be the polygonal curve with vertices $p_i = x(s_i) \in \mathbb{R}^N$ and segments $[p_i, p_{i+1}] \subset \mathbb{R}^N$

(a) Show that if X is a smooth immersed curve, then the sums of the "external angles" of P_ε approximate the total curvature of X

$$\sum_i c_i(P_\varepsilon) \to \int_0^l curv^\perp(x(s))ds \text{ for } \varepsilon \to 0,$$

$s \in [0, l]$ is arc length parameter on X.

(b) Use this approximation for the definition of the curvature of X as the density of the weak limit of the measures $\sum_i c_i \delta(p_i)$, where $\delta(p_i)$ are Dirac's δ-measures at the points p_i.

(c) Similary define curvature measures of *piecewise smooth* immersed curves and also localy convex planar curves.

(d) Prove *Fenchel total curvature* $\geq 2\pi$*-Inequality* for closed smooth immersed curves

$$\int_{S^1} curv^\perp(x(s))ds \geq 2\pi$$

where X is paramerized by the unit circle S^1 and then generalise this to all closed curves. [8]

(e) Construct smooth embedded curves $X_{d,\varepsilon} \subset \mathbb{R}^3$ for all $d = 1, 2, ...$ and $\varepsilon > 0$ with total curvatures $2\pi + \varepsilon$, which have linking numbers d with a straight line in $\mathbb{R}^3$.

[7] One can decompose P to $\approx \log_2 k$ triangles.

[8] See [Chern] and also sections 7-9 for related matters.

(f) Let r be the normal projection from the 3-space to the line $\mathbb{R} = \mathbb{R}_r$ in space defined by a unit vector $r \in S^2 \subset \mathbb{R}^3$. Let $N_r(X)$ be the number of critical points of the composed function $S^1\mathbb{R}_r$ for $s \mapsto r \circ x(s)$.

Show that
$$\int_{S^2} N_r(X)dr = 4\int_{S^1} curv^\perp(x(s))ds$$

Hint. Apply Crofton's formula to the spherical curve $s \mapsto x'(s) \in S^2$.

(e) Prove *Fáry–Milnor theorem.* If the total curvature a closed embedded $X \subset \mathbb{R}^3$ is strictly less than 2π then X is unknotted.

2 Products of Spheres, Clifford's sub-Tori with Small Curvatures and Petrunin Inequality

The product X of spheres $S^{n_i}(R_i) \subset \mathbb{R}^{N_i=n_i+1}$, $i = 1, ...m$,

$$X = S^{n_1}(R_1) \times S^{n_2}(R_2) \times ... \times S^{n_m}(R_m) \subset \mathbb{R}^{N=(n_1+n_2+...+n_m)+m},$$

has the curvature equal to the maximum of $1/R_i$, $i = 1...., m$, and if

$$R_1^2 + R_2^2 + ... + R_m^2 \leq 1,$$

then X is contained in the unit ball in $\mathbb{R}^N$. (If $R_1^2 + R_2^2 + ... + R_m^2 = 1$, then X is contained in the unit sphere $S^{N-1}(1) = \partial B^N(1) \subset \mathbb{R}^N$.)

For example, the product of m-copies of S^n admits an embedding to the unit ball in $\mathbb{R}^{mn+m}$, where

$$curv^\perp\big((S^n)^m \subset B^{mn+m}(1)\big) = \sqrt{m}$$

The main instance of this is the *Clifford n-torus*, that the product of n circles imbedded to the unit $2n$ ball, such that

$$curv^\perp\big(\mathbb{T}^n \subset \partial B^{2n}(1)\big) = \sqrt{n}.$$

It is conceivable that the above (Clifford's) products of spheres $S^{n_1}(R_1) \times S^{n_2}(R_2) \times ... \times S^{n_m}(R_m) \subset \mathbb{R}^N$ are *conv-curv$^\perp$-extremal,* where this seems realistic for $m < min_i n_i$, but we have no idea, for instance, if there are immersions of n-tori to $B^{2n}(1)$ with $curv^\perp < \sqrt{n}$.

Yet, if $N >> n$, then the n-torus can be immersed to the unit ball $B^N(1)$ with *unexpectdly small* curvature.

2.A. $\sqrt{3}$-Clifford Sub-Torus Theorem. (Section ?) **[a]** If N is much greater than n, then the Clifford torus

$$\mathbb{T}^N \subset S^{2N-1} \subset B^{2N}(1),$$

contains an n-subtorus $\mathbb{T}^n_o \subset \mathbb{T}^N$, such that the normal curvature of this n-torus the ambient Euclidean space $\mathbb{R}^{2N} \supset B^{2N} \supset \mathbb{T}^n_0$ satisfies

$$\left[\tfrac{3n}{n+2}\right]_{\mathbb{T}^n} \qquad curv^\perp\big(\mathbb{T}^n_o \subset B^{2N}(1)\big) \leq \sqrt{\frac{3n}{n+2}}.$$

One has a poor bound on the best (i.e. the smallest) N for this purpose, (something like 10^{10^n}, see section 13) but

[b] *if $N \geq 8n^2 + 8$, then, there exists a a locally isometric (with respect to the Euclidean metrics in $\mathbb{R}^n$ and $\mathbb{T}^N$) map, that is a group homomorphism*

$$g : \mathbb{R}^n \hookrightarrow \mathbb{T}^N \subset B^{2N}(1),$$

such that

$\left[\frac{3n}{n+2}\right]_{\mathbb{R}^n}$.
$$curv^{\perp}\big(\mathbb{R}^n \hookrightarrow B^{2N}(1)\big) \leq \sqrt{\frac{3n}{n+2}}.$$

[c] It follows that *for all $\varepsilon > 0$, there exists a sub-torus* $^{-}$

$$\mathbb{T}^n_{\varepsilon} \subset \mathbb{T}^N \subset B^{2N}(1),$$

such that

$\left[\frac{3n}{n+2} + \varepsilon\right]_{\mathbb{T}^n}$
$$curv^{\perp}\big(\mathbb{T}^n_{\varepsilon} \subset B^{2N}(1)\big) \leq \sqrt{\frac{3n}{n+2}} + \varepsilon.$$

2.B. $\sqrt{3}$-Immersion Corollary. *Let $f : X \hookrightarrow \mathbb{R}^m$ be an immersion then, for all $\epsilon > 0$, there exist an immersion (actually an embedding) f_ϵ to the unit ball B^{16m^2+16m} with curvature*

$$curv^{\perp}\big(X \overset{f_\varepsilon}{\subset} B^{16m^2+16m}(1)\big) \leq \sqrt{\frac{3m}{m+2}} + \varepsilon.$$

Proof. Let λ be a large constant, $\lambda >> 1/\epsilon$, *scale the manifold* $X \overset{f}{\hookrightarrow} \mathbb{R}^m$ by λ and compose the scaled map $\lambda \cdot f : X \hookrightarrow \mathbb{R}^m$ with the map $g : \mathbb{R}^m \hookrightarrow \mathbb{T}^N \subset B^{2N}(1)$ from the above **[b]**.

Then, if one one wishes, one slightly perturbs the resulting immersion $X :\to\hookrightarrow \mathbb{T}^N$. and makes it an embedding.

On sharpness of $\left[\frac{3n}{n+2}\right]$. It is not hard to show that the Euclidean curvatures of all Clifford subtori $\mathbb{T}^n \subset \mathbb{T}^N \subset \mathbb{R}^{2N}$ (these $\mathbb{T}^n$ are very special submanifolds in $B^{2N}(1) \supset \mathbb{T}^N$)) satisfy $curv^{\perp}_{\mathbb{R}^{2n}}(\mathbb{T}^n) \geq \sqrt{\frac{3n}{n+2}}$, but the following is not so obvious.

2.C. Petrunin's $\sqrt{3}$-Inequality. (Section 14.2) *All immersions $\mathbb{T}^n \hookrightarrow B^N(1)$ satisfy*

$$curv^{\perp}\big(\mathbb{T}^n \hookrightarrow B^N(1)\big) \geq \sqrt{\frac{3n}{n+2}} \text{ for all } n \geq 1 \text{ and all } N.$$

It is unclear what is, in general, the geometry of immersions $\mathbb{T}^n \hookrightarrow B^N(1)$ with $curv^{\perp} \approx \sqrt{3}n$[9] depending on the ambient dimension N. Conceivably the n-tori admit no immersions $\mathbb{T}^n \hookrightarrow B^N(1)$ with $curv^{\perp} \leq \sqrt{3}$ for $N << n^2$, but we have no means to rule out such immersions, say for for $N \leq 3n$ and $n \geq 4$.

[9]Anton Petrunin told me that there exist extremal tori in $B^N(1)$. which are not contained in S^{N-1}.

3 Focal Radius and $+\rho$-Encircling

.

Let Y be a complete Riemannian manifold, let $X \hookrightarrow Y$ be a smooth embedded or immersed submanifold, let $x_0 \in X$, let $\nu_0 \in T^{\perp}_{x_0}(X)$ be a unit normal vector at the point x_0 and $\gamma_\nu \hookrightarrow Y$ be a geodesic ray issuing from x_0 in the ν_0-direction.

Define ν_0-*focal radius* $rad^{\perp}_{\nu_0}(X)$ as the supremum of $r \geq 0$, such that the the segment $[x_0, y] \subset \gamma_0$ *locally minimises* the length of curves in Y between y and X, that is all curves, which are sufficiently close to the segment $[x_0, y]$ in C^0-topology and which join y and X, have *length* $> r$.

Then let

$$rad^{\perp}_{x_0}(X) = \inf_{\nu_0 \in T_{x_0}(X)} rad^{\perp}_{\nu_0}(X) \text{ and } rad^{\perp}(X) = \inf_{x_0 \in X} rad^{\perp}_{x_0}(X).$$

Equivalently, *the focal radius of* $X \hookrightarrow Y$ *is equal to the supremum of* r, *such that the normal exponential map* $\exp T^{\perp}(X) \to Y$ *is an immersion on the* r*-ball subbundle* $B^{\perp}_X(r) \subset T^{\perp}(X)$.

Observe the following.

3.A. Eclidean Reciprocity. The focal radius of a submanifold in a Euclidean space is equal to reciprocal of its normal curvature:

$$rad^{\perp}_x(X \hookrightarrow \mathbb{R}^N) = \frac{1}{curv^{\perp}_x(X \hookrightarrow \mathbb{R}^N)}.$$

3.B. Focal Radius in S^n. Focal radii of *submanifolds in the* R*-spheres* and in the Euclidean spaces satisfy the following relation:

$$rad^{\perp}_{\mathbb{R}^N}(X) = 2R \sin \frac{1}{2} rad^{\perp}_{S^{N-1}}(R)(X),$$

which agrees with the Pythagorean formula for the curvature of immersions $X \hookrightarrow S^N(R)$ from 1.1.C:

$$\left(curv_{S^{n-1}}(X \hookrightarrow S^N(R))\right)^2 = \left(curv^{\perp}_{\mathbb{R}^N}(X \hookrightarrow \mathbb{R}^N)\right)^2 - 1/R^2.$$

For instance,

• the spherical focal radii of an equatorial subsphere (with zero spheriacal curvature) in the unit sphere $S^{N-1}(1)$ is equal to $\pi/2$, while their Euclidean focal radius is equal to one;

• the spherical focal radii of a subsphere with spherical radius $\pi/4$ is also $\pi/4$, and the spherical curvature is equal to one, while the Euclidean curvature is $\sqrt{2}$ with agreement with he identity $\frac{\sin \pi}{4} = 1/\sqrt{2}$.

Exercises. (i) Let $x_0 \in X$ be a *local maximum* point in $X \subset Y$ for the distance function $x \mapsto dist_Y(x, y_0)$ for some $y_0 \in Y$. Show that

$$dist(x_0, y_0) \geq rad^{\perp}_{x_0}(X).$$

(ii). Let $rad^{\perp}_{x_0}(X) \geq r$ and let $B(R) \subset Y$ be an R-ball, which contains a (small) neighbourhood $V_+0 \subset X$ of x_0 and such that the boundary sphere $S(R) = \partial B(R)$ contains x_0. Show that:

• $R \geq r$,

• if $R = r + \varepsilon$ for a small $\varepsilon \geq 0$, then the sphere $S(R)$ is *smooth* at the point x_0,

• if $S(R)$ is smooth at x_0, then the *radial component* of the second fundamental form of X at x_0 is greater than that of $S(R)$,

$$\langle \mathrm{II}_X(\tau,\tau),\nu\rangle \geq \langle \mathrm{II}_{\mathrm{S(R)}}(\tau,\tau)\nu\rangle,$$

where ν is the inward looking unit normal vector to $S(R)$ at x_0 and $\tau \in T_{x_0}(X) \subset T_{x_0}(S(R))$.

(If the sphere $S(R)$ is convex at x_0, then $0 \leq \langle \mathrm{II}_{\mathrm{S(R)}\,X}(\tau,\tau),\nu|\rangle = \sqrt{\|\mathrm{II}_{\mathrm{S(R)}\,X}(\tau,\tau)\|}$.)

3.C. Defintion of $+\rho$-Encircling $\mathbf{X}_{+\rho} = \mathbf{T}^{\perp}_{\rho}(\mathbf{X})$ Given an immersed $X \hookrightarrow \mathbb{R}^N$ let $T^{\perp}_{\rho}(X) \to \mathbb{R}^N$, $\rho > 0$, be the *normal exponential* (tautological) map from the ρ-spherical normal bundle of X to $\mathbb{R}^N$, where this "ρ-spherical normal bundle $T^{\perp}_{\rho}(X)$" is the set of vectors normal to X of length ρ.

For instance if $X \hookrightarrow \mathbb{R}^N$ is an embedding and $\rho > 0$ is small then the image of this map is equal the *the boundary of the ρ-neighbourhood of* X, denoted

$$X_{+\rho} = \partial U_\rho(X) = \{y \in \mathbb{R}^N\}_{dist(y,X)} = \rho.$$

In general, if $X \overset{f}{\hookrightarrow} \mathbb{R}^N$ is an immersion and if $\rho < \left(curv^{\perp} X \hookrightarrow \mathbb{R}^n\right)^{-1}$ then the exponential map is also an immersion and we abbreviate this by writing

$$X_{+\rho} \overset{f_{+\rho}}{\hookrightarrow} \mathbb{R}^N$$

and observe the following.

(a) If $X \hookrightarrow \mathbb{R}^N$ is contained in R-ball, then $X_{+\rho} \hookrightarrow \mathbb{R}^N$ is contained in the $(R+\rho)$- ball, where the relation $\rho = \frac{r}{c} - \rho$, implies that

$$curv^{\perp}\left(X_{+\rho} \hookrightarrow B^N(1)\right) = 1/\rho \leq 1 + 2c = 1 = 1 + 2 \cdot curv^{\perp}(X).$$

(b) The focal radius and the curvature of $X_{+\rho}$ satisfy the following mutually equivalent relations

$$[\rho] \qquad rad^{\perp}(X_{+\rho}) = \min(\rho, rad^{\perp}(X) - \rho)$$

and

$$[\rho^{-1}] \qquad curv^{\perp}(X_{+\rho} \overset{f_{+\rho}}{\hookrightarrow} \mathbb{R}^N) = \max\left(\rho^{-1}, \left(\left(curv^{\perp}(X \hookrightarrow \mathbb{R}^n)\right)^{-1} - \rho\right)^{-1}\right).$$

3.D. $[\mathbf{1} + \mathbf{2c}]$-Example. Let $curv^{\perp}\left(X \hookrightarrow B^N(1)\right) \leq c$ and move X to the smaller ball $B^N(r)$ by scaling $X \mapsto X' = rX$ for $r = 1 - \rho$, for some $0 < \rho < 1/c$.

Then $X'_{+\rho}$ is contained in the unit ball and

$$\left(curv^{\perp}\left(X'_{+\rho} \hookrightarrow B^N(1)\right)\right)^{-1} \geq \min\left(\frac{1}{\rho}, \left(\frac{r}{c} - \rho\right)^{-1}\right).$$

Riemannian Remark. The definition of $X_{+\rho}$ makes sense for an immersed submanifold X in a Riemannian manifold Y if the normal exponential map

$$\exp^{\perp}_{\rho} : T^{\perp}_{\rho}(X) \hookrightarrow Y$$

is an immersion, e.g. if Y is complete with non-positive sectional curvature κ and also for $\kappa(Y) \leq 1$ and $\rho < \pi$. Here the above [ρ] remains true but [ρ^{-1}] doesn't.

This suggests that the reciprocal of the focal radius of X, may serve a replacement for the normal curvature for Riemannian submanifolds

$$curv_x^{foc}(X \hookrightarrow Y) = \frac{1}{rad_x^\perp(X \hookrightarrow Y))}.$$

Y be a complete Riemannian manifold, $X \subset Y$ a smooth immersed submanifold and let us *define* the focal curvature of X in Y as the reciprocal of the focal radius of X,

$$curv_x^{foc}(X \hookrightarrow Y) = \frac{1}{rad_x^\perp(X \hookrightarrow Y))}.$$

3.1 Maximum Principle by Exercises

Definition of maxrad$^\perp$. Let Y be a metric space, let $X \subset Y$ be a subset and let $x_0 \in X$.

Define $maxrad_{x_0}^\perp(X)$ as the *infimum of the numbers* R, such that there exists a point $y_0 \in Y$ such that $dist(x_0, y_0) \leq R$ and the distance function $x \mapsto dist_Y(x, y_0)$ assumes *local maximum* at x_0.

3.1.A. Consult (i) from the previous section and show that R

$$maxrad_{x_0}^\perp(X) \geq rad_{x_0}^\perp(X),$$

for smooth submanifolds X in Riemannian manifolds Y.

3.1.B. Show that

$$maxrad_{x_0}^\perp(X) = rad_{x_0}^\perp(X) \text{ for } dim(X) = 1,$$

for smooth 1-submanifolds (curves) X in Riemannian manifolds Y, provided the normal exponential map $\exp : T^\perp x_0(X) \to Y$ is immersion on the R-ball $B_{0=x_0}^{N-n}(T_{x_0}^\perp(X)$. Show that the condition $dim(X) = 1$ is necessary.

3.1.C. Show that if a compact subset $X \subset Y$ is *contained in an R-ball* $B_{y_0}(R) \subset Y$, then

$$\inf_{x \in X} maxrad_x^\perp(X) \leq R$$

3.1.D. Show that the inequality $\inf_{x \in X} maxrad_x^\perp(X) \leq R$ remains valid for smooth immersed *complete, possibly non-compact,* submanifolds $X \hookrightarrow Y$, provided $curv^\perp(X) < \infty$.

(The condition $curv^\perp(X) = \sup_x curv^\perp(X) < \infty$ is necessary: there are examples [Roz 1961] of complete surfaces X in the unit 3-ball with negative Gauss curvatures, hence with $maxrad_x(X) = \infty$ for all $x \in X$.)

3.1.E. Let $D(\rho) \subset B^N(1) \subset \mathbb{R}^N$, $N \geq 3$ be the *boundary of the convex hull of a truncated unit ball*, where $D(\rho)$ is equal to the union of a spherical cap $C^{N-1}(\rho) \subset S^{N-1}(1) = \partial B^N(1)$, $0 < \rho < \pi$ and a flat $(n-1)$-ball $B^{N-1}(r = \sin\rho) \subset B^N(1)$,

$$D(\rho) = C^{N-1}(\rho) cup B^{N-1}(r),$$

where ρ is the radius of $C^{N-1}(\rho)$ regarded as a ball in the spherical geometry in $S^{N-1}(R)$, and where the (edge-like) intersection E of the two parts of $D(\rho)$, $\bar{}$

$$E(r) = C^{N-1}(\rho) \subset S^{N-1}(R) \cap B^{N-1}(r) = (\partial C^{N-1} = \partial B^{N-1}$$

is an $(N-1)$-sphere contained in $S^{N-1}(R)$ of (Euclidean) radius r.

Let $x_0 \in E_r$ and show that

$\bullet_{conv}$ if $\rho \leq \frac{\pi}{2}$ then $maxrad^{\perp}_{x_0}(D(\rho)) = r = \sin\rho$,

$\bullet_{concv}$ if $\rho \geq \frac{\pi}{2}$ then $maxrad^{\perp}_{x_0}(D(\rho)) = R$.

3.1.F. Non-Smooth Maximum Principle. Let $X \subset B^N(R) \subset \mathbb{R}^N$ be a closed connected subset in an R ball, such that

$$maxrad^{\perp}_{x}(X) \geq r \text{ for some } r \leq R \text{ qnd all } x \in X.$$

Prove that if $r = R$, then the intersection $X \cap \partial B^N(R) \subset S^N(R) = \partial B^N(R)$ is non-empty and show that *no connected component* of this intersection $X \cap \partial B^N(R) \subset S^N(R) = \partial B^N(R)$ is *contained in a spherical cap*

$$C^{N-1}\left(\rho < \frac{\pi}{2}r\right) \subset S^{N-1}(R).$$

Consequently, this intersection has no isolated points.

Moreover,

the topological dimensions of all connected components of $X \cap \partial B^N(R)$ *satisfy*

$$dim(comp(X \cap \partial B^N(R)) \leq 1.$$

3.1.G. Show that there exists a smooth convex (topologically spherical) rotationally symmetric surface in the unit 3-ball, $X \subset B^N(1) \subset \mathbb{R}^3$, which is *not equal to the boundary sphere* $S^2(1) = B^3(1)$ and such that $maxrad^{\perp}_{x}(X) \geq 1$ for all $x \in X$.

3.1.H. Generalize the above to subsets X (e.g. smooth submanifolds) in balls $B(R)$ in Riemannian manifolds Y, where the boundary of B, as well as the boundaries of concentric balls of radii $0 < r \leq R$ are smooth and where the inequality $\rho \geq \frac{\pi}{2}$ should be be replaced by $\rho \geq \delta = \delta(B) > 0$.

Thus show that if a compact connected subset $X \subset B(R)$ satisfies

$$maxrad^{\perp}_{x}(X) \geq R$$

for all $x \in X$, then

- the intersection $X \cap \partial B^N(R)$ is non-empty,
- the connected components of this intersection satisfy

$$dim(compX \cap \partial B^N(R)) \leq 1,$$

- no connected component of $X \cap \partial B^N(R))$ can be diffeomorphic to segments $[0,1]$ and/or $(0,1]$.

Consequently,

if *all geodesics* γ in a smoothly immersed closed submanifold in a ball $B^N(R)) \subset Y$ satisfy $rad^{\perp}(\gamma \hookrightarrow Y) \geq R$, then X *is contained in the boundary* $\partial B(R)$.

3.1.I. Let X be a smoothly immersed *complete connected* submanifold in a ball $B(R)) \subset Y$, such that the intersection X with the boundary sphere $S(R) = \partial B(R)$ is*nonempty* and such that each point $x_0 \in X \cap S(R)$ admits a neighbourhood $X_0 \subset X$ such that radial component of the second fundamental form of X at all $x \in X_0$ is non greater than that of the concentric sphere $S(r)$, which contains x,

$$\langle \mathrm{II}_X(\tau,\tau),\nu\rangle \le \langle \mathrm{II}_{S(r)}(\bar\tau,\bar\tau),\nu\rangle,$$

where τ and ν and $\bar\tau \in T_x(S(r)$ is the normal projection of $\tau \in T_x(X) \subset T_x(Y) \supset T_x(S(r))$ to $T_x(S(r))$.

Then X is contained in the boundary of the ball, $X \subset \partial B(R)$.

Hint. Prove convexity of a $\phi(dist(x,S(R))$ for a suitable function $\phi(d)$. (Compare with ✪ in section 1.)

Question What should be a comprehensive "non-smooth 'maximum principle", which would incorporate all we know in the smooth case?

3.2 Topologically Defined Focal Radius

Let $X^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface, let $x \in X$ and observe that the normal curvature $curv^\perp_x(X)$ is equal to the infimum of the curvatures c of the spheres $S^n_\pm(1/c)$, which are:

(i) tangent to X at x,

(ii) the balls bounded by these spheres do not intersect (small) neighbourhoods of $f(x)$ in $f(X)$ minus $f(x)$ itself,

(iii) do not mutually intersect away from x.

Generalise this to smooth submanifolds $X^n \subset \mathbb{R}^N$ for all $N \ge n+1$ as follows.

Let $\mathcal{B}(c)$ be a family of balls $B^N_y(1/c)R^N$ with centers $y \in \mathbb{R}^N$ such that

(i′) all balls from $\mathcal{B}(c)$ contain $x \in X$,

(ii′) the balls do not intersect (small) neighbourhoods of x_0 in X minus x_0 itself,

(iii′) for all $\varepsilon > 0$, there exists a family of points in $\mathbb{R}^N$ continuously parametrized by $\mathcal{B}(c)$, say

$$\phi_\varepsilon : \mathcal{B}(c) \ni B \to \mathbb{R}^N,$$

such that

- $\phi_\varepsilon(B) \in B$ for all $B \in \mathcal{B}(c)$,
- $dist(\phi_\varepsilon(B), x_0) \le \varepsilon$ for all $B \in \mathcal{B}(c)$,
- the set $\mathcal{B}(c)$ contains an $(N-n-1)$-cycle the ϕ-imagef this cycle. is non-trivilally linked with X for all sufficiently small ε.[10]

Then show that $curv^\perp_x(X)$ is equal to

● *the infimum of* $c > 0$, *such that a family* $\mathcal{B}(c)$ *with all these properties exists.*

The ●-definitions of c and $\inf c$ apply to *non-smooth topological* submanifolds $X\mathbb{R}^N$. Yet if this ●-"curvature" $curv^{\bullet}_x(X) = 1/\inf c$ is uniformly bounded for $x \in X$, then X is C^1-smooth, moreover, it is $C^{1,1}$-smooth–the partial derivatives are *Lipschitz*.

4 Products of Spheres in B^{n+1} with Small Curvatures

4.A. Products of Spheres Represented by Hypersurfaces Let X be a

[10]Think of X as a relative n-cycle in the pair $(B^N_x(2\varepsilon), \partial(B^N_x)(2\varepsilon))$.

product of m spheres and $k \geq m-1$. Then $X_m \times S^k$ admits a codimension one embedding to the unit ball with normal curvature $1+2\sqrt{m}$.

Proof. Imbed X to $B^{N+m}(1) \subset B^{N+k+1}(1)$ for $N = dim(X)$ with curvature $c = \sqrt{m}$ (see 1.A), let $\rho = 1+2\sqrt{m}$ and observe that $X'_{+\rho} \subset B^{N+k+1}(1)$, (this is the boundary of the ρ-neighbourhood of $X' \subset B^{N+k+1}(1)$ in the present case) is diffeomorphic to $X \times S^k$. Since $curv^\perp(X'_{+\rho}) \leq 1+2c$ (see 3.D), the proof follows.

Two Examples and one Theorem.

($\bullet_1$) *Products of two spheres admit codimension one embeddings to the unit balls with normal curvatures 3:*

$$[2/3]\times[1/3]. \qquad curv^\perp\big(S^{n_1}\times S^{n_2} = S^{n_1}_{+1/3}(2/3) \subset B^{n_1+1+n_2}(1)\big) = 3,$$

($\bullet_2$) *Products of three spheres $S^{n_1}\times S^{n_2}\times S^{n_3}$, e.g. 3-tori $\mathbb{T}^3$, admit codimension one embeddings to the unit balls with curvatures $1+2\sqrt{2} < 4$.*

We don't know answers to the following questions:

are there immersions $S^{n_1}\times S^{n_2} \hookrightarrow B^{n_1+n_2}(1)$ with $curv^\perp < 3$?

are there immersions immersions $S^{n_1}\times S^{n_2}\times S^{n_3} \hookrightarrow B^{n_1+n_2+n_3}(1)$ with $curv^\perp < 1+2\sqrt{2}$.

But the situation changes starting from $m = 4$ and $C = 1+3\sqrt{2} = 5.24264...$. with the following.

4.B. Codimension one Immersion Theorem. *Let X be a compact orientable n-manifold, which admits an immersion to $\mathbb{R}^{n+1}$, e.g. X is (diffeomorphic to) a product of spheres S^{n_i} of dimensions n_i, $\sum_i n_i = n$.*

Then, for all $\varepsilon > 0$, the product $S^{20n^2}\times X$ admits an immersion f_ε to the $(20n^2+n+1)$-ball, such that

$$(<\mathbf{4.5}) \qquad curv^\perp\big((S^N\times X)\overset{f_\varepsilon}{\hookrightarrow} B^{20n^2+n+1}(1)\big) \leq 1+2\sqrt{\frac{3(n+1)}{n+3}}+\varepsilon < 4.5.$$

Proof. The $\sqrt{3}$-immersion corollary 1.C with $m = n+1$ delivers an immersion $X \to B^{20n^2}(1)$ with $curv^\perp \leq \sqrt{\frac{3(n+1)}{n+3}}+\varepsilon$ and the manifold X'_ρ as in $[1+2c]$-example (1.G) does the job since it is diffeomorphic to $X\times S^{20n^2}$ in the present case.

$[X = \mathbb{T}^n]$**-Case.** If $N >> n$, then the $\sqrt{3}$-Clifford sub-torus theorem **1.C** implies that $S^N\times\mathbb{T}^n$ admits an immersion to the $(N+n+1)$-ball, such that

$$curv^\perp\big((S^N\times X)\overset{f_\varepsilon}{\hookrightarrow} B^{N+n+1}(1)\big) \leq 1+2\sqrt{\frac{3n}{n+2}}.$$

Embedding Remark. Unlike how it is in ($\bullet_1$) and ($\bullet_2$), the construction of f_ε in 1.K creates self-intersection of $S^k\times X$ in the ball.

Sharpness Conjectures. The constant $1+2\sqrt{\frac{3n}{n+2}}$, probbaly, is optimal for tori $\mathbb{T}^n$ of dimension $n\geq 3$

We also conjecture that there are *no embeddings* $\mathbb{T}^n\times S^k \to B^{n+k+1}(1)$ with $curv^\perp \leq 1+2\sqrt{\frac{3n}{n+2}}+\varepsilon$ for all $n\geq 3$ and $\varepsilon < 1/n^2$.

But it is hard to say if the constant $\sqrt{\frac{3(n+1)}{n+3}}$ for general orientable $X^n\mathbb{R}^{n+1}$ can be improved, even to $\sqrt{\frac{3n}{n+2}}$.

Also it is unclear what to expect in this regard from *non-orientable immersed* hypersurface $X^n \hookrightarrow \mathbb{R}^{n+1}$

Products of Equidimensional Manifolds. The codimension one immersion theorem doesn't deliver immersions of products of equidimensional manifolds with "interesting" curvature bounds, while by arguing as in $(\bullet_1)$ and $(\bullet_2)$ we show the following.

$(\bullet_3)$ *The product of* $(m+2)$ *copies of* S^m *admits an embedding to the ball* $B^{m(m+2)+1}(1)$ *with* $curv^\perp \leq 1 + 2\sqrt{m+1}$.

For instance, (as in $\bullet_2$) the 3-torus embeds to the unit 4-ball, such that

$$curv^\perp\big(\mathbb{T}^3 \subset B^4)\big) \leq 1 + 2\sqrt{2} < 4.$$

Conjectrally, *the constant* $1+2\sqrt{m+1}$ *is optimal for all* $m = 1, 2, ...4$, *possibly, not only for embedding but also for immersions*

$$(S^m)^{m+2} \hookrightarrow B^{m(m+2)+1}(1).$$

5 Extremality, Rigidity, Stability: Spheres and Veronese Varieties

The natural candidates for *extremal immersions* $X \hookrightarrow B^N$, which implement maximal topological complexity with minimal curvatures are the most symmetric ones that are immersions, which are equivariant under large isometry groups G acting on X and B^N

For instance the standard ($O(n)$-equivariant) embedding $S^n \hookrightarrow B^N$ is extremal by ❂ Example. 1.A: *the* n*-dimensional spheres of radius one, are the only closed submanifolds with* $curv^\perp \leq 1$ *in* $B^N(1)$. (If $n = 1$ these may be multiple coverings of the circle).

Rigidity and Stability. Most (all?) sharp geometric inequalities are accompanied by the rigidity/stability of the extremal objects[11].

To establish stability for $S^n \subset B^N$ we observe that the maximum principle argument used in 1.A. equally applies to *complete* (for the induced Riemannian metrics) manifolds $C^{1,1}$-immersed to $B^N(1)$ and that the space of $C^{1,1}$-|immersions $curv^\perp \leq const$ of immersed complete manifolds to the ball is compact.

Thus we conclude that there exists $\varepsilon > 0$, such that if a closed immerses submanifold satisfies

$$curv^\perp\big(X^n \hookrightarrow B^N(1)\big) \leq 1 + \varepsilon \text{ and } n \geq 2,$$

then X can be obtained by a δ-small C^1-perturbation of a unit n-sphere $S^n \subset B^N$, where $\delta \to 0$ for $\varepsilon \to 0$.

[11]See stability Gr. for a general discussion

A priori, this ε could depend on n and N, but when this argument is applied to geodesics in S^n, it provides an effective, albeit rough, bound on ε, e.g. $\varepsilon = 0.01$ (See below and section 12 for Petrunin's sharp result.)

Immersions to Tubes. The maximum principle applied to closed immersed n-submanifolds in "unit tubes" $B^N(1) \times \mathbb{R}^k \subset \mathbb{R}^{N+k}$ shows that

$$curv^\perp\big(X^n \hookrightarrow B^N(1) \times \mathbb{R}^k\big) \geq 1 \text{ for } k \leq n+1,$$

where extremal X, i.e. where $curv^\perp\big(X^n \hookrightarrow B^N(1) \times \mathbb{R}^k = 1$ for $k \geq 1$ are by no means unique. (see section 11)

About Mean Curvature. The maximum principle argument also applies to immersed n-submanifolds X in B^N with $mean.curv \leq n-1$ and shows that these X lie in S^{N-1}, where they are *minimal*, i.e. have zero mean curvatures.

An abundance of minimal surfaces in S^{N-1} makes it plausible that all n-manifolds admit and $curv^\perp(X) \leq const = const(n)$, say $const(n) = 100^n$, for given $\varepsilon, \delta > 0$. [12]

5.A. Veronese Manifolds. Besides n-spheres, there are other $O(n+1)$-equivariant immersion $S^n \hookrightarrow B^N(1)$, where the most interesting ones are the (quadratic) *Veronese maps.*

These are (minimal) isometric immersions of the n-spheres of radii $R_n = \sqrt{\frac{2(n+1)}{n}}$ to the unit balls, which factors through embeddings of the projective spaces $\mathbb{R}P^n = S^n(R_n)/\{\pm 1\}$ to the balls $B^{\frac{m(m+3)}{2}}$, where these embedding have *amazingly small* curvatures:

$$curv(Ver_n) = curv^\perp\left(\mathbb{R}P^n_{Ver} \hookrightarrow B^{\frac{n(n+3)}{2}}\right) = \sqrt{\frac{2n}{n+1}}, \text{ e.g.}$$

$$curv(Ver_2) = curv\left(\mathbb{R}P^2_{Ver} \hookrightarrow B^5\right) = 2\sqrt{\frac{1}{3}} < 1.155,$$

Observe that the radii R_n of the Veronese n-spheres, which covers $\mathbb{R}P^n_{Ver}$, satisfy

$[2/curv^\perp]$ $$R_n = \frac{2}{curv(Ver_n)}.$$

Conjecture.

$$curv^\perp(X^n, B^N) < \sqrt{\frac{2n}{n+1}} \implies X =_{diffeo} S^n.$$

The "homeo-version" of this proven by Petrunin for $n = 2$. (See [Petrunin 2023] and section 12, where we also explain the above and say more about Veronese maps and their generalizations.)

[12] Possibly the case $n = 2$ can be approached with the techniques from [Nadirashvili 1996] and its generalizations.

6 Exercises: Hypersurfaces Inscribed in Convex Sets

Given a subset $V \subset \mathbb{R}^{n+1}$, let $ext_{+r}(V)$ denote the *r-neighbourhood of* V, that is the subset of points in $\mathbb{R}^{n+1}$ within distance $\leq r$ from V.

$$ext_{+r}(V) = \{y \in \mathbb{R}^n\}_{dist(y,V)\leq r} \subset \mathbb{R}^{n+1},$$

and let

$$int_{-r}(V) \subset V$$

be the complement of the interior of the r-exterior of the complement of $\mathbb{R}^{n+1} \setminus V$, that is equal to the set of points in V with distance $\geq r$ from the boundary of V,

$$int_{-r}(V) = \{v \in V\}_{dist(v,\partial V)\geq r} \subset V.$$

Clearly,

$$ext_{+r}(int_{-r}(V)) \subset V \text{ and } int_{-r}(ext_{+r}(V)) = V.$$

Let $R = R(V)$ denote the *in-radius of* V, that is the maximal distance from the boundary of V in V,

$$R = inrad(V) = \sup_{v\in V} dist(v, \partial V)$$

and let

$$cntr(V) = int_{-r}(V)$$

be the set of the centers of the R-balls in V, that is the subsets of $v \in V$ with $dist(v, \partial V) = R = inrad(V)$.

Let $V \subset \mathbb{R}^{n+1}$ be a *compact convex domain*, e.g. the $(n+1)$-cube $\square^{n+1} = [-1,1]^{n+1}$ or an $(n+1)$-simplex $\triangle^{n+1}$.

Then, clearly, the r-*interior* of V is convex and if $r = R = inrad(V)$ then $int_R(V)$ called *the central locus* in V,

$$int_R(V) = cntr(V)$$

is a *non-empty compact convex subset in* V *of dimension* $\leq n = dim(V) - 1$.

For instance, if V is a cube or a simplex, then $cntr(V)$ consists of a single point and $ext_{+r}(int_{-r}(V))$ is equal to the (unique maximal) ball inscribed into V.

(If V is a general $(n+1)$-dimensional rectangular solid then $int_{-r}(V)$ is a subsolid of certain dimension $0, 1, ..., n$.)

6.A. ***inrad*-Convex Exercises.** (a) Let the boundary of V be $C^{1,1}$-*smooth*[13] (e.g. piecewise C^2-smooth) with curvature bounded by a constant c,

$$curv^{\perp}(\partial V \subset \mathbb{R}^{n+1}) \leq c.$$

Show that if $r \leq \frac{1}{c}$, then, the r-balls $B \subset \mathbb{R}^{n+1}$ tangent to ∂V either are fully contained in V or lie outside V, meeting W at a single contact point between the boundaries of B and V; consequently:

$$ext_{+R}(cntr(V)) = V \text{ for } R = inrad(W).$$

[13] Locally, the hypersurface $\partial V \subset \mathbb{R}^{n+1}$ is representable by the graph of a C^1-function with bounded measurable second derivatives.

(b) *Let* $X \hookrightarrow \mathbb{R}^{n+1}$ *be a* C^2*-smooth compact immersed hypersurface* in $\mathbb{R}^{n+1}$ and let

$$W = conv(X)$$

be the *convex hull* of (the image of) $X \hookrightarrow \mathbb{R}^{n+1}$.

Show that the boundary of W is $C^{1,1}$*-smooth*[14] with curvature bounded by that of X,

$$curv^{\perp}(\partial W \subset \mathbb{R}^{n+1}) \leq curv^{\perp}(X \hookrightarrow \mathbb{R}^{n+1}).$$

(c) *Sphericity.* Let $V \subset \mathbb{R}^{n+1}$ be a convex bounded domain, e.g. a polytope, such as $(n+1)$-cube $\square^{n+1} = [-1,1]^{n+1}$ or an $(n+1)$-simplex $\triangle^{n+1}$, and let $X \overset{f}{\hookrightarrow} V$ be a C^2-smooth immersion, where X is a closed n-manifold.

Apply (a) and (b) to the convex hull $W = conv(X) \subset V$ of X and show that if

$$inrad(V) = R \leq \frac{1}{curv^{\perp}(X \hookrightarrow V)},$$

then, in fact,

$$inrad(V) = \frac{1}{curv^{\perp}(X \hookrightarrow V)}.$$

Fithermore, if $cntr(V)$ consists of a single point $o \in V$, (e.g. $V = \square^{n+1}$ or $V = \triangle^{n+1}$), show that the image of the immersion f is contained the R-ball centred at o for $R = inradV$.

Consequently, (see 1.3.A)

the image of $X \overset{f}{\hookrightarrow} V$ *is equal to the* R*-sphere centered at* $o \in V$.

(d) *Stability.* Argue as in section 1.3 and, assuming as above that $cntr(V)$ consists of a *single point* $o \in V$, show that the (only) R-sphere in V is stable:

there exists an $\varepsilon = \varepsilon(V) > 0$, *such that all immersed closed hypersurfaces* $X \hookrightarrow V$ *with* $curv^{\perp}(X) \leq R + \varepsilon$ *are* δ*-close in the* C^1*-topology to to the* R*-sphere* $S^n_o(R)$, *where* $\delta \underset{\varepsilon \to 0}{\to} 0$.

More on Stability. Unlike 1.3.B, this ε is sensitive to dimension.

For instance if V is the regular unit simplex then $\varepsilon(\triangle^{n+1}) \sim \varepsilon_0/n$ and if it is the cube $\square^{n+1} = [-1,1]^{n+1}$, then $\varepsilon(\square^{n+1}) \sim \varepsilon_0/\sqrt{n}$.

On $dim(cntr(V) > 0$. If $int_{-r}(V)$ *has positive dimension*, then there are many non-spherical C^2-immersed (and even more $C^{1,1}$) hypersurfaces in V with curvatures $\leq \frac{1}{inradV}$, see section 11.

On $\mathbf{dim(cntr(V)) > 0}$. Let $dim(cntr(V)) = k > 0$, let $Z \subset \mathbb{R}^{n+1}$ be an affine k-dimensional subspace which contains the (convex!) subset $cntr(V) \subset \mathbb{R}^{n+1}$ and let $B^{n+1}_Z(R) = ext_{+R}(Z) \subset \mathbb{R}^{n+1}$ be the R-neighbourhood of Z.

7 Bowl Inequalities

Let $X \hookrightarrow \mathbb{R}^N$ be an immersed complete (e.g. closed) connected n-dimensional submanifold in the Euclidean N-space, let $x_0 \in X$, let $T = T_{x_0} \subset \mathbb{R}^N$ be the tangent space to X at x_0 (represented by an affine subspace in $\mathbb{R}^N$) and let $P_{x_0} : X \to T^n_0$ be the normal projection map.

[14] Locally, the hypersurface $\partial W \subset \mathbb{R}^{n+1}$ is representable by the graph of a C^1-function with bounded measurable second derivatives.

Let $U_{x_0} \subset X$ be the maximal connected neighbourhood of x_0, such that the normal projection $P = P_{x_0}$, from U_{x_0} to T_{x_0} is a one-to-one diffeomorphism onto a domain $V_+x_0 \subset T_{x_0} = \mathbb{R}^n$, which is *star convex* with respect to x_0.

Clearly such a U_{x_0} exists and unique.

Let $\underline{S} = S^n(R)$ an n-sphere of radius R, which is tangent to X at the point x_0, (such spheres $\underline{S} = \underline{S}_\nu$ are parametrised by the unit normal vectors $\nu \in T^\perp_{x_0}(X)$) let $\underline{P} : \underline{S} \to T_{x_0}$, be the normal projection map and observe that the corresponding neighbourhood $\underline{U}_{x_0} \subset \underline{S}$ is the hemisphere $\underline{S}_+$ that is the ball $B_{x_0}(\frac{\pi}{2}R) \subset \underline{S} = S^n$ around x_0.

Let $d(x) = dist_T(P(x), x_0)$ and let $\underline{d}(s) = dist_T(\underline{P}(s), x_0) = R \sin \frac{1}{R} dist_{\underline{S}}(s, x_0)$ be the corresponding function for the sphere $\underline{S}$.

Let $h(x) = dist(x, y = P(x))$, $x \in X$, and let $\underline{h}(s) = R \cos \frac{1}{R} dist_{\underline{S}}(s, x_0)$ be the corresponding function for the sphere $\underline{S}$.

Remark. Both d–functions and both h-functions have their gradients *bounded by one*, in fact,

$$\|grad_{\underline{S}}(d(s))\|^2 + \|grad_{\underline{S}}(\underline{h}(s))\|^2 = 1 \text{ and } \|grad_X(d(x))\|^2 + \|grad_X(h(x))\|^2 \le 1,$$

The gradients of both d-functions have unit norms at x_0,[15], they don't vanish in the interiors of the domains U_{x_0} and $\underline{U}_{x_0}$ correspondingly; $grad(\underline{h})$ vanishes on the boundary. of $\underline{U}_{x_0}$ and $\underline{U}_{x_0}$ vanishes at at least 2 ponts at the boundary of U_{x_0}.

The gradients of the h-functions have *norms* < 1 in (the interiors of) domains U_{x_0} and $\underline{U}_{x_0}$ correspondingly, and these norms. are equal to one the boundaries of these domains.

In fact, $\underline{U}_{x_0}$ is the same as the maximal connected neighbourhood of x_0, where $\|grad_X(h)\| < 1$ and the P-image of which is star convex.

The following proposition, says that U_0 lies at least as close in the C^1-metric to T_{x_0} as $\underline{S}_+$.

7.A. Hemisphere Comparison Inequalities. Let

$$curv^\perp(X) \le curv^\perp(\underline{S}) = 1/R.$$

Then:

The gradient of the h-function on X,

$$h : x \mapsto dist(x, P(x))$$

for $x \in U_{x_0}$ is bounded by that for the $\underline{h}$-function on $\underline{S}_+$

$$\|grad(h)\| \le \|grad(\underline{h})\| \text{ for } dist_X(x, x_0) \le dist_{\underline{S}}(s, x_0)\| \text{ and } s \in \underline{S}_+$$

Consequently, the domain $U_{x_0} \subset X$ *contains an open R-ball* centered at x_0.

$\bullet_d$ The gradient of the d-function on X in the radial direction is bounded from below by that for $\underline{d}$:

if $s \in \underline{S}_+$ and a unit vector $\tau \in T_x(X)$ which is tangent to a geodesic segment γ in X issuing from x_0 and termnating at x satisfy

$$length(\gamma) \le dist(s, x_0)$$

[15]These functions are non-differentiable at x_0 but the norms of their gradients continously extend to one at x_0.

then

$$\langle grad(d), \tau\rangle \geq \|grad(\underline{d}(s))\|,$$

Consequently, the P-images in T of the r-balls from $U_{x_0} \subset X$ centered at x_0, contain the $\underline{P}$-images of the corresponding spherical balls from $\underline{S}_+$,

$$P(B_{x_0}(r)) \supset B_{x_0}\left(R\cdot \sin\frac{1}{R}r\right) \subset T \text{ for all } r \leq \tfrac{\pi}{2}R,$$

$\bullet^{-1}_h$ The inverse function $h^{-1}(y)$, $y \in B_{x_0}(R) \subset T$, and the norm of its gradient are bounded by $\underline{h}^{-1}(y)$, and $\|grad(\underline{h}^{-1}(y))\|$ correspondingly.

7.B. Out of Ball Corollary. Let $\underline{B}^N(R) \subset \mathbb{R}^N$ be a ball, such that the boundary sphere $\underline{S}^{N-1}(R) = \partial\underline{B}^N(R)$ is tangent to X at x_0, i.e.

$$T_{x_0}(\underline{S}^{N-1}(R)) \supset T_{x_0}(X).$$

If $curv^\perp(X) \leq 1/R$, then the subset $U_{x_0} \subset X$ doesn't intersect the interior of this ball. Thus, U_{x_0} lies in the closure of the complement of the union of the R-balls tangent to X at x_0.

7.C. Spherical Bowl Theorem. Let $U_{x_0}(+r) \subset X$ be the r-neighbourhood of U_{x_0} in X. Then the gradient of the function $d(x) = dist_T(P(x), x_0)$ doesn't vanish in the interior of the complement $U_{x_0}(+R) \smallsetminus U_{x_0}$ and the P-mage of the complement $U_{x_0}(+r) \smallsetminus U_{x_0}$, $r \leq R$, doesn't intersect the interior of the ball $B_{x_0}(R-r) \subset T$.

Proof The bounds on the gradients of the functions h in the hemisphere comparison inequalities follow from the angular arc inequality. 1.1.E, while the bowl theorem follows from these inequalities applied to X at x_0 and at all ponts $x \in \partial U_{x_0}$.

If $dim(X) = 1$, then the bowl theorem, where the proof[16] becomes especially transparent [17] implies the following.

7.D. Circular Bow Inequality $[dist_X \geq dist_A]$**.**[18] Let a planar circular arc $A \subset \mathbb{R}^2$ (a segment of a circle) and a smooth spatial curve $X \hookrightarrow \mathbb{R}^N$ satisfy:

$$length(X) = length(A) = l \text{ an]d } curv^\perp(X) \leq curv^\perp(A).$$

Then the distance between the endpoints of X is *greater than or equal to that* in A, where the equality holds if and only if X is *congruent to* A.

Remark This inequality applied to geodesic segments from X yields most essential geometrical properties X with no use of the full (n-dimensional) spherical bowl theorem.

Example. Let $X \subset \mathbb{R}^N$ is a closed curve of length 2π. If $curv^\perp(X) \leq 1$, then the Euclidean distances between opposite points $x, x_{opp} \in X$ are ≥ 2, where an equality $dist(x_0, (x_0)opp = 2$ implies that X is circlular.

Exercises. (a) Show that all closed curves of length 2π in the Euclidean space contain pairs of opposite points $x, x_{opp} \in X$,n(i.e. with the X-distance π between them), such that $dist(x, x_{opp}) \leq 2$.

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[16] Hopf Schimd
[17] he abive Hopf Schimdt, oter proofs
[18] According to [Sch 1921] this goes back H. A. Schwarz, 1884.

(b) Let $X \subset \mathbb{R}^N$ be a smooth simple[19] curve, such that the two ends of X, say $x_1, x_2 \in X$, are positioned in two parallel hyperplanes,

$$x_1 \in H_1 \subset \mathbb{R}^N \text{ and } x_2 \in H_2 \subset \mathbb{R}^N,$$

where X is *tangent* to these hyperplanes at *the points* x_1, x_2.

If *no tangent line* to X, except for these at the points $x_1 x_2$ is parallel to H_i, $i = 1, 2$, if there exists a point $x_\circ \in X$, such the tangent to X at $x_\circ$ is *normal to* H_i and if

$$curv^\perp(X) \leq c,$$

then the distance between the hyperplanes is at least π/c,

$$dist(H_1, H_2) \geq \pi/c,$$

where the equality implies that X is a circular arcs with curvature c and length π/c.

(c)*Remark.* If we allow $C1,1$-*curves*, then the extremal X are *unions of pairs* of circular arcs of length $\pi/2c$, where the (normal) curvature (vector function) of X, may be *discontinuous at the middle* of X.

Moreover, if also allow X to be tangent H_i at some intervals $X_i \subset X$ around the ends x_i and nowhere else (the subset $X \parallel subsetX$, where the of tangent lines to X are parallel to H_i, consist of *two* components), then gthe inequality still $dist(H_1, H_2) \geq \pi/c$, holds valid and the extremal X, where the equality holds, contains, besides two circular $\pi/2c$-arcs, two straight segments that are $X_i \subset H_i$, $i = 1, 2$.

7.1 High Dimensional Applications of the Circular Bowl inequality

Basic geometry properties of immersed n-submanifolds X in Euclidean spaces with

$$curv^\perp(X \overset{f}{\hookrightarrow} \mathbb{R}^N) \leq c,$$

can be reduced to the case $n = 1$ applied to the geodesic segments from X, which are, by the definition of the normal curvature $curv^\perp(X)$, are *curves in* $\mathbb{R}^N$ *with* $curv^\perp \leq c$.

The circular bow inequality applied to geodesics in immersed n-submanifolds

$$X \overset{f}{\hookrightarrow} \mathbb{R}^N, \; dim(X) = 1, 2, ...n, ...$$

yields the following.

7.1.A. $[2\sin]_{\mathbf{bow}}$ and $2\sin]_{dist}$ Inequalities. Let $\gamma \hookrightarrow X$ be an (not necessary minimising) *geodesic segment*[20] between two points $x_0, x_1 \in X$. If the normal curvature of X is bounded by $1/R$ and if $length(\gamma) = l \leq 2\pi R$, then

Then the Euclidean distance between these points is bounded from below:

$$[2\sin]_{bow} \qquad dist_{\mathbb{R}^N}\big(f(x_0), f(x_1)\big) \geq 2R \sin \frac{l}{2R}$$

[19]"Simple" means *homeomorphic to* $[0,1]$.

[20]Recall "Geodesic" refers to the induced (inner) Riemannian metric in X,

and, the equality implies that the f-image of γ is a circular ark in a plane in $\mathbb{R}^N$.

If X is connected and the induced metric in X is *complete* (e.g, X is compact without boundary), then

$$[2\sin]_{dist} \qquad dist_{\mathbb{R}^N}\mathbb{R}^N\big(f(x_0), f(x_1)\big) \geq 2\sin\left(\frac{dist_X(x_0,x_1)}{2}\right)$$

for all $x_0, x_1 \in X$, *such that* $dist(x_0, x_1) \leq 2\pi$.

Remarks. The $[2\sin]_{dist}$-inequality for infinitesimally close points x_0, x_1 is equivalent to the inequality $curv^\perp \leq R$.

The $[2\sin]_{bow}$-inequality holds for immersions to (complete simply connected) manifolds Y with *non-positive* sectional curvatures. (see ..)

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7.1.B 2π-Injectivity Let $TB_x(r) \subset T_x(X)$ gent space be the r-ball in the tangent space at a point $x \in X$ and let $\exp_x : TB_x(r) \to X$ be the exponential map. If $r < \pi$, then the composition of this map with our immersion $f : X \hookrightarrow \mathbb{R}^N$ is one-to one.

Here are two obvious sub-corollaries.

2π-Geodesic Loop Inequality. Geodesic loops γ in X have $length(\gamma) \leq 2\pi$.

2π-Diameter Inequality. If the *intrinsic diameter*, i.e. the diameter with respect to to the induced Riemannian metric, of $X \overset{f}{\hookrightarrow} \mathbb{R}^N$, satisfies

$$diam_{int}(X) < 2\pi,$$

then X is *embedded* to $\mathbb{R}^N$: the map f is one-to-one.

This inequality is sharp: the equality holds for $S^n_{Ver}(R_n) \to \mathbb{R}P^n_{Ver} \hookrightarrow B^{\frac{n(n+3}{2}}(1)$ by the above $[\frac{2}{curv^\perp}]$

$$diam_{int}(S^n_{Ver}) = \pi R_n = \frac{2\pi}{curv^\perp(S^n_{Ver})}.$$

Question. Are Veronese the only ones with this property? (Compare with [Petr 2024] and also with section 12)

Exercise. Let $X^n \overset{f}{\hookrightarrow} B^N(R)$ be an immersion, such that all geodesics in X issuing from a point $x_0 \in X$ have their Eucldiean curvatures $curv^\perp$ bounded by one.

Show that the image $f(X) \subset B^N(R)$ is equal to an equatorial n-sub-sphere in $S^{N_1}(R) = \partial B^N(R)$ and if X is connected and $dim(X) \geq 2$ then the immersion f is an embedding.

8 Optimal Control, Bow Theorem and Arm Lemma

The circular bow inequality $[dist_X \geq dist_A]$ from section 7 represents the solution of the following variational problem for curves, now denoted $y(s)$, $s \in [0, l]$, of length l issuing from the origin in $\mathbb{R}^N$, i.e. $y(0) = 0$, and such that the curvatures of these curves are bounded by a positive constant c,

$$curv^\perp(y(s)) \leq c.$$

If $l \leq 2\pi c$, then, according to this inequality,

the minimum of the distance from the second end of $y(s)$ to the origin, that is $\|y(l)\|$, is achieved by planar circular arcs of curvature c.

More generally Let $h(y)$ be a piece-wise smooth function on a Riemannian manifold Y, let $\tau_0 \in T_{y_0}(Y)$ be a unit tangent vector and let $\sigma \geq 0$ be a positive Borel measure on the segment $[0, l]$, e.g. $\sigma = c(s)ds$, $s \in [0, l]$, for a bounded measurable function, c(s) or $\sigma = \sum_i c_i \delta(s_i)$ as in 1.1.I.

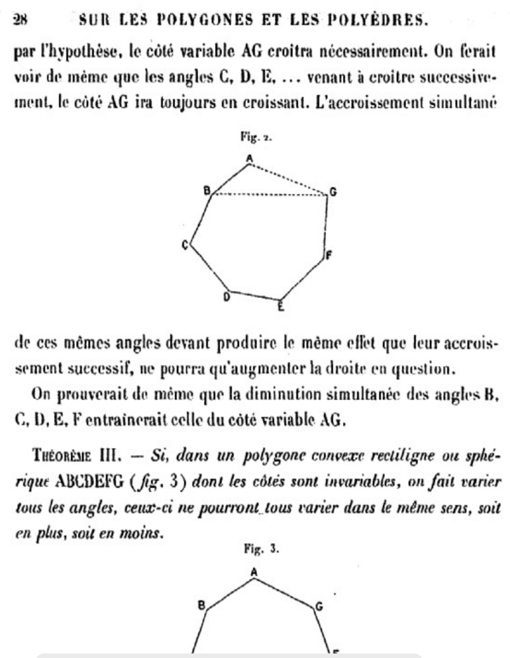
Find an isometric, i.e length preserving, immersion $[0, l] \to Y$, written as $s \mapsto y(s)$, such that

$$y(0) = y_0,\ y'(0) = \tau_0 \text{ and } curv^{\perp}(y(s) \leq c(s),$$

which minimises $h(y(l))$.

This is an instance of *an optimal control problem*[21] where solutions are often piecewise smooth rather than smooth according to Fel'dbaum's *n-interval theorem* from the optimal systems control theory.[22]

The variational principle behind this theorem suggests an effortless proof of the classical combinatorial antecedent of the bow theorem.



28 SUR LES POLYGONES ET LES POLYÈDRES.

par l'hypothèse, le côté variable AG croîtra nécessairement. On ferait voir de même que les angles C, D, E, ... venant à croître successivement, le côté AG ira toujours en croissant. L'accroissement simultané

Fig. 2.

de ces mêmes angles devant produire le même effet que leur accroissement successif, ne pourra qu'augmenter la droite en question.

On prouverait de même que la diminution simultanée des angles B, C, D, E, F entraînerait celle du côté variable AG.

THÉORÈME III. — *Si, dans un polygone convexe rectiligne ou sphérique* ABCDEFG (*fig.* 3) *dont les côtés sont invariables, on fait varier tous les angles, ceux-ci ne pourront tous varier dans le même sens, soit en plus, soit en moins.*

Fig. 3.

8.A. Cauchy(1813)-Legendre(1794) Arm Lemma.[23] Let $Q \subset \mathbb{R}^N$ be a polygonal curve with vertices $q_1, q_2, ...q_k \in \mathbb{R}^N$ and segments $s_i = [q_i, q_{i+1}] \subset \mathbb{R}^N$, where the external angles between consequitive edges are bounded by positive

[21] Think of piloting a jet plane, where acceleration must be limited by a couple of G for your comfort.https://en.wikipedia.org/wiki/Optimal_control.

[22] see Optimal Control Systems https://www.scribd.com/document/390018919/Optimal-Control-Systems-Feldbaum-pdf and https://encyclopediaofmath.org/index.php?title=Pontryagin_maximum_principle

[23] See [Sab 2004] and references to the contributions by Legendre, Cauchy and Steinitz.

numbers $0 \le c_i \le \pi$, that is

$$\angle_{q_i} = \angle(s_{i-1}, s_i) \ge \pi - c_i, \ i = 2, ..., k-1.$$

Let $P \subset \mathbb{R}^2$ be a locally convex polygonal curve in the plane with vertices $p_1, p_2, ...p_k$, where the segments have the same lengths l_i as those in P,

$$\|p_i - p_{i+1}\| = l_i = \|q_i - q_{i+1}\|$$

and the external angles at the vertices p_i are equal c_i.

$$\angle_{p_i} = \pi - c_i, \ i = 2, ..., k-1.$$

If the curve P is *convex*, that is if the union of this curve with the segment $[p,p]$ makes a *closed convex* curve, then then the end points q_1 and q_k of Q lies *further apart* than these of P,

$[dist_Q \ge dist_P]$ $$dist(q_1, q_k) \ge dist(p_1, p_k),$$

where *the equality implies that Q is congruent to P.*

Proof. Let $Q = Q_{extr} \subset \mathbb{R}^N$ be a polygonal curve which *minimises* the distance $dist(q_1, q_k)$ among all curves with

$$length(s_i) = l_i \text{ and } \angle_{p_i} \ge \pi - c_i.$$

1. *Extremality* $\Longrightarrow$ *Rigidity.* [24] If the angle of an extremal curve Q at some vertex point lies strictly between 0 and $\pi - c_i$,

$$0 < \angle_{q_{i_0}} < \pi - c_{i_0},$$

and if $dist(q_1, q_k) > 0$ then the points q_{i_0}, q_1, q_k are *collinear.*

Otherwise, the spheres around q_{i_0} of radius $R_0 = \|q_{i_0} - q_k\|$ and around q_1 of radius $R_1 = \|q_1 - q_k\|$ would be transversal,

$$S^{N-1}_{q_{i_0}}(R_0) \pitchfork S^{N-1}_{q_1}(R_1),$$

and a small variation of this angle along with small rotation of the part of Q following q_{i_0} around the edge $[q_{i_0}, q_{i_0+1}]$ would decrease the distance between q_1 and q_k.

2. *Vertex cut off:* $[k-1 \Longrightarrow k]$[25] Let the angle of a (not necessarily extremal) Q at some point be minimal possible,

$$\angle_{q_{i_0}} = \pi - c_{i_0},$$

let Q' be the curve obtained by removing p_{i_0} and by joining p_{i_0-1} and p_{i_0+1} by an edge and let P' be a similarly truncated curve P. Then the curve P' is convex and the (old as well as new) external angles of Q' remain bounded by those of P'.

If $dist(q_1, q_k) > 0$, then induction on k and Lemma 1 reduce the arm theorem to where either all vertices are collinear or $k = 3$ and where the validity of the lemma is obvious.

[24] Compare with Connelly

[25] compare p. 28 in [Cauchy] and with section 3 in Zaremba

3. *Terminal Edge Cut off:* $[(j-1)\frac{l_k}{m} \implies j\frac{l_k}{m}]$. Let m be a positive integer, such that $\delta = \frac{l_k}{m} > 0$, where l_k is the length of the terminal edge in P, is smaller than the distance from p_1 to this edge,

$$\delta = \frac{l_k}{m} < dist[p_{k-1}, p_k],\ \ l_k = \|p_k - p_{k-1}\|$$

Let $P_j \subset P$, $j = 0, 1, ..., m$, be obtained from P by cutting away the terminal $l_k - (m-j)\delta$ segment of length $l_k - (m-j)\delta$ from the edge $[p_{k-1}, p_k] \subset P$; thus $P_m = P$ and P_0 is a polygon with $k-1$ vertices.

Let $Q_j \subset Q$ be the corresponding parts of Q and observe that (validity of) the inequality $[dist_Q \geq dist_P]$ for Q_j implies that the distance between the end vertices in Q_{j+1} doesn't vanish, hence, by **1** and **2**, Q_{j+1} satisfy $[dist_Q \geq dist_P]$ for Q_j as well, and since P_0 has only $k-1$ vertices, the proof of $[dist_Q \geq dist_P]$ for $P = P_m$ is conclude by induction in j and k.

Finally notice that the congruence of Q and P in the case where $dist(q_1, q_k) = dist(q_1, q_k)$ follows by tracking strictness of angle inequalities in **2** of this a argument.

8.B. Euclidean C^2-Bow Corollary (A. Schur 1920, E. Schmidt 1925) Let $y(s)$, $0 \leq s \leq l$, be a smooth curve in $\mathbb{R}^N$ parametrized by the arc length and let

$$curv^{\perp}(y(s)) = \|y'(s)\| \leq c(s),\ \ c(s) \geq 0.$$

Let $a(s) = a_c(s)$ be a planar curve with

$$curv^{\perp}(a(s)) = c(s).$$

(Such a curve, which is locally convex, is unique up to congruence.)

If $a(s)$ is convex, that is if the union of the image $a[0, l] \subset \mathbb{R}^N$ and the straight segment $[a(0), a(l)] \subset \mathbb{R}^2$ constitutes a (planar) closed convex curve (of length $l + \|a(l) - a(0)\|$), then $y(s)$ satisfies the $[dist_Y \geq dist_A]$ inequality:

$$dist(y(0), y(l)) \geq dist(a(0), a(l)).$$

In fact, this follows from the above $[dist_Q \geq dist_P]$ by approximation (see section 1.1.I) of smooth curves by the polygonal ones.

Remarks. (a) There is a calculus proof of this inequality, commonly attributed to E.Schmidt [Hopf 1946], which we reproduce in the next section.

(b) The approximation argument works both ways: "C^2-bow" implies "polygonal arm".

(c) The approximation argument also delivers the bow inequality for arbitrary convex curves $a(s)$, a direct proof of which is presented in [Sull 2007]

8.C. *Spherical and Hyperbolic Arms and Bows.* (i) The above proofs of arm and bow inequalities extends verbatim to curves in Riemannian manifolds Y with constant sectional curvatures, spheres and hyperbolic spaces,[26]where it also applies to minimization of distance functions $h(y)$ for curves with arbitrary initial conditions $(y(0), y'(0))) \in T)(Y)$

Also the spherical bow inequality reduces to the Euclidean one [Connely 1982, p. 31] as follows.

[26]According to [Ni 2023] a proof of the hyperbolic bow inequality is presented in a 1985 preprint by C. L. Epstein, which I was unable to locate on the web.

Given positive numbers r_i, $i = 1, ..., k$, let us denote the corresponding "radial lift" of points $q_i \in S^{N-1} = S^{N-1}(1)$ to $\mathbb{R}^N \supset S^{N-1}(1)$ by

$$q_i \mapsto \tilde{q}_i = r_i \cdot q_i \in \mathbb{R}^N,$$

write the distances $\tilde{d}$ between $\tilde{q}_i$ as $\tilde{d} = \Phi(d) = \Phi_{\{r_i\}}(d)$, i.e.

$$dist_{\mathbb{R}^N}(\tilde{q}_i, \tilde{q}_j) = \Phi(dist_{S^{N-1}}(q_i, q_j)),$$

similarly express the *angles* $\tilde{\alpha}$ *between the segments* $[\tilde{q}_{i-1}, \tilde{q}_i], [\tilde{q}_i, \tilde{q}_{i+1}] \subset \mathbb{R}^N$ as the function in the angles α between the corresponding *geodesic segments* in the sphere S^{N-1}

$$\angle_{\tilde{q}_i} = \Phi_\angle(\angle_{q_i}),$$

and observe that the functions $\Phi(d)$ and $\Phi_\angle(\alpha)$, (determined by $\{r_i\}$) are *monotone increasing.*

If points $p_i \in S^2 \subset S^{N-1}$ (consequently joined by geodesic segments) form a *convex* polygonal curve $P \subset S^{N-1}$, then there exist numbers r_i, such that the polygonal curve $\tilde{P} \subset \mathbb{R}^N$ with vertices $\tilde{p}_i = r_i \cdot p_i \in \mathbb{R}^N$ is *planar convex.*

Indeed, since P, being convex, is contained in a hemisphere, say $S_+^{N-1} \subset S^{N-1}$, the *radial* projection ρ from P to the hyperplane $T \subset \mathbb{R}^N$, which is tangent to S_+^{N-1} at the centre, sends P to a planar convex polygonal curve $\tilde{P} = \rho(P) \subset T$ and the required r_i are taken from the relations $\rho(p_i) = r_i \cdot p_i$.

Then, due to monotonicity of the functions Φ and $\Phi_\angle$, the Euclidean arm lemma for $\tilde{p}_i = \rho(p_i), \tilde{q}_i r_i \cdot q_i \in \mathbb{R}^N$ with these very r_i yields the spherical lemma for $p_i, q_i \in S^{N-1}$.

8.1 Semi-Circle Lemma and Calculus Proof of the Bow Inequality

Let us look closer at the minimization problem for $h(y(l))$ for smooth curves $y(s)$, $s \in [0, l]$ in $Y = \mathbb{R}^N$ (see the beginning of the pevious section), such that

$$y(0) = y_0 \text{ and } y'(0) = \tau_0 \text{ and } curv^\perp(y(s)) \leq c(s).$$

and where $h(y)$ is a linear function on $\mathbb{R}^N$.

Since $Y = \mathbb{R}^N$, this can be reformulated in terms of the derivative $z(s) = y'(s) \in S^{N-1}(1)$ as minimization of problem for the integral

$$\int_0^l \langle z(s), grad(h) \rangle ds$$

under constraint

$$\|z'(s)\| \leq c(s).$$

Let $h(y)$ be a *unit* linear function, i.e. $\|\mathsf{grad}(h)\| = 1$ and let $y(s)$ start at the origin, $y(0) = 0$ and satisfy

$$curv^\perp(y(s)) \leq 1.$$

Represent $\mathsf{grad}(h)$ by a point in the unit sphere, say $g \in S^{N-1}$, and let $z(0) \in S^{N-1}$ represent the initial derivative $z(0) = y'(0)$.

8.1.A. Semi-Circle Lemma. The derivative $z(s)$ of the extremal $y(s)$, which minimises $h \circ y(l)$, follows the shortest geodesic arc in S^{N-1} from $z(0)$ to $-g$ with constant speed c for

$$s \leq c^{-1} dist_{S^{N-1}}(z(0), -g)$$

and $z(s)$ is constant equal $-g$ for

$$s \geq c^{-1} dist_{S^{N-1}}(z(0), -g)$$

and the extremal curves $y(s)$ are planar circular arcs, of length $\leq \pi/c$, which may be followed by straight segments for large l.

8.1.B. Semi-Circle Example/Corollary. Let the ends of a smooth immersed curve $y(s)$, $0 \leq s \leq l$, such that $curv^{\perp}(y(s)) \leq 1$, lie in two parallel hyperplanes $H_0, H_l \subset \mathbb{R}^N$, which are tangent to $y(s)$ at the corresponding ends $y(0)$ and $y(l)$.

If no tangent to the curve $y(s)$. at an interior point $s \in (0.l)$ is parallel to H_0 and if there exists a point $s_{\perp} \in [0, l]$, where the tangent to the curve is normal H_0, then

$$dist(H_0, H_l) \geq 2,$$

where the equality $dist(H_0, H_l) \geq 2$ implies that the $[s_1, s_2]$-part of the curve for an interval $[s_1, s_2] \subset [0, l]$ is congruent to a planar semicircle of unit radius.

Lemma 8.1. If $curv^{\perp}(y(s) \leq c(s)$, where $c(s) \geq 0$ is a non-constant (bounded measurable), then arguing as in 8.1.A we obtain the folloing.

8.1.B. Minimal displacement Lemma. The derivative $z_{ext}(s)$ of the extremal $y_{ext}(s)$, which minimises $h \circ y(l)$ follows the shortest geodesic segments from $z(0)$ to $-g$ with (now variable) speed $c(s)$ for

$$\int_0^s c(s) ds \leq dist_{S^{N-1}}(z(0), -g),$$

where (this part is irrelevant for the proof of the bow inequality below) continues with constant $z(s) = -g$ for s for

$$\int_0^s c(s) ds \geq dist_{S^{N-1}}(z(0), -g).$$

It follows that
$[\mathbf{a_{ext}}]$ *if* $y'(0) = -grad(h)$ *and*

$$\int_0^l c(s) ds \leq \pi = dist_{S^{N-1}}(g, -g),$$

then the extremal curves $y_{ext}(s) = \int_0^s z_{ext}(s) ds$ *are a planar*[27] *locally convex curves, call them* $a(s) = a_c(s)$, *such that*

$$curv^{\perp}(a(s)) = c(s),$$

where, as we know, all these a-curves are mutually congruent.

Now we are ready to prove "bow inequality", that is, recall, the bound

$$dist(y(0), y(l)) \geq dist(a(0), a(l))$$

[27] "Planar means contained in a 2-plane in $\mathbb{R}F^N$.

for smooth curves $y(s)$, $s \in [0,l]$ in $\mathbb{R}^N$, such that

$$curv^{\perp}(y(s))\|y'(s)\| \leq c(s)$$

and such that the corresponding (locally convex) curves $a(s)$ with $curv^{\perp}(a(s)) = c(s)$ are convex.

This is done by dividing both curves $a(s)$ and $y(s)$ into halves by a point $s_\circ \in [0,l]$, such that

(1) The total curvatures of the halves of both curves are bounded by π,

$$\int_0^{s_\circ} c(s)ds, \int_{s_\circ}^{l} c(s)ds \leq \pi.$$

We apply $[\mathbf{a_{ext}}]$ to these halves and to the linear function

$$h(y) = h_\circ(y) = \langle y, y'(s_\circ)\rangle, \;\; y \in \mathbb{R}^N.$$

which is characterised by the equality $grad(h)(y(s_\circ)) = y'(s_\circ)$ and obtain two inequalities

$$h_\circ y(s_\circ) - h_\circ y(0) \geq h_\circ a(s_\circ) - h_\circ a(0)$$

and

$$h_\circ y(l) - h_\circ y(s_\circ) \geq h_\circ a(l) - h_\circ a(s_\circ),$$

where the abbreviation $h_\circ y(s)$ stands for $h_\circ(y(s))$, etc. and where we use same notation $h_\circ$ for the planar linear function $x \mapsto \langle x, a'(s_\circ)\rangle$.

These two add up to

$$h_\circ y(l) - h_\circ y(0) \geq h_\circ a(l) - h_\circ a(0),$$

where the first term is bounded by

$$h_\circ y(l) - h_\circ y(0) \leq dist(y(0), y(l)),$$

since $\|grad(h_\circ)\| \leq 1$. Therefore,

$$h_\circ a(l) - h_\circ \circ a(0) \leq dist(y(0), y(l)).$$

(2) Finally, let $s_\circ \in [0,l]$ be a point, where the tangent to the (image of the) curve $a(s)$ at $s = s_\circ$ is *parallel to the straight segment* between the ends of this curve, that is $[a(0), a(1)] \subset \mathbb{R}^2$.[28] Then, by convexity of $a(s)$,

$$h_\circ a(l) - h_\circ a(0) = dist(a(0), a(l))$$

and since, also by convexity of $a(s)$, the total curvatures of the both halves of $a(s)$, hence of $y(s)$, are bounded by π, (1) applies. Then confronting the above "blue" relations yields the bow inequality

$$dist(y(0), y(l)) \geq dist(a(0), a(l)).$$

[28]Such a point, e.g. the one which maximises the distance between pairs of points $a(s) \in \mathbb{R}^2$, $s \in [0,l]$, and $x \in [a(0), a(1)]$, exists on all smooth immersed curves.

On the Error Term in the Bow Inequality. One can sharpen the bow inequality (at least the circular one for $\int curv^{\perp} \leq \pi$) with a bound on the total curvature, where the extremal ones, i.e. with minimal $dist(y(0), y(l))$ are circular arcs extended by straight segments at one of the ends. Also one can evaluate non-planarity of curves by non-additivity of angles, e.g. in decompositions of polygonal curves into triangles.

EXAMPLES.

9 Angular Bow InequaIity and non-Angular Corollaries

Let Y and A be Riemannian manifolds and $y(s)$ and $a(s)$, be smooth curves of length l in these manifolds parametrized by the arc length $s \in [0, l]$, where $a(s)$ is a *convex ark* (as it is explained below) in A and where

the normal curvature of $y(s)$ is everywhere bounded by that of $a(s)$:

$$curv^{\perp}(y(s)) \leq curv^{\perp}(a(s) \text{ for all } s \in [0, l].$$

9.A. Angular Bow Inequality for Negative Curvature.

Let both manifolds be complete simply connected with nonpositive sectional curvatures, let $dim(A) = 2$ and

$$sect.curv(Y) \leq \inf_{a \in A} sect.curv(A, a).$$

Let the ark $a(s)$ be *locally convex and "acute":* the angles α_1 and α_2 between geodesic segments $[a(s_1), a(s_2)] \subset A$ and the curve $a(s)$ at both ends are $\leq \pi/2$, where the curve and the segments are oriented in same direction , where short segments are acute. (Arks in the unit circle are "acute" for $length \leq \pi$.)

Then *angles β_0 and β_l between geodesic segment $[y(0), y(l)] \subset Y$ and the curve $y(s)$ are bounded* by the corresponding angles between the segment $[a(0), a(l)] \subset A$ and $a(s)$,

$[\beta \leq \alpha]$ $$\beta_0 \leq \alpha_0 \text{ and } \beta_l \leq \alpha_l.$$

Proof. Let us enumerate the relevant properties of the distance functions $d = d(a_1, a_2) = dist_A(a_1, a_2)$ on A and $D = dist_Y(y_1), y(2))$ on$Y$ restricted to our curves.

(1) Since both manifolds complete simply connected with nonpositive sectional curvatures, the functions d and D are smooth away the diagonals.

What is relevant for our purpose, is that they are

smooth at the pairs $(a(s_1), a(s_2)) \in A \times A$ and the pairs $(y(s_1), y(s_2)) \in Y \times Y$ respectively for $s_2 > s_1$.

(2) Let

$$\alpha_{s_1}[\overrightarrow{s_1 s_2}] = \angle(a'(s_1), -grad_{s_1}(d)),$$

where $a'(s) = \frac{da(s)}{ds} \in T_{a(s)}(A)$ stands for the derivative of a, be the angle between the a-curve at the point $a(s_1)$ and the directed geodesic segment [29]

[29]Smoothnss of d imy the uniqueness of the minimal segment.

$\overrightarrow{[a(s_1), a(s_2)]} \subset A$ and let

$$\alpha_{s_2}[\overrightarrow{s_1 s_2}] = \angle(a'(s_2), grad_{s_2}(d))$$

be the angle between this curve with same segment at the second end $a(s_2)$.

Similarly, define such angles for the y-curve and denote them

$$\beta_{s_1}[\overrightarrow{s_1 s_2}] \text{ and } \beta_{s_2}[\overrightarrow{s_1 s_2}].$$

Thus our objective is the inequalities

$$\beta_{s_i}[\overrightarrow{s_1 s_2}] \leq \alpha_{s_i}[\overrightarrow{s_1 s_2}] \text{ for } i = 1, 2.$$

Recall that the inequality [$sect.curv(Y) \leq 0$] and [smoothness of the distance function $D(y_1, y_2)$] imply that

($\ast$) the R-spheres in Y are *greater than R-spheres in* $\mathbb{R}^N, n = dim(Y)$; in fact the exponential map $\exp_{y_1} : T_{y_1}(Y) \to Y$ is *distance increasing.*

This, or rather the corresponding contraction property of the differential of the (rescaled) inverse exponential map, can be expressed in terms of the distance function as follows.

Let

$$y_2 \overset{\chi}{\mapsto} grad_{y_2}(D(y_1, y_2))(y_1) \in T_{y_1}(Y), \ y_1, y_2 \in Y$$

and let

$$\chi' = \chi'_{y_1, y_2} : T_{y_2}(Y) \to T_{y_1}(Y)$$

be the differential of χ

Then ($\ast$) is equivalent to the bound of the norm χ' by the inverse distance function:

$$(\ast') \qquad \|\chi'_{y_1, y_2}\| \leq D^{-1}(y_1, y_2).$$

Flat Example. If $Y = \mathbb{R}^N$, then $\chi'(t) = t \cdot \frac{1}{D(y_1, y_2)}$, $t \in \mathbb{R}^N = T_{y_1}(\mathbb{R}^N) = T_{y_2}(\mathbb{R}^N)$.

The conditions [$sect.curv(Y) \leq 0$]&[smoothness of D] imply the following stronger geometric property than ($\ast$):

($\ast\ast$) the R-spheres $S(R) \subset Y$ are "more convex" than R-circles in $\mathbb{R}^2$: the curvatures of these spheres at all unit tangent vectors $\tau \in T(S(R))$ are $\geq 1/R$, where the implication $(\ast\ast) \implies (\ast)$ is seen with the metric definition of curvature (see sect1.1.E).

The above was meant to motivate the following relation between the norms of the the (linear) maps χ' in X and in A which follows from the inequality

$$sect.curv(Y) \leq \inf_{a \in A} sect.curv(A, a).$$

(3) The norms of the maps χ' in X are bounded by these in A as follows

$$\|\chi'_{y(s_1), y(s_2)}\| \leq \|\chi'_{a(s_1) a(s_2)}\| \text{ for } D(y(s_1), y(s_2) \geq d(a(s_1), a(s_2),$$

where this inequality is strict, $\|\chi'_{y(s_1), y(s_2)}\| < \|\chi'_{a(s_1)(s_2)}\|$, for $D(y(s_1), y(s_2) > d(a(s_1), a(s_2)$.

Corollary. The derivatives of the above defined angles. α and β satisfy

$$\alpha'_{s_1}[\overrightarrow{s_1s_2}] = \chi' \cdot \sin\alpha_{s_2}[\overrightarrow{s_1s_2}]$$

and

$$\|\beta_{s_1}[\overrightarrow{s_1s_2}]\| \le \chi'(s_1,s_2)\cdot\sin\beta_{s_2}[\overrightarrow{s_1s_2}] \text{ for } D(y(s_1,y(s_2)\ge d(a(s_1,a(s_2),$$

where χ' in both cases is for A, i.e. $\chi' = \chi'_{a(s_1)a(s_2)}$, where
this χ' is *strictly monotone decreasing in the distance* $d(a(s_1)a(s_2))$
which makes
the β-inequality strict for for $D(y(s_1,y(s_2)>d(a(s_1,a(s_2)$.

*Remark.*This "sin" plays no special role in our proof of the inequality $[\beta\le\alpha]$ and it can be replaced by another positive monotone strictly increasing function on the segment $[0,\pi/2]$.

(4) The inequality $sect.curv(Y)\le\inf_{a\in A} sect.curv(A,a)$ impies that the curvatures of the spheres in X are greater or equal than of the spheres of same radii in A, nd $a(s_2)$.

I fact we shall need this bound only for the spheres $S_{y(s_1)}\uparrow^{y(s_2)}$ centred at $y(s_1)\in Y$ which contains $y(s_2)$, and will use it at the points $y(s_2)$, where "greater or equal" applies to the curvatures of these spheres at all their unit tangent vectors, $\tau\in T_{s_2}(S_{y(s_1)}\uparrow^{y(s_2)})y(s_2)$:

$$curv^{\perp}_{\tau}(S_{y(s_1)}\uparrow^{y(s_2)})y(s_2)\ge curv^{\perp}(S_{a(s_1)}\uparrow^{a(s_2)})a(s_2).$$

(The sphere $S_{a(s_1)}\uparrow^{a(s_2)}$ is 1-dimensional with a $\pm$single tangent vector at $a(s_2)$.)

(5) The *derivatives* of the distance functions $D(s_2)=D_{s_1}(s_2)=D(y(s_1),y(s_2))$ and $d(s_2)=d(a(s_1),a(s_2))$ in the s_2 variable are *monotone decreasing* in the angle $\beta_{s_2}=\beta_{s_2}[\overrightarrow{s_1s_2}]$. In fact,

$$D'(s_2)=\cos\beta_{s_2} \text{ and } d'(s_2)=\cos\alpha_{s_2}$$

and by symmetry of the distance function,

$$D'(s_1)=-\cos\beta_{s_2} \text{ and } d'(s_2)=-\cos\alpha_{s_2}$$

for $D(s_1)=-D_{s_2}(s_1)=D(y(s_1),y(s_2)$ and $d(s_1)=d_{s_2}(s_1)=d(a(s_1),a(s_2)$.

(6) Finally we turn to the contribution of the $curv^{\perp}$-curvatures of the curves $a(s)$ and $y(s)$ to the derivatives of the angles $\alpha(s_2)=\alpha_{s_2}[\overrightarrow{s_1s_2}]$ and $\beta(s_2)=\beta_{s_2}[\overrightarrow{s_1s_2}]$ and thus to the second derivatives of the distance functions $d(a(s_1),a(s_2))$ and $D(y(s_1),y(s_2))$ in the second variable.

Let $c_a(s_2)=curv^{\perp}(a(s_2))$ be the normal curvature of the curve $a(s)$ in A at the point $a(s_2)\in A$ and $c_y(s_2)=curv^{\perp}(y(s_2))$ be this for $y(s)$ in Y.

Let $c_d(s_2)$ be the normal curvature of the circle $S^1=a(s_1)\uparrow^{a(s_2)})$ (with centre $a(s_1)$ and radius $dist_A(a(s_1),a(s_2))))$ at the point $a(s_2)$ and let $c_D(s_2)$ be the normal curvature of the sphere $S^{N-1}=S_{y(s_1)}\uparrow^{y(s_2)})$ (with centre $y(s_1)$ and radius $dist_Y(y(s_1),y(s_2)))$ at the unit tangent vector $\tau\in T_{y(s_2}(S^{N-1})$ that is the normalized normal projection of the unit tangent vector to the curve $y(s)$ at $s=s_2$, (the derivative $y'(s_2)\in T_{y(s_2)}(Y)$) to the tangent space $T_{y(s_2)}S^{N-1}\subset T_{y(s_2)}(Y)$.

Then

$$\alpha'(s_2) = c_a(s_2) - c_d(s_2), \text{ and } \|\beta'(s_2)\| \le c_y(s_2) - c_D(s_2),$$

Therefore, our assumptions $curv^{\perp}(y(s_2) \le curv^{\perp}(y(s_2)$ and $sect.curv(Y) \le \inf_{a\in A} sect.curv(A,a)$ imply that

the s_2*-derivatives of the* β*-angles* of the curve $y(s)$ in the manifold Y *are bounded by these of the* α*-angles* of (the convex "acute" arc) $a(s)$ in the surface A,

$[\beta' \le \alpha']$ $$\|\beta'(s_2)[\overrightarrow{s_1s_2}]\| \le \alpha'(s_2)[\overrightarrow{s_1s_2}]$$

for all pairs of points $0 \le s_1 \le s_2 \le l$, such that $D = d$, i.e. where

$$dist_Y(y(s_1), y(s_2) = dist_Y(y(s_1), y(s_2)).$$

The proof of the $[\beta \le \alpha]$ inequality is concluded with the following.

9.B. ***Elementary Calculus Lemma***. Let $d(s_1,s_2)$ and $D(s_1,s_2)$, $0 \le s_1, s_2 \le l$, be positive symmetric functions, which are strictly positive and smooth for $s_1 \ne s_2$ and which are approximately equal to $|s_1 - s_2|$ at the diagonal $\{s_1 = s_2\}$,

$$d(s_1, s_1 + \varepsilon) = |\varepsilon| + O(\varepsilon^3) \text{ and also } D(s_1, s_1 + \varepsilon) = |\varepsilon| + O(\varepsilon^3).$$

If these functions satisfy the above (1)-(6) relations, where the α- and β-angles are defined as the derivatives of these functions, then these angles satisfy the $[\beta \le \alpha]$ inequality, where either of the equalities $\beta_0 = \alpha_0$ or $\beta_l = \alpha_l$ implies that the two functions are equal, $d(s_1,s_2) = D(s_1,s_2)$. ely curved metrics

Corollaries

Since the angles α_s and β_s are represent the derivatives of the distance functions $d(a(0), a(s)$ and $D'(y(0), y(s)$, the $\beta \le \alpha$ implies that $dist_Y(y(0), y(l) \ge dist_A(a(0), a(l)$ for convex "acute" arks .

More interestingly, the same holds for some non-acute arks which can be divided two acute segments by a point $s_\circ$.

Thus we arrive at the following

9.C. (De)composed Bow Inequality. Let A and Y be complete simply connected manifolds with non-positive sectional curvatures let

$$sect.curv(Y) \le \inf_{a\in A} sect.curv(A,a)$$

and let

$$curv^{\perp}(y(s)) \le curv^{\perp}(a(s) \text{ for all } s \in [0,l].$$

Let the curve $a(s)$ be locally convex and let $s_\circ \in [0,l]$ be a point, such that the two parts $a[0,s_\circ]$ and $a[s_\circ,l]$ of the curve are "acute":

if either $s_1,s_2 \le s_\circ$ or $s_1,s_2 \ge s_\circ$, then the angles between the geodesic segment $[a(s_1), a(s_2)]$ at its ends with the curve $a(s)$ are acute.

Then

$[dist_Y \ge dist_A]$ $$dist_Y(a(0), a(l)) \ge dist_A(a(0), a(l)),$$

provided one of the following two conditions is satisfied.

Condition 1 $$dist_A(a(0), a(s_\circ)) = dist_A(a(s_\circ), a(l)),$$

Condition 2 $$dist_A(a(0), a(s_\circ)) \geq l - s_\circ.$$

In fact, the angular inequality bow inequality "$[\beta \leq \alpha]$" shows that the angle $\angle_y(s_\circ)$ between the geodesic segments $[y(s_\circ), y(0)]$ and $[y(s_\circ), y(l)]$ in Y is greater or equal than the the angle $\angle_a(s_\circ)$ between $[a(s_\circ), a(0)]$ and $[a(s_\circ), a(l)]$ in A and the proof follows by the standard comparison theorems for geodesic triangles.

9.D. ***Subcorollary: Riemannian Circular Bow Inequality.*** Let A and Y be complete simply connected manifolds, where A has constant sectional curvature κ and $sect.curv(Y) \leq \kappa$, where for $\kappa > 0$ we require that all geodesic segments of length $< \pi/\sqrt{\kappa}$ are distance minimising.

Let the curve $a(s)$ be (planar) convex with constant curvature $c \geq 0$ and

$$curv^\perp(y(s)) \leq c.$$

Then

$[dist_Y \geq dist_A]$ $$dist_Y(a(0), a(l)) \geq dist_A(a(0), a(l)),$$

Proof. If A and $a(s)$ have constant curvatures (each of its own kind), then the above two conditions are satisfied and the case of $\kappa \leq 0$ follows.

9.E. Non-Simply Connected Generalization and $\kappa > 0$. The above argument applies to arc-length parametrized curves $[0, l] \ni s \mapsto y(s) \in Y$ in manifolds Y, such that

• every subsegment $[0, l] \supset [s_1, s_2] \to Y$ admits a homotopy by curves with fixed ends and with length$\leq s_2 - s_1$ to a geodesic segment $[y(s_1), y(s_2)]$ in Y and there is only one segment achievable by such a homotopy, where the length of this segment takes the role of the distance D between s_1 and s_2;

• the exponential map $\exp_{y(s)}$ is defined on the balls $B(R) \subset T_{y(s)}(Y)$ of radii $R \leq \max(s, l - s)$ for all $s \in [0, l]$, where this map is an immersion which is strictly locally convex on the R-spheres.

For instance, these conditions are satisfied by complete manifolds Y with $sect.curv(Y) \leq \varepsilon^2$ and curves $y(s)$ of length $l \leq \pi/2\varepsilon$ and the β-angles of these curves are bounded by those of convex acute arcs $a(s)$ in the 2-sphere $S^2(1/\varepsilon^2)$, if $curv^\perp(a(s)) \geq curv^\perp(y(s))$.

Now, if the distance between the ensds of a curve $C \in Y$ is understood as the length of the geodecsic obtained by shortening homotopy of C the above argument delivers the so modified Riemannian Circular Bow Inequality remain valid for all complete manifolds Y with $sect.curv(Y) \leq \kappa$. and all $-\infty < \kappa < \infty$

Alternatively if one insists on manifolds being simply connected and on keeping true metric distances, then the condition $sect.curv(X) \leq \kappa$ for $\kappa > 0$ must be augmented by $inj.rad(X) \geq \pi/\sqrt{\kappa}$.

Then the problem reduces to the case of $\kappa = 0$ by taking the $1/\sqrt{\kappa}$-cone $Cone(X)$, which albeit is singular, has $sect.curv \leq 0$ in the sense of Alexandrov's $CAT(\kappa)$-theory, where the above analytic proof can be carried over with minor modifications.

Remark. The circular bow inequality in the $(N-1)$-sphere $S^{N-1} \subset \mathbb{R}^N$ effortlessly follows from that for $\mathbb{R}^N$, since curves of constant (geodesic) curvature in the sphere are *planar*, i.e. contained in planes in $4\mathbb{R}^N$, *Exercise.* Show that (domains in) spheres are the only smooth submanifolds in $\mathbb{R}^N$, where curves of constant geodesic)] curvature are *planar.*[30]

10 Hypersurfaces in Balls and Spheres.

Hypersurfaces with Unit Curvature in $B(2)$ and in $B(2+\delta)$. Let the image of an immersion $f : X \hookrightarrow \mathbb{R}^{n+1}$, $n = dim(X)$, such that $curv^\perp(X) \leq 1$, is contained in the ball $B^{n+1}(2)$.

10.A. ***∞-Extremality.*** If $n = 1$ and the *degree of the Gauss map* $S^1 = X \to S^1(1) \subset \mathbb{R}^2$ *equals zero*, then (this was stated in 1.C) the image $f(X) \subset B^2(1)$ equals the union of *two unit circles* which tangentially meet at the center of the disk $B^2(2)$.

10.B. ***Star Convex Rigidity.*** If $n \geq 2$, then either $f(X)$ is star convex and the radial projection $X \to S^n(2)$ is a diffeomorphism, or $f(X)$ is equal to a unit n-sphere $S^n_{y_o}(1)$, where the centre of this sphere is positioned half way from the boundary of the ball $B^{n+1}(2)$, i.e. $\|y_o\| = 1$.

10.C. ***Star Convex Stability Switch.*** There exists $\delta > 0.01$ (probably $\delta > 0.2$), such that if $n \geq 2$ and the image $f(X)$ is contained in the ball $B^{n+1}(2\varepsilon))$ then $f(X) \subset B^{n+1}(2+\delta)$ is star convex with respect to some point in $B^{n+1}(2+\varepsilon)$.

Proof. If $f(X)$ is not star convex with respect to the centre of the ball $B^{n+1}(2)$ then some radial ray is tangent to $f(X)$ at some point $y_0 = f(x_0) \in f(X)$ and the semi-circle lemma 8.1.A implies that y_0 is equal to the centre of $B^{n+1}(2)$ and that $f(x)$ equals a unit sphere passing through y_0.

This proves (b) while the stability of the bow proof argument (see section 8.1) yields an approximate unit sphere in $B^{n+1}(2+\varepsilon)$ and (c) follows as well.

Remark/Example. The boundary X_{+1} of the ρ-neighbourhood for $\rho = 1$ of a circular ark S with radius 2 has curvature bounded by 1. If such an S is slightly shorter than half circle, then, because of "shorter", X_{+1} can be fit to the ball of radius $3 - \epsilon$ and X_{+1} and it is non-star convex because of "slghtly".

Question Do δ and ϵ ever meet or there is a definite gap between their possible values?

10.D. Hypersurfaces in S^{n+1}. It is unknown if there are closed connected n-manifolds X non- diffeomorphic to spheres S^n, which admit immersions to the unit balls B^{n+1} with $curv^\perp(X \hookrightarrow B^{n+1}(1)) < 3$.

Nor do we know if there exist codimension two immersions of these X with $curv^\perp(X \hookrightarrow B^{n+2}(1)) < \sqrt{2}$.

Conjecturally, the only non-spherical immersions with *"critical curvatures"*, that are 3 for codimension 1 and $\sqrt{2}$ for codimension 2, are the standard embeddings[31] of $S^{n_1} \times S^{n_2}$ to $B^{n+1}(1))$ and to $B^{n+2}(1))$, where the latter are Clifford's and the former are encircling of round spheres.

But the (first) critical curvature is unquestionably is *equal to one* for hypersurfaces *in the unit spheres.* Here there are non-spherical submanifold with unit curvatures, namely Cliffords product of spheres

$$S^{n_1}(1/\sqrt{2}) \times S^{n_2}(1/\sqrt{2}) \subset S^{n_1+n_2+1}(1),$$

which have curvature $curv^\perp = 1$ and there is nothing of the kind for smaller curvature:

[30] I didn't solve this exercise.

[31] If either n_1 or n_2 is equal to 1, one may have covers of these embedded X.

❆*Closed connected orientable*[32] *immersed hypersurfaces* $X \hookrightarrow S^{n+1}(1)$ *with* $curv^{\perp} < 1$ *are diffeomorphic to the* n*-sphere* S^n.(Compare [Ge 2021].)

More generally, let X be a smooth closed connected n-manifold, let Y be a closed connected simply connected Riemannian $(n+1)$-manifold, let $f : X \to Y$ be a cooriented (two-sided) immersion and $f^{\perp}_{\pm t} : X^n \hookrightarrow Y(1)$, $t \geq 0$, be (the composition of f with) the normal exponential map $\exp^{\perp} : X \times \mathbb{R}_{\pm} \to Y$ at $t \in \mathbb{R}_+$.

Define two *one-sided focal radii*: $rad_{\pm}{}^{\perp}(X)$ that are the suprema of $t > 0$ for which the maps $\mathrm{f}^{\perp}_{\pm t}$ are immersions.

Notice that $rad^{\perp}(X) = \min(rad^{\perp}_{+}(X), rad^{\perp}_{-}(X)$.

If $sect.curv(Y) \geq 1$ *and*

$$rad^{\perp}_{+}(X) + rad^{\perp}_{-}(X) > \pi/2,$$

e.g. $rad^{\perp}(X \hookrightarrow Y) \geq \pi/4$, X *is diffeomorphic to the sphere* S^n.

Proof. Observe following [Ge 2021] that if $rad_{+}{}^{\perp}(X) > t_1$ and $rad_{-}{}^{\perp}(X) \geq t_2$, then the (kind of composed) map

$$(f^{\perp}_{+t_1})_{-(t_1+t_2)} : X \to Y$$

is defned and it satsfies

$$(f^{\perp}_{+t_1})_{-(t_1+t_2)} = f^{\perp}_{-t_2}.$$

It follows that if $rad^{\perp}_{+}(X) + rad^{\perp}_{-}(X) > \pi/2$, then the map $f^{\perp}_{-t_2} : X \to Y$ is an immersion (by the definition of $rad^{\perp}$) and if

$$sect.curv(Y) \geq 1 \text{ and } t_1 + t_2 > \pi/2$$

this immersion is *locally strictly concave.*[33]

In fact, if $sect.curv(Y) \geq 1$ and $f_* : X \to Y$ is an immersion, such that $rad^{\perp}_{-}(f_*(X)) > \pi/2$, then the map $(f_*)_{-t} X \to Y$ is a locally strictly concave immersion in the range $\pi/2 < t < rad^{\perp}_{-}(X)$ by the Hermann Weyl tube curvature formula.[34]

Then Gromoll-Meyer's *concave contraction* adjusted to immersions[35] and followed by smoothing delivers a regular homotopy of the (locally concave) immersion $f_{t_2} = f^{\perp}_{-t_2} : X \to Y$, say $\bar{f}_t : X^n \to Y$, where $t_2 \leq t < t_{\bullet}$, for some $t_{\bullet} > t_2$, where $\bar{f}_{t_2} = f_{t_2}$ and such that the locally concave immersions $\bar{f}_t$ become *concave embeddings*[36] for t close to $t_{\bullet}$, which eventually converge to a constant map for $t \to t_{\bullet}$.

Since "small" closed convex hypersurfaces in (all compact Riemannian manifolds) Y are diffeomorphic to spheres, the proof follows. [37]

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[32] This must be redundant. Anyway this is relevant only for those even n, where $\mathbb{R}P^N$ admits an immersion to $\mathbb{R}^{n+1}$.

[33] A smooth immersion $f : X \to Y$ is locally strictly convex/concave if the second fundamental form of $f(X)$ is positive/negative definite. "Convex" is distinguished from"concave" for families of immersions f_t: *convexity* indicates i*increase* of the induced metric in X and concavity corresponds to decrease of this metric. Thus the boundaries X of convex sets are convex for outward deformations of X and concave for inward deformations. Similarly, we attribute convexity and concavity to families of non-smooth hypersurfaces.

[34] Here and in the contraction argument below we follow [Esch 1886] and [Gro 1990].

[35] This argument needs $dim(X) = n \geq 2$, which can be assumed in the present case.

[36] These are boundaries of (geodesically convex subsets in Y with inward directed normal fields on them.

[37] Ge assumes that f is an embedding and he proves the existence of a diffeomorphism $X \cong_{diff} S^n$ for $n \neq 4$ by the h-cobordism theorem.

Remark. The above applies to X which satisfy a *point-wise version* of the inequality $rad^{\perp}_{+}(X) + rad^{\perp}_{-}(X) > \pi/2$,that is $rad^{\perp}_{+}(X,x) + rad^{\perp}_{-}(X,x) > \pi/2$ for all $x \in X$, $rad^{\perp}_{\pm}(X,x)$ is the maximal r,where the immersion condition on the exponential map is required only at thepoint $x \in X$.

Problem. Show that *the only topologically non-spherical smoothely immersed closed connected hypersurfaces X with $curv^{\perp}(X) \leq 1$ in unit spheres are Cliffords $S^{n_1} \times S^{n_2} \subset S^{n_1+n_2+1(1)}$*.

More generally, if a closed n-manifold, which is non-diffeomorphic to S^n, is immersed to a simply connected $(n+1)$-manifold Y with $sect.curv(Y) \geq 1$, such that $rad^{\perp}_{+}(X)+rad^{\perp}_{-}(X) \geq \pi/2$, or if $rad^{\perp}_{+}(X,x)+rad^{\perp}_{-}(X,x) \geq \pi/2\ \forall x \in X$, then $Y = S^{n+1}(1)$, conjecturally and X is Clifford's product of spheres.[38]

Remarks. (a) Since the sectional curvature of the induced metric in X is non-negative by Gauss' formula, $\bullet_{\leq}$ the only non-spherical surfaces in (1) with $curv^{\perp} \leq 1$ are flat tori, which are, by a simple argument, are coverings of Clifford's tori.

Exercise[39]. Let W be a compact Riemannian $(n+1)$-manifold with two boundary components $\partial_{\pm}(W)$ with the distance $dist(\partial_{-}, \partial_{+}) \geq \frac{\pi}{2} + \varepsilon$, e.g. V equals the ρ-neighbourhood, $\rho = \frac{\pi}{4} + \varepsilon/2$, of a hypersurface $X \subset S^n$ with $curv^{\perp}(X) < 1$.

Let $sect.curv(W) \geq 1$ and let W admit a locally isometric immersion to a complete $n+1$-manifold Y with $sect.curv(Y) > 0$.

Show that there exists an embedded n-sphere $S^n \subset W$, which separates the two boundary components of W.

11 Immersed Submanifolds in Bands and in Tubes

Let an n-dimensional manifold X be immersed to the k-*tube* $B^N_{\mathbb{R}^k}(R)$ *of radius* R,

$$X \overset{f}{\hookrightarrow} B^N_{\mathbb{R}^k}(R) = B^N(R) \times \mathbb{R}^k \subset \mathbb{R}^{N+k},$$

(where $B^N(R) = B^N_0(R) \subset \mathbb{R}^N$ is the R-ball).

Let $p : X \to \mathbb{R}^k_{ax} = \{0\} \times \mathbb{R}^k$ be the projection of $X \hookrightarrow B^N_{\mathbb{R}^k}(R)$ to the central axes of the tube, let

$$\mathcal{K} = \mathcal{K}(p) \subset T(X) \hookrightarrow T(B^N_{\mathbb{R}^k}(R))$$

be the *kernel of the differential* $dp : T(X) \to T(B^N_{\mathbb{R}^k}(R))$.

Let

$$\Sigma = \Sigma(p) = \{x \in X\}_{rank(\mathcal{K}_x)>0} \subset X$$

be the support of $\mathcal{K}$.[40]

[38]Consulting [Ge 2021] [Luis Guijarro and Frederick Wilhelm, Focal Radius, Rigidity, and Lower Curvature Bounds, 2017],[Grisha Perelman. Proof of the soul conjecture of cheeger and gromoll. Journal of Differential Geometry, 1994] [D. Gromoll and K. Grove, A generalization of Berger's rigidity theorem for positively curved manifolds], is instructive .

[39]Compare with [Ge 2021] and section 3.7.3(F) in [Gro 2021]

[40]If $k < n$, then $\Sigma = X$ and $rank(\mathcal{K}_x(p)) = n - k$, for generic maps $p : X^n \to \mathbb{R}^k$ and generic points $x \in X$. If $k \geq n$, then ether p is an immersion, i.e. $\Sigma = \varnothing$, or $dim(\Sigma(p)) = 2n - k - 1$ for generic p and and $rank(\mathcal{K}_x(p)) = 1$ at generic $x \in \Sigma$.

Let the induced Riemannian metric in X be *geodesically complete*, e.g. X is compact without boundary, and let

$$\gamma_\tau(l) \hookrightarrow X \hookrightarrow B^N_{\mathbb{R}^k}(R),\ \tau \in \mathcal{K}_x$$

be the geodesic segment of length l issuing from $x \in \Sigma$ in the τ-direction, where τ is a *non-zero vector* in the vector (sub)space $\mathcal{K}_x \subset T_x(X), x \in \Sigma$.

If

$$curv^\perp(X \hookrightarrow B^N_{\mathbb{R}^k}(R) \subset \mathbb{R}^{N+k}) \le 1/R,$$

then the half circle lemma applied to the curves $\gamma_{\pm\tau}(\frac{1}{2}\pi R)$ in the R-tube $B^N_{\mathbb{R}^k}(R)$ and to the hyperplane $H = H_{\perp\tau} \subset \mathbb{R}^{N+k} \supset B^N_{\mathbb{R}^k}(R)$, which contains $f(x) \in B^N_{\mathbb{R}^k}(R)$ and is normal to τ imply the following.

$[\frac{\pi}{4} * \frac{\pi}{4}]$ Either $\Sigma = \varnothing$, i.e. $p : X \to \mathbb{R}^k$ is an immersion, (in this case one may have $curv^\perp(X) < 1/R$) the curves $f(\gamma_{\pm\tau}(\frac{1}{2}\pi R))$ and $f(\gamma_{\pm\tau}(\frac{1}{2}\pi R))$ are composed of quarter's of planar circlers, both of which reach the boundary of the tube and where the (normal) curvature vectors of these curves are parallel to the central axes $\mathbb{R}^k_{ax}$ of the tube, i.e. the planes containing these curves are perpendicular to the N-(sub)space, which contains the ball $B^N(R)$ (which is normal to $\mathbb{R}^k_{ax}$). Thus, $f(X) \subset B^N_{\mathbb{R}^k}(R)$ intersect the boundary of $B^N_{\mathbb{R}^k}(R)$ at at least two points.

$[\frac{\pi}{2}]$ If an immersion f is C^2-smooth, so is the πR-curve in the tube made of $f(\gamma_\tau(\frac{1}{2}\pi R))$ and $f(\gamma_{-\tau}(\frac{1}{2}\pi R))$. This necessarily makes this curve a planar half circle.

In general, $C^{1,1}$-curves composed of circular arc of same curvature $1/R$ are not always planar arks themselves.

However if the geodesic segments γ_τ issuing from a point x in all directions $\tau \in T_x(X)$ in an n-dimensional $C^{1,1}$-immersed n-manifolds $X \hookrightarrow \mathbb{R}^N$, are planar circular arcs of same curvatures $c > 0$, an if $n \ge 2$, then $x \in \Sigma \subset X^n$, serves as the centre of a geodesic R-hemisphere $(S^m_+)_x \subset X$ of dimension $m = rank(\mathcal{K})$, such that the map f isometrically sends $(S^m_+)_x$ to an equatorial m-hemisphere in the $(N-1)$-sphere $S^{-1}_{p(x)}$, where the boundary of this hemisphere is contained in the boundary of the tube $B^N_{\mathbb{R}^k}(R)$.

Exercises. (a). Recall that the real projective spaces of dimension $n = 2^l$, admit no immersions to $\mathbb{R}^k$ for $k \le 2n-2$, and show that they admit no immersions to the tubes $B^N_{\mathbb{R}^k}(R)$ with $curv^\perp(f) < 1/R$.

(b) Let a closed connected n-manifold X, $n \ge 2$, be immersed to a (cylindrical) $(1,R)$-tube

$$X \xrightarrow{f} B^N_{\mathbb{R}^1}(R) \subset \mathbb{R}^{N+1}.$$

(b$_1$) Show that *the only critical points* $x \in X$ of the function $p : X \to \mathbb{R} = \mathbb{R}^1_{ax}$, i.e. where $rank(\mathcal{K}_x) = n$, are the maximum x_{max} and the minimum x_{min} ponts of p, and that the f-images of both of them in the tube are positioned on the axial line $\mathbb{R}^1_{ax} = \{0\} \times \mathbb{R}^1$, where they serve as the centers of n-hemispheres $(S^n_+)_{max}(R)$ and $(S^n_+)_{min}(R)$, both of radius R and where both are contained in the $f(X) \subset B^N_{\mathbb{R}^1}(R)$ and where the spherical ($S^{n-1}(R)$) boundaries of them are contained in the boundary of the tube.

(b$_2$) Show that the $(n-1)$-hemispheres $(S^{n-1}_+)_{f(x)}(R)$ in the tube tangent at their centres $y = f(x)$ to the (topologically $(n-1)$-spherical) fibers of the map p for all non-critical points $x \in X$ *continuously* depend on x .

Observe that the resulting continuous map from the $(n-2)$-sphere bundle U over $X \setminus \{x_{max}, x_{min}\}$ of unit vectors tangent to the fibres to X, say $\Phi : U \to X$, sends U to the intersection of X with the boundary of the tube.

(b_3) Show that the image of the immersion $f : X \to B^N_{\mathbb{R}^1}(R)$ equals the union of the two hemi-spherical cups $(S^n_+)_{max}(R)$ and $(S^n_+)_{min}(R)$ and a region between them *contained in the boundary of the tube.* that is equal the Φ-image of $X \setminus ((S^n_+)_{max}(R) \cup (S^n_+)_{min}(R)$

Hint Start with case $n = 2$

(i) Let $n \geq 2$ and $N = n$and show that f is an embedding, the image of which is equal the $+R$-encircling of a segment in the central line $\mathbb{R}^1$ in $B_{\mathbb{R}^1}(R)^N$, that is a (convex) region between to half-R-spheres normal to this line, which is equal in the present case to the boundary of the convex hull of $f(X) \subset B_{\mathbb{R}^1}(R)^N$. (Unless the half-$R$-spheres have a common boundary, this region is only *piecewise* C^2.)

(ii) Let $n \geq 2$ and $N > n$. Show that f is an embedding into the $+R$-encircling of a central segment $[a,b] \subset \mathbb{R}^1$ in $B_{\mathbb{R}^1}(R)^N$, where this image contains two n-hemispheres of radius R and a cylindrical region between them which is fibered over $[a,b]$, where the fibers are equatorial $(n-1)$-subspheres in the $(N-1)$-spheres $S^{N-1}_y(R) \subset \partial B_{\mathbb{R}^1}(R)^N$, $y \in [a,b]$.

∞-Circles In Bands. [41] Let $f : S = S^1 \to \mathbb{R}^2$ be a $C^{1,1}$-immersion with curvature $curv^\perp(f) \leq 1$. If the Gauss map $G_f : S \to S^1(1)$ has zero degree, then *width* of the image [42]$f(S) \subset \mathbb{R}^2$ is at least 4, where the equality $width(f(S)) = 4$ implies that

the image $f(S)$, contains the left and the right halves of the figure 8, where these halves are composed of pairs of semicircles of unit radii.

Proof. Let $\tilde{G}_f : S \to \mathbb{R}$ be a lift of G_f to the universal cover $\mathbb{R} \to S^1(1)$ and let the function $\tilde{G}_f(s)$ assumes its *minimum* at $s_0 \in S$.

Then there exist (exactly) two *disjoint minimal* segments $S_1, S_2 \subset S$ in the circle, say $S_1 = [s_0, s_1]$ and $S_2 = [s_0, s_2]$, such that $G_f(s_i) = -G_f(s_0)$, $i = 1, 2$, where "minimal" signifies that they contain no points s where $G_f(s) = -G_f(s_0)$, except for their second ends s_i and "disjoint" means $S_1 \cap S_2 = \{s_0\}$.

Thus, the Gauss map sends S_1 and S_2 to two disjoint semicircle in S^1 "disjoint" means hat the two meet only at their end points.

Let $H_0, H_1, H_2 \subset \mathbb{R}^2$ be *three lines tangent to* $f(S^1)$ at the points s_0, s_1 and s_2 correspondingly, which are, by their construction, *mutually parallel* and where, by the minimality of $\tilde{G}(s_0)$ and minimality of the segments S_i, the only tangent lines to the curves $f(S_i)$, which, are parallel to H_0, H_1, H_2, are these lines themselves.

Also. observe that the *tangents* to these curves at the points $(s_\circ)_i$, where the Gauss map values $\tilde{G}_f(s)_i \in S^1(1)$ lie in the centre of the π-segment $[G_f(s_0), -[G_f(s_0)]$, are *normal* to H_i.

Now semi-circle example 8.1.B applies and the proof completed by applying the above to the *maximum* as well the minimum point of the function $\tilde{G}_f$.

EXAMPLES.

[41]The proof of this presented below was pointed out to me by Anton Petrunin. (compare circles in 1.A and with "crcles in discs" from the previous section).

[42]The width of a subset X in the Euclidean space is the supremum of the widths of the bands between parallel hyperplanes in the space, which contain X.

(a) Let $X_o \hookrightarrow \mathbb{R}^k \subset \mathbb{R}^{N+k}$ be a smooth closed immersed submanifold with $curv^\perp \leq 1/2$.

Then the 1-encircling[43] $X_{o+1} = (X_o)_{+1} \hookrightarrow \mathbb{R}^{N+k}$ of $X_o \hookrightarrow \mathbb{R}^k \subset$ is an smooth immersed hypersurface in the tube $B^N_{\mathbb{R}^k}(R) = B^N(R) \times \mathbb{R}^k \subset \mathbb{R}^{N+k}$,

(b) If $N = 1$ and $B^1_{\mathbb{R}^k}(R)$ is a band of width 2R rather than a "tube" and if $dim(X) = k$ then X_{o+1} contain a flat domain inside the boundary of the band. the There is much freedom in deforming with curvature $\leq 1/R$, while keeping X_{o+1} within the band, for all $k = dim(X)$, but especially for $k = 1$.

This is quite different from what happens submanifolds of dimensions≥ 2 in tubes $B^N_{\mathbb{R}^1}(R)$ and a similar rigidity is probably present whenever $dim(X) > k$.

(c) Let $X_\circ \hookrightarrow \mathbb{R}^2$ be the figure ∞ curve made of two unit circles (as in 1.C) and let $S^1 \times S^{n-1} = X_{\circ\circ} \hookrightarrow \mathbb{R}^{n+1}$ be obtained by rotating $X_\circ$ around an axes $A \subset \mathbb{R}^2 \subset \mathbb{R}^{n+1} = \mathbb{R}^{2+(n-1)}$.

If this axes is normal to the line between the centers of the circles, then the image of the immersion $X \hookrightarrow \mathbb{R}^{n+1}$ is contained in the unit tube $B^{n+1}_{\mathbb{R}^2}(1)$ and if $dist(A, X_\circ) \geq 1$ then $curv^\perp(X_{\circ\circ}) \leq 1$. This $X_{\circ\circ} \hookrightarrow \mathbb{R}^{n+1}$ is C^1-smooth and piecewise C^2 smooth as in X_{o+1} (b) but the geometry of $X_{\circ\circ}$ is significantly different from that of X_{o+1}.

These (a)(b)(c) reasonably well represent immersed hypersurfaces with curvatures one in the unit "tubes". $B^{n+1}_{\mathbb{R}^k}(1)$,especially for $k = 1$, where all immersions of closed n-manifolds to $B^{n+1}_{\mathbb{R}^k}(1)$ for $n \geq 2$ are embedding, which are 1-encirclings (boundaries of 1-neighbourhoods) of segments in the line $\mathbb{R}^1$.

12 Veronese Revisited

Besides invariant tori, there are other submanifolds in the unit sphere S^{N-1}, which have small curvatures and which are transitively acted upon by subgroups in the orthogonal group $O(N)$.

The generalized Veronese maps are a *minimal equivariant isometric* immersions of spheres to spheres, with respect to certain homomorphisms (representations) between the orthogonal groups $O(m+1) \to O(m+1)$,

$$ver = ver_s = ver_s^m : S^m(R_s) \to S^m = S^{m_s} = S^{m_s}(1),$$

where

$$m_s = (2s + m - 1)\frac{s + m - 2)!}{s!(m - 1!} < 2^{s+m} \text{ and } R_s = R_s(m) = \sqrt{\frac{s(s + m - 1)}{m}},$$

for example,

$$m_2 = \tfrac{m(m+3)}{2} - 1,\ R_2(m) = \sqrt{\tfrac{2(m+1)}{m}} \text{ and } R_2(1) = 2,$$

(see [DW1971]If $s = 2$ these, called *classical Veronese maps*, are defined by taking squares of linear functions (forms) $l = l(x) = \sum_i l_i x_i$ om $\mathbb{R}^{m+1}$,

$$Ver : \mathbb{R}^{m+1} \to \mathbb{R}^{M_m},\ M_m = \frac{(m+1)(m+2)}{2},$$

[43] "R-Encircling" is a generalization of "boundary of the R-neighbourhood" for embeddings, see section 3.

where tis $\mathbb{R}^{M_m}$ is represented by the space $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^{m+1})$ of quadratic functions (forms) om $\mathbb{R}^{m+1}$,

$$Q = \sum_{i=1,j=1}^{m+1,m+1} q_{ij} x_i x_j.$$

The Veronese map, which is (obviously) equivariant for the natural action of the orthogonal group group $O(n+1)$ on $\mathcal{Q}$, where, observe, this action fixes the line $\mathcal{Q}_\circ$ spanned by the form $Q_\circ = \sum_i x^2$ as well as the complementary subspace $\mathcal{Q}_\diamond$ of the *traceless forms* Q, where the action of $O(n+1)$ is irreducible and, thus, it has a *unique, up to scaling* Euclidean/Hilbertian structure.

Then the normal projection[44] defines an equivariant map to the sphere in $\mathcal{Q}_\diamond$

$$ver : S^m \to S^{M_m-2}(r) \subset \mathcal{Q}_\diamond,$$

where the radius of this sphere, a priori, depends on the normalization of the $O(m+1)$-invariant metric in $\mathcal{Q}_\diamond$.

Since we want the map to be isometric, we either take $r = \frac{1}{R_2(m)} = \sqrt{\frac{m}{2(m+1)}}$ and keep $S^m = S^m(1)$ or if we let $r = 1$ and $S^m = S^m(R_2(m))$ for $R_2(m) = \sqrt{\frac{2(m+1)}{m}}$.

Also observe that the Veronese maps, which are not embeddings themselves, factor via embeddings of projective spaces to spheres

$$S^m \to \mathbb{R}P^m \subset S^{M_m-2} \subset \mathbb{R}^{M_m-1} = \mathcal{Q}_\diamond,\ M_m = \frac{(m+1)(m+2)}{2}.$$

Curvature of Veronese. Let is show that
CURvature of veronese by Petrunin formula

$$curv^\perp_{ver}\left(S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)\right) = \sqrt{\frac{R_2(1)}{R_2(m)} - 1} = \sqrt{\frac{m-1}{m+1}}.$$

Indeed, the Veronese map sends equatorial circles from $S^m(R_2(m))$ to planar circles of radii $R_2(m)/R_2(1)$, the curvatures of which in the ball B^{M_m-1} is $R_2(1)/R_2(m) = 2\sqrt{\frac{m}{m+1}}$ and the curvatures of these in the sphere,

$$curv^\perp(S^1 \subset S^{M_m-2}(1)) = \sqrt{curv(S^1 \subset B^{M_m-1}(1))^2 - 1} = \sqrt{\frac{4m}{m+1} - 1} = \sqrt{\frac{3m-1}{m+1}}$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)$ itself

$\sqrt{R_2(1)/R_2(m)} = \sqrt{\frac{2m}{m+1}}$, and the curvatures of these in the sphere,

$$curv^\perp(S^1 \subset S^{M_m-2}(1)) = \sqrt{curv(S^1 \subset B^{M_m-1}(1))^2 - 1},$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)$itself. QED.

It may be hard to prove (conjecture in section 1) that Veronese manifolds have the smallest possible curvatures among non-spherical m-manifold in the unit ball: *if a smooth compact m-manifold X admits a smooth immersion to the*

[44]The splitting $\mathcal{Q} = \mathcal{Q}_\circ \oplus \mathcal{Q}_\diamond$ is necessarily normal for all $O(m+1)$-invariant Euclidean metrics in $\mathcal{Q}$.

unit ball $B^N = B^N(1)$ with curvature $curv^\perp(X \hookrightarrow B^N) < \sqrt{\frac{2m}{m+1}}$, then X is diffeomorphic to S^m.

It is more realistic to show that the Veronese have smallest curvatures among submanifolds $X \subset B^N$ *invariant under subgroups in $O(N)$, which transitively act on X.*

Remark. Manifolds X^m immersed to S^{m+1} with curvatures < 1 are diffeomorphic to S^n, see 5.5, but, apart from Veronese's, we can't rule out such X in S^N for $N \geq m+2$ [45] and, even less so, non-spherical X immersible with curvatures $< \sqrt{2}$ to $B^N(1)$, even for $N = m+1$.

It seems hard to decide this way or another, but it may be realistic to try to prove *sphericity of simply connected* manifolds immersed with curvatures < 1 to $S^N(1)$ for all N.

The curvatures of Veronese maps can be also evaluated with the *Gauss formula,* (teorema egregium), which also gives the following formula for curvatures of all ver_s:

$m = 2$ $1 - 2c^2 = 1/3$, $2c^2 = 2/3$ $c\sqrt{1/3}$

$C = \sqrt{1 + 1/3} = 2/\sqrt{3}$

From Veronese to Tori. The restriction of the map $ver_s : S^{2m-1}(R_s) \to S^{N_s}$ to the Clifford torus $\mathbb{T}^m \subset S^{2m-1}(R_s)$ obviously satisfies

$$curv^\perp_{ver_s}(\mathbb{T}^m) \leq A_{2m-1,s} + \frac{\sqrt{m}}{R_s} = \sqrt{3 - \frac{5}{2}m + \varepsilon(m,s)}$$

for

$$\varepsilon(m,s) = \frac{2}{4m^2} - \frac{4m-2}{s(s+2m-2)} + \frac{5(2m-1)}{2ms(s+2m-2)} - \frac{2m-1}{(ms(s+2m-2))^2}.$$

This, for $s >> m^2$, makes $\varepsilon(m,s) = O\frac{1}{m^2}$

Since $N_s < 2^{s+2m}$,

starting from $N = 2^{10m^3}$

$$curv^\perp_{ver_s}(\mathbb{T}^m) < \sqrt{3 - \frac{5}{2}m}.$$

where it should be noted that

the Veronese maps restricted to the Clifford tori are $\mathbb{T}^m$-equivariant

and that

this bound is *weaker than the optimal one* $\frac{||y||^2_{l_4}}{||y||^2} \geq \sqrt{3 - \frac{3}{m+2}} + \varepsilon$ from the previous section.

Remarks. (a) It is not hard to go to the (ultra)limit for $s \to \infty$ and thus obtain an

equivariant isometric immersion ver_∞ of the Euclidean space $\mathbb{R}^m$ to the unit sphere in the Hilbert space, such that

$$curv^\perp_{ver_\infty}(\mathbb{R}^m \hookrightarrow S^\infty) = \sqrt{\frac{(m-1)(2m+1)}{(m+1)^2}} = \sqrt{2 - \frac{5}{m+1} + \frac{2}{(m+1)^2}},$$

[45]Hermitian Veronese maps from the complex projective spaces $\mathbb{C}P^m$ to the spaces $\mathcal{H}_n$ of Hermitian forms on $\mathbb{C}^{m+1}$ are among the prime suspects in this regard.

where equivariance is understood with respect to a certain unitary representation of the isometry group of $\mathbb{R}^m$.

Probably, one can show that this ver_∞ realizes the *minimum* of the curvatures among all equivariant maps $\mathbb{R}^m \to S^\infty$.

(b) Instead of ver_s, one could achieve (essentially) the same result with a use of compositions of the classical Veronese maps, $ver : S^{m_i} \to S^{m_{i+1}}$, $_{i+1} = \frac{(m_i+1)(m_i+2)}{2} - 2$,

$$S^{m_1} \hookrightarrow S^{m_2} \hookrightarrow ... \hookrightarrow S^{m_i},$$

starting with $m_1 = 2m - 1$ and going up to $i = m$. (Actually, $i \sim \log m$ will do.)

12.1 Petrunins Veronese Rigidity Theorem

Large Simplex Property.(Compare with section 5 in pet.) Let the curvature of a complete [46] connected n-submanifold in an n-ball of radius r be bounded by one,

$$curv^\perp(X \hookrightarrow B^N(r)) \leq 1,$$

and let $x_0, ..., x_m \in X$ be $m+1$ points (e.g. m=n), such that

$$dist_X(x_i, x_j) = \pi, 0 \leq i < j \leq m.$$

Then

$$r \geq \sqrt{\frac{2m}{m+1}}.$$

In fact, the Euclidean distances between x_i are ≥ 2 by $[2\sin]_{bow}$inequality, the minimal ball which contains these point cant be smaller than the ball circumscribed about regular m-simplex with the edge length 2 by the Kirszbraun theorem.

Petrunin's two Balls Covering and the Sphere Theorem. Let the f-mage of X be contained in the ball of radius $r < 2/\sqrt{3}$ and let $x_-, x_+ \in X$ be two points joint by a geodesic segment of length π. Then the two geodesic balls $B_{x_\pm}(\pi) \subset X$ cover X.

It follows that X is homeomorphic to the sphere and, except for $n = 1$, the map $f : X \hookrightarrow \mathbb{R}^N$ is an embedding.

Proof. The above for $m = 2$ shows that the boundaries of these balls don't intersect and since these boundaries are connected for $n \geq 2$ the balls do cover X.

Petrunin's Veronese Planes Rigidity Theorem. If the image $f(X)\mathbb{R}^N$ is contained the ball $B^N(2/\sqrt{3})$ and is not homeomorphic to the sphere then f is an embedding and all geodesic segment in $f(X)$ are planar (contained in planes).

Consequently, X is either (congruent to) a *Veronese plane or its complex, quaternionic or Cayley numbers counterpart.*

Proof . Track the two balls covering argument in the extremal case with the bow rigidity at you hand or consult [Petr 2024].

Embedding Remark. Petrunin requires that f is embedding, but this seems ? unneeded for his argument.

[46]"Complete" refers to the induced Riemannian metric .

13 Hilbert's Rational Spherical Designs and Optimal Tori

Let

$$E:\mathbb{R}^N\to B^{2N}(1)\subset\mathbb{R}^{2N}$$

be the composition of the Clifford embedding $\mathbb{T}^n\subset B^{2N}$ and the exponential (locally isometric covering) map

$$\mathbb{R}^N=T_0(\mathbb{T}^N)\overset{\exp}{\to}\mathbb{T}^N.$$

A simple computation shows [Gro 2022] that the Euclidean curvature of E on the line $\bar{x}\subset\mathbb{R}^N$ generated by a non-zero vector $x\subset\mathbb{R}^n$ is

$$(\star)\qquad curv^{\perp}(\bar{x}\overset{E}{\hookrightarrow}\mathbb{R}^N)=\left(\frac{\|x\|_{L_4}}{\|x\|_{L_2}}\right)^2,$$

where $x=(x_1,...,x_N)$ for the standard Euclidean (corresponding to the cyclic torical) coordinates x_i and

$$\|x\|_{L_p}=\sqrt[p]{\frac{\sum_1^N|x_i|^p}{N}}.$$

Let $P(n,4)$ be the linear space of homogeneous polynomials of degree 4 on $\mathbb{R}^n$, this has dimension $\binom{n+4}{n}=\frac{n(n-1)(n-2)(n-3)}{24}$, and let

$$V_+4:\mathbb{R}^n\to P(n,4),\ V_+4:(c_1,...,c_n)\mapsto(c_1x_1+...c_nx_n)^4$$

be the *4th degree Veronese map.*

Then A $(n-1)$-spherical N-multi-set, that is map from a set Σ of cardinality N to the unit sphere $S=S^{n-1}\subset\mathbb{R}^n$ written as $\sigma\overset{\mathcal{D}}{\mapsto}s(\sigma)$, is called is a called *a design of degree* 4 *and cardinality* N *in* $S=S^{n-1}$ if

the center of mass of the N*-multi-set* V_+4D *in the image* $V_+4(S^{n-1})\subset P_(n,4)$ *is equal to the center of mass of* $V_+4(S^{n-1})$ *itself with respect to the usual spherical measure* or, equivalently, if

$$\frac{1}{N}\sum_{\sigma\in\Sigma}l^4(\mathcal{D}(\sigma))=\int_S l^4(s)ds$$

for all linear functions l on $S=S^{n-1}$, where ds is the normalised (i.e, of the full mass one) spherical measure.

Yet another way to characterise the design property of a muti-set D on S^{n-1} of cardinality N is via the tautological map

$$\mathbb{R}^n=\mathbb{R}_D\hookrightarrow\mathbb{R}^N$$

from the Euclidean n-space of linear functions $l(s)$ on S^{n-1} to the space $\mathbb{R}^N$ of (all) functions on Σ.

In these term $\mathcal{D}$ is a *a design (of degree* 4 *and cardinality* N in $S=S^{n-1}$) if and only if – this follows by the standard Γ-formulas for the $\int_S l^p(s)ds$-integrals, the L_2 and the L_4 norms on the non-zero vectors $x \in \mathbb{R}^N$ which are contained in $\mathbb{R}_{\mathcal{D}}$ satisfy:

$$\frac{\|x\|_{L_4}}{\|x\|_{L_2}} = \sqrt[4]{\frac{3n}{n+2}}$$

Thus, in view of ★,

every Design $\mathcal{D}$ of degree 4 and cardinality N on S^{n-1} defines a homomorphism (which is a locally isometric immersion), call it $E_{\mathcal{D}}$, from $\mathbb{R}^n = \mathbb{R}_{\mathcal{D}}$ to the Clifford N-torus, such that the curvature of $E_{\mathcal{D}}$ in the ball $B^{2N}(1) \supset \mathbb{T}^N$ satisfies:

$$curv^{\perp}\left(\mathbb{R}^n \overset{E_{\mathcal{D}}}{\hookrightarrow} B^{2N}(1)\right) = \sqrt{\frac{3n}{n+2}}.$$

A design D is *rational* if all points in D are rational.

Hilbert's Lemma.[47] *If $N >> n$, then S^{n-1} contains a rational design of cardinality N.*

Proof. Use three simple facts.

(i) the center of mass $\mathbf{c_o} \in P(n,4) = \mathbb{R}^{\binom{n+4}{n}}$ lies in the *interior* of the convex hull of the image $V_+4(S^{n-1}) \subset P(n,4)$

(ii) $\mathbf{c_o}$ is a *rational* point in $P(n,4)$,

(iii) rational points in S^{n-1} are dense

and proceed in four steps;

(1) Because of (i) and (iii) there exit finitely many *rational* points $s_i \in S^{n-1}$, $i = 1, ..., M$, such that the convex hull of these ponts contains $\mathbf{c'_o}$.

(2) Because of rationality of $\mathbf{c_o}$, there exist *rational* numbers $p_i \geq 0$, $p_1 + ... + p_M = 1$, such that $p_1 V_+4(s_1) + ... p_M V_+4(s_M)1 = \mathbf{c_o}$.

(3) Let Q be the common denominator of these numbers and write them as $\frac{P_i}{Q}$ for integer P_i, $i = 1, ..., M$, where $P_1 + ... + P_M = Q$.

(4) Let D be the multi-set in S^{n-1}, which consists of the points s_i, each taken with multiplicity P_i.

Then the center of mass of V_+4D is

$$\frac{1}{Q}\sum_i P_1 V_+4(s_i) = \sum_i p_i V_+4(s_i) = \mathbf{c_o}.$$

QED.

A. $2n^2$**-Designs**. The number N delivered. by the above proof is very big, a rough estimate is $N \leq$ but non-rational designs are known to exist for much smaller N.

For instance *If n is a power of 2, then there exists a design of cardinality $N = 2n^2 + 4n$.* [48]

[K1995] H. Konig, Isometric imbeddings of Euclidean spaces into finite dimensional lp -spaces, Banach Center Publications (1995) Volume: 34, Issue: 1, page 79-87.

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[47] In his solution of the Waring problem, Hilbert uses this lemma (for all even degrees) in the form of an identity $\sum_{i=1}^N l(x_j)^{2d} = (\sum_{j=1}^N (x_j^2))^d$ for some linear form l_i with rational coefficients.

[48] This was stated and proved in a written message by Bo'az Klartag to me. Also, Bo'az pointed out to me that the Kerdock code used in [Kon 1995] yields designs for $N = 4^k$ and $N = \frac{n(n+2)}{2}$.

homomorphism, (which is a locally isometric immersion) from the Euclidean n-space to the Clifford N-torus in the ball B^{2N} for $N = 8(n^2+n)$, such that the normal Euclidean curvature of this immersion is

$$(\star\star) \qquad curv^\perp\big(\mathbb{R}^n \hookrightarrow B^{16(n^2+n)}(1)\big) = \sqrt{\frac{3n}{n+2}}$$

Since rational points are dense in the sphere, we conclude to the extence of subtori $\mathbb{T}^n_\varepsilon \subset \mathbb{T}^{8(n^2+n)}$, such that

$$(\star\ \star\ \star). \quad curv^\perp\big(\mathbb{T}^n_\varepsilon \hookrightarrow B^{16(n^2+n)}(1)\big) \leq \sqrt{\frac{3n}{n+2}} + \varepsilon \text{ for all } n \text{ and all } \varepsilon > 0.$$

[K1995] H. Konig, Isometric imbeddings of Euclidean spaces into finite dimensional lp -spaces, Banach Center Publications (1995) Volume: 34, Issue: 1, page 79-87

if $N >> n$ as in Hilbert's Lemma, then there exist n-subtori $\mathbb{T}^n \subset B^{2N}$, fsuch that

$$curv^\perp\big(\mathbb{T}^n \hookrightarrow B^{2N}\big) = \sqrt{\frac{3n}{n+2}}.$$

Example/Non-Example. Regular pentagons serve as designs of cardinality five and degree four on the circle; these are irrational and there is no apparent *simple* rational design on S^1.

14 Link with the Scalar Curvature via the Gauss Formula

The $curv^\perp$ problem came up in the context of Riemannian geometry of manifolds X with *positive scalar curvatures* [Gro 2017], where

the scalar curvature of an X at $x \in X$, denoted $Sc(X,x)$, is the sum of the values of the sectional curvatures κ at the $n(n-1)$ (ordered) orthonormal bivectors in $T_x(X)$, for $n = dim(X)$.[49]

For instance, scalar curvatures of surfaces are equal to twice their sectional (Gauss) curvatures.

Spheres Example. The n-spheres of radii R in the Euclidean space $\mathbb{R}^{n+1}$ (which have constant sectional curvatures $1/R^2$), satisfy:

$$Sc(S^n(R)) = n(n-1)/R^2 \text{ for all } n.$$

Additivity. It follows from the definition that the scalar curvature is additive under Riemannian products,

$$Sc(X_1 \times \underline{X}) = Sc(X_1) + Sc(\underline{X}).$$

[49] One knows that $Sc(X,x) > 0$ if and only if the volume of the ball $B_x(\varepsilon) \subset X$ is smaller than the volume of the ε-ball in $\mathbb{R}^n$, *provided $\varepsilon > 0$ is sufficiently small*: $\varepsilon \leq \varepsilon(X,x) > 0$. Albeit looking explanatory, this is only an illusion of understanding the geometric meaning of the inequality $Sc(X) > 0$.

For instance, the scalar curvature of the n-th power of the unit 2-sphere is

$$Sc(\underbrace{S^2 \times S^2 \times ... \times S^2}_{n}) = 2n = Sc(S^{2n}(R = \sqrt{2n-1})$$

This also shows that the topology of manifolds with positive scalar curvatures of dimensions $n \geq 4$, can be arbitrary complicated[50] for

$$Sc(X \times S^2(\varepsilon)) \underset{\varepsilon \to 0}{\to} +\infty \text{ for } \textit{all compact} \text{ Riemannian manifolds } X.$$

Yet, there are limits to this complexity: there are compact manifolds of all dimensions, which admit no metrics with $Sc > 0$, called ∄PSC, where the three basic examples are as follows.

Basic ∄PSC Manifolds[51]

14. A. Lichnerowicz Theorem. The (Kummer) surface defined by the equation $z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$ in the complex projective space $\mathbb{C}P^3$ and, more generally orientable spin manifolds with non vanishing $\hat{A}$ genus (dimensions of these are multiples of 4) admit no Riemannian metrics with $Sc > 0$.

Proved in 1963 with the first (1963) *Atiyah–Singer index theorem for the Dirac operator.*

14. B. Hitchin theorem: there exist manifolds Σ homeomorphic (but non-diffeomorphic!) to the spheres S^n for all $n = 8k+1, 8k+2$, $k = 1, 2, 3...$, which admit no metrics with $Sc > 0$.

Proved in 1974 with the second (1971) *Atiyah–Singer index theorem.*

14.C. Geroch Conjecture. n-Tori admit no metrics with $Sc > 0$.

Proposed in 1975, *proved* in 1979 by Schoen-Yau for $n \leq 7$ with via *minimal hipersurfaces* by induction on n and by Gromov-Lawson in 1980 for all n with the *index theorem for the Dirac operators twisted with almost flat bundles.* **14. D. Product Manifolds.** Products of the above manifolds, e.g. of tori by Hitchins spheres are also ∄PSC.

This is proven with the *index theorem* for the (generalized) Dirac operators.

Sectional Curvature Remarks. Although the inequality $Sc > 0$ is much weaker then $sect.curv > 0$ (which is equivalent to geodesic triangles having the sums of the angles $> \pi$) *no alternative proofs* of non-existence of metrics with $sect.curv > 0$ on manifolds from **A** and **B** are available, while the $sect.curv > 0$ (and $Ricci > 0$) version of **C** follows by an elementary argument relying on the geometry of geodesics in X.

(The ancient Bonnet-Myers theorem says that $Ricci(X) \geq \kappa > 0 \implies diam(X) \leq \sqrt{1/\kappa}$, which rules out closed manifolds with infinite universal coverings, such as tori.)

Turning to Constant Sectional Curvature. If one requires the strongest possible condition of this kind, namely the sectional curvature to be constant as well as positive, then everything about X appears 100% transparent.

[50]Three manifolds with $Sc > 0$ are not too simple either : *connected sums lens spaces* and copies of $S^1 \times S^2$ admit metrics with $Sc > 0$ by a theorem by Schoen and Yao.

[51]See [Gro 2021] for a survey on topological properties and examples of ∄PSC Manifolds.

Indeed, one knows. that these metric are *locally spherical*; hence all simply connected n-manifold X with $sect.curv(X) = \kappa > 0$ admit *locally isometric immersions to $S^n(R)$ for $R = \sqrt{1/\kappa}$.*

Consequently,

the universal coverings of closed (compact without boundaries) manifold X with $sect.curv(X) = \kappa$ are isometric to $S^n(R)$. This is the end of the story.

Yet, this may be hard to believe, there are *non-trivial links between geometry and topology* of manifolds X with *constant sectional curvatures* if these X have *non-empty boundaries*, where the available proofs of these properties rely on the scalar curvature inequality $Sc(X) \geq n(n-1)/R^2$ and where one doesn't know how to exploit to full power of the condition $sect.curv = const = 1/R$ (see section 15)

14.1 Gauss Formula and Petrunin's Curvature

Let $X \subset Y$ be a smooth n-dimensional submanifold in a Riemannian N-manifold, e.g. in $Y = \mathbb{R}^N$ and let II= II(X,x)II=II(τ_1, τ_2) be the *second fundamental form* (corresponding to the *shape operator*) of X at $x \in X$, where $\tau_1, \tau_2 \in T_x(X)$ are tangent vectors to X and the form II takes values in the normal space $T_x^\perp(X)$ and where $\mathrm{II}(\tau, \tau)$ is equal to the second derivative of the geodesic in X issuing from x with the velocity τ.

The normal curvature of $X \subset Y$ at $x \in X$, in these terms is

$$curv_x^\perp = \sup_{\|\tau\|=1} \|\mathrm{II}(\tau, \tau)\|.$$

The l_2-norm of II at x is

$$\|\mathrm{II}\|_{\mathrm{l}_2}^2 = \sum_{\mathrm{i}_1, \mathrm{i}_2 = 1, \ldots \mathrm{m}} \|\mathrm{II}(\tau_{\mathrm{i}_1}, \tau_{\mathrm{i}_2})\|^2,$$

where $\{\tau_i\}$, $i = 1, ..., n = dim(X)$, is a frame of orthonormal vectors in the tangent space $T_x(X)$.

We shall need the simple inequality

$$\|\mathrm{II}\|_{l_2}^2 \leq kn \cdot curv^\perp(X)^2,$$

which is useful for $k < n$. One can also show that $\|\mathrm{II}\|_{l_2}^2 \leq n^2 \cdot curv^\perp(X)^2$, for all k but the following inequality. will serve us better.

Petrunin curvature $\Pi = \Pi_x(X \subset Y)$ is the average of

$$\|\mathrm{II}(\tau, \tau)\|^2$$

over the unit vectors $\tau \in S_x^{m-1} \subset T_x(X)$, where clearly,

$$\frac{(curv_x^\perp)^2}{n-1} \leq \Pi_x \leq (curv_x^\perp)^2$$

and where the equality $\frac{(curv_x^\perp))^2}{n-1} = \Pi_x$ holds if the form II has rank one and $\Pi_x = (curv_x^\perp)^2$ if $\|\mathrm{II}\|_{l_2}^2 = \|mean.curv(X, x)\|^2$.

For instance, if $codim(X) = 1$, the latter means that all principal curvatures X at x are mutually equal.

More interestingly [Petr 2023])

$$\sqcap = \tfrac{2}{n(n+2)}\left(||\mathrm{II}||^2_{l_2} + \tfrac{1}{2}\|mean.curv^\perp\|^2\right)$$

or

$$\|mean,curv\|^2 - \|\mathrm{II}\|^2_{l_2} = \frac{3}{2}mean,curv^2 - \frac{n(n+2)}{2}\sqcap,$$

which is proven with the *same* Γ-function formula for the integrals of polynomials of degree four on S^{n-1}, which goes along with spherical designs and used for construction of immersions $\mathbb{T}^n \to \mathbb{R}^N$ with $curv^\perp = \sqrt{3n/(n+2)} + \varepsilon$.

(One wanders if there is a geometric reason for this, e.g. a "Riemannian curvature averaging formula" of some kind.)

For instance, if $n = dim(X) = 2$, $N = dim(Y) = 3$ and α_1 and α_2 dnote the principal curvatures of X at x, then

$$curv^\perp(X,x) = \max(|\alpha_1|, |\alpha_2|),$$

$$||\mathrm{II}||^2_{l_2} = \alpha_1^2 + \alpha_2^2,$$

$$\|mean.curv^\perp\| = |\alpha_1 + \alpha_2|$$

and

$$\sqcap = \frac{1}{4}(\alpha_1^2 + \alpha_2^2) + \frac{1}{8}(\alpha_1 + \alpha_2)^2 = \frac{3}{8}(\alpha_1^2 + \alpha_2^2) + \frac{1}{4}\alpha_1\alpha_2;$$

if $X = S^2 \subset Y = \mathbb{R}^3$, where $\alpha_1 = \alpha_2 = 1$, this makes $\sqcap = 1$ as well.

Gauss Formula. Let Y have constant sectional curvature κ and let $Sc_{|n} = Sc_{|n}(Y) = nk(k-1)$. Then the scalar curvature of X satisfies:

$$Sc(X,x) = Sc_{|n} + ||mean.curv^\perp(X,x)||^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2},$$

where by Petrunin's formula

$$Sc(X,x) = Sc_{|n} + \frac{3}{2}\|mean.curv(X,x)\|^2 - \frac{n(n+2)}{2}\cdot\sqcap,$$

Hence, the inequality $Sc_{|m}(Y) \geq \sigma_n$ implies that

$$Sc(X) \geq \sigma_n - ||\ \mathrm{II}(X,x)||^2.$$

Therefore

$$[kn] \qquad Sc(X) \geq \sigma_n - kn\cdot curv^\perp(X)^2$$

for $k \leq n$ and

$$Sc(X) \geq \sigma_n - n^2 curv^\perp(X)^2.$$

for all k, where Petrunin's formula yields better, in fact optimal, inequality for $k >> n$

$$Sc(X) \geq \sigma_n - \frac{n(n+2)}{2}\sqcap \geq \sigma_n - \frac{n(n+2)}{2}curv^\perp(X)^2.$$

It follows that *if the manifold X is ∄PSC, i.e. it admits no metric with $Sc > 0$, then*

$$curv^\perp(X) \geq \sqrt{\sqcap} \geq \sqrt{\frac{2\sigma_n}{n(n+2)}}\ \textit{for all } k \textit{ and } N = n + k = dim(Y),\ Y \hookleftarrow X,$$

and

$$curv^\perp(X) \geq \sqrt{\frac{\sigma_n}{kn}} \text{ for } k < n/2.$$

Examples and Corollaries.

Let X be an n-dimensional $\nexists$PSC manifold, e.g. the n-torus $\mathbb{T}^n$, Hitchin's exotic n-sphere Σ^n or a product $\Sigma^m \times \mathbb{T}^{n-m}$.

$(*_{S^{n+k}})$ Then immersions from X to the unit sphere satisfy

[A] $$curv^\perp\big(X \hookrightarrow S^{n+k}(1)\big) \geq \sqrt{\frac{n-1}{k}}$$

and

[B] $$curv^\perp\big(X \hookrightarrow S^{n+k}(1)\big) \geq \sqrt{\sqcap} \geq \sqrt{\frac{2n-2}{n+2}}.$$

Inequality **[A]** is better than **[B]** roughly for $k \leq n/2$, while Petrunin's **[B]** takes over for larger N, where it is, as we known (see sections 2 and 14) , optimal for $k >> n^2$.

14.2 Petrunin's $\sqrt{3}$ Extremality Theorem

The above doesn't directly apply to immersions to the Euclidean balls, since these have $Sc_{|n} = 0$, where the Gauss and Petrunin formulas for the induced metric g, reduce to

[a] $$Sc(g) = ||mean.curv^\perp||^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2}$$

and

[b] $$Sc(g) = \frac{3}{2}\|mean.curv\|^2 - \frac{n(n+2)}{2}\sqcap.$$

Yet, inequality **[A],** applied to the image of $X \hookrightarrow B^N(1)$ in S^N under the radial projection of of the unit ball in tangent hyperplane $B^N \subset \mathbb{R}^N = T_s(S^N) \subset \mathbb{R}^{n+1} \supset S^N$ to S^N shows that

$[\frac{\mathbf{1}}{\mathbf{8}-\varepsilon}]$ $$curv^\perp\big(X \hookrightarrow B^{n+k}(1)\big) \geq \sqrt{\frac{n-1}{(8-\varepsilon_{n,k})k}}.$$

for some (moderately small) $\varepsilon_{n,k} > 0$.

This is crude, but in the $\sqcap$-case Petrunin proves the sharp $curv^\perp$-inequality

[B⋆] $$curv^\perp\big(X \hookrightarrow B^N(1)\big) \geq \sqrt{\frac{3n}{n+2}}.$$

for all n-dimensional $\nexists$PSC manifolds X, all n and N.

This is done by showing that *if*

$$curv^\perp\big(X \overset{f}{\hookrightarrow} B^N(1)\big) < \sqrt{\frac{3n}{n+2}},$$

then a conformal change of the induced metric g on X has positive scalar curvature. Namely, if $n \geq 3$[52]*, then*

$$Sc(u^{\frac{4}{n-2}}g) > 0 \text{ for } u(x) = \exp -l\tfrac{1}{2}\|f(x)\|^2 \text{ and } l = \tfrac{3}{4}\cdot\tfrac{n-2}{n-1}\cdot n.$$

Remark. One might think, that Petrunin's argument with the Gauss formula

$$Sc(g) = \|mean.curv\|^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2} \geq \|\mathrm{mean.curv}\|^2 - \mathrm{k}(\mathrm{curv}^\perp)^2$$

rather than Petrunin's

$$Sc(g) = \frac{3}{2}\|mean.curv\|^2 - \frac{n(n+2)}{2}\Pi \geq \frac{3}{2}\|mean.curv\|^2 - \frac{n(n+2)}{2}(curv^\perp)^2$$

would improve the above inequality $[\frac{1}{8-\varepsilon}]$.

In fact, if one uses Petrunin's formula for the Laplace operator $\Delta = \Delta_g$ applied to the above function $u(x)$ on X:

$$-\frac{\Delta u}{u} = lrc\cdot|H| + (ln - l^2r^2s^2),$$

where $H = mean.curv\big(X \overset{f}{\hookrightarrow} B^{n+k}(1)\big)$, $r = r(x) = \|f(x)\|$, and $c = c(x)$, $s = s(x)$ are function (cos and sin of certain angles), which are bounded in the absolute values by one, $|c|,|s| \leq 1$, one arrives at the following version of $[\frac{1}{8-\varepsilon}]$. :

$$curv^\perp\big(X \hookrightarrow B^{n+k}(1)\big) \geq \sqrt{\frac{n}{k(8+(4/(n-2)))}}$$

This is no better $[\frac{1}{8-\varepsilon}]$. but can be slightly improved with the inequalities $c^2+s^2 \leq 1$ and $r^2+s^2 \leq 1$ proved in [Petr 2023r under the assumption $curv^\perp \leq 2$.

14.3 Lower Bounds on $curv^\perp(X \hookrightarrow Y)$ for Manifolds Y with $Sc_n \geq \sigma_n$.

Let us define the n-dimensional scalar curvature $Sc_n(Y)$ for general Riemannian manifolds Y of dimension $N \geq n$, that is a function on the tangent n-planes $T^n_y \subset T(Y)$ in Y, which is eual to the sum of the sectional curvatures κ of Y on the bivectors in T^n_y at y.

Equivalently, $Sc_n(Y,T_y)$ is the scalar curvature of the submanifold $\exp(T_y) \subset Y$ at y, that is is the germ of the image of the exponential map from T_y to Y.

Then the Gauss' and Petrunin's formulas for the scalar curvature of $X \hookrightarrow Y$ remains as they were for manifolds Y with constant sectionl curvatres

$$Sc(X,x) = Sc_{|m}(Y,T_x(X)) + ||mean.curv^\perp(X,x)||^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2},$$

and

$$||mean.curv^\perp(X,x)||^2 - ||\ \mathrm{II}(X,x)||^2_{l_2} = ||\tfrac{3}{2}mean.curv^\perp(X,x)||^2 - \tfrac{n(n+2)}{2}\Pi.$$

---

[52]If $n = 2$ then the average value of Π is $\geq \sqrt{\frac{3}{2}}$, see [Petr 2023]

Thus, the above inequalities **[A]** and **[B]** concerning immersions of n-manifolds X to the unit sphere S^{n+k} generalize to immersions to $(n+k)$-dimensional manifolds Y, such that $Sc_n(Y) \geq n(n-1)$:

$[\mathbf{A_Y}]$
$$curv^{\perp}(X \hookrightarrow Y) \geq \sqrt{\frac{n-1}{k}}$$

and

$[\mathbf{B_Y}]$
$$curv^{\perp}(X \hookrightarrow Y) \geq \sqrt{\Pi} \geq \sqrt{\frac{2n-2}{n+2}}.$$

Example. Let $Y = S^{n+k_0}(R)(1) \times H^l_{-1}$, where the sphere $S^{n+k_0}(R)$ has constant curvature $+1/\rho^2$ and H^l_{-1} is the hyperbolic space with the sectioanal curvature -1 and let $n \geq l+2$. Then

$$Sc_n(Y) \geq \frac{1}{\rho^2}(n-l)(n-l-1) - l(l-1)$$

and the two above inequalities hold with $k = k_0 + l$, if

$$\rho^2 \leq \frac{(n-l)(n-l-1)}{n(n-1)+l(l-1)}.$$

For instance, if $l = 2$, and $n \geq 4$ one needs $\rho^2 \leq \frac{1}{7}$. for this purpose.

Notice in conclusion, that neither
the above inequalities $[\frac{1}{8-\varepsilon}]$ and Petrunin's $[\mathbf{B}\star]$ for immersion to unit balls nor such inequalities from the previous sections based on the $\frac{2p}{n}$inequalities admit (not at lest obvious) counterparts for these Y.

15 Second Link with the scalar Curvature: Width Inequalities for Riemannian Bands

15.A. Example: Torical $\frac{2\pi}{n}$-Inequality. [53] Let V be a Riemannian manifold homeomeorphic to the product of the n-torus by the unit interval $V = \mathbb{T}^n \times [-1,+1]$, such that $Sc(V) \geq \sigma > 0$. Then the distance between the two components of the boundary of V is bounded as follows:

$$dist(\mathbb{T}^n \times \{-1\}, \mathbb{T}^n \times \{+1\}) \leq 2\pi\sqrt{\frac{n}{\sigma(n+1)}}.$$

(See section 16.1 for a few words about the proof.)

15.B. Corollary: No Wide Torical Bands in the Spheres. *If a Riemannian $(n+1)$-manifold V homeomorphic to $\mathbb{T}^n \times [-1,+1]$ admits a locally isometric immersion to the $(n+1)$-sphere of radius R then*

$$dist(\mathbb{T}^n \times \{-1\}, \mathbb{T}^n \times \{+1\}) \leq \frac{2\pi R}{n+1}.$$

[53]See [Gro 2021] for an account on known results in geometry of manifolds with $Sc \geq 0$, which are formulated in sections 16.1, 16.2, 16.3 without further references.

15.C. Large Normal Curvature Sub-corollary. *Let*

$$f : \mathbb{T}^n \hookrightarrow B^{n+1}(1)$$

be a smooth immersion from the n-torus to the unit Euclidean $(n+1)$-ball $B^{n+1} \subset \mathbb{R}^{n+1}$. Then the curvature of f is bounded from below by:

$$curv^{\perp}\left(\mathbb{T}^n \overset{f}{\hookrightarrow} B^{n+1}(1)\right) \geq \frac{n+1}{\pi} - 1.$$

Proof of **15.B** $\implies$ **15.C.** Let

$$E_f : \mathbb{T}^N \times \mathbb{R}^1 \to \mathbb{R}^{n+1} \supset B^{n+1}(1)$$

be the *normal exponential map*, i.e. such that the restriction $E_f | \mathbb{T}^N \times \{0\} = f$ and where E_f isometrically sends the lines $\{t\} \times \mathbb{R}^1$, $t \in \mathbb{T}^n$, to the straight lines in $\mathbb{R}^{n+1}$ normal to the immersed torus $f(\mathbb{T}^n) \subset \mathbb{R}^{n+1}$ at the points $f(t) \in f(\mathbb{T}^n)$.

If $curv(f) < c$, then, (this is the same as it is for circles of radii $1/c$ in the plane) the map E_f is an *immersion* on $\mathbb{T}^N \times [-r, r] \subset T^N \times \mathbb{R}^1$ for $r = 1/c$, while the image of $f(\mathbb{T}^n)$ is contained in the ball $B^{n+1}(1 + r)$.

Let

$$\mathbb{R}^{n+2} \subset S_+^{n+1}(1 + r) \overset{p}{\to} \mathbb{R}^{n+1} \supset B^{n+1}(1 + r)$$

be the normal projection from the hemisphere, compose E_f on $\mathbb{T}^N \times [-r, r]$ with the inverse map to p and let

$$\tilde{E} : p^{-1} \circ E_f : \mathbb{T}^N \times [-r, r] \to S_+^{n+1}(1 + r).$$

Since the projection p is *distance decreasing*, the spherical distance between the two components of the boundary of $T^N \times [-r, r]$ with respect to the Riemannian metric $\tilde{g}$ in $\mathbb{T}^N \times [-r, r]$ induced by $\tilde{E}$ from the spherical metric in $S_+^{n+1}(1 + r)$ V is bounded from below by $2r$. Then **D** applied to

$$(\mathbb{T}^N \times [-r, r], \tilde{g}) \overset{\tilde{E}}{\to} S_+^{n+1}(1 + r) \subset S^{n+1}(1 + r)$$

shows that

$$\tilde{d} = dist_{\tilde{g}}(\mathbb{T}^n \times \{-r\}, \mathbb{T}^n \times \{+r\}) \leq \frac{2\pi(1 + r)}{n + 1}$$

and since $\tilde{d} > 2r = 2/c$ the inequality $c \geq \frac{n+1}{\pi} - 1$ follows. QED.

Exercise. Generalise the large normal curvature sub-corollary to immersions of tori to products of balls:

$$curv^{\perp}\left(\mathbb{T}^{n+k} \overset{f}{\hookrightarrow} B^{n+1}(1) \times B^k(R)\right) \geq \frac{n+1}{\pi} - 1.$$

for all $k = 0, 1, 2, ...$ and all $R \geq 0$.

On Low Dimensions. The inequality $curv^{\perp}\left((\mathbb{T}^n \hookrightarrow B^{n+1}(1)\right) \geq \frac{n+1}{\pi} - 1$ may be asymptotically optimal for $n \to \infty$ but its performance for small n is poor.

For instance, if $n \leq 5$ then $\frac{n+1}{\pi} - 1 < 1$ and our inequality is weaker than $curv^{\perp}(X^n \hookrightarrow B^{n+k}(1) \geq 1$, which follows for all closed n-manifolds X and all n, k by the obvious "maximal principle" argument.

Furthermore, since

$$curv^{\perp}\big((X^n \hookrightarrow B^{n+1}(1)\big) > 2$$

for all *non-spherical* X (this is elementary, see section ...), our $\big(\geq \frac{n+1}{\pi} - 1\big)$-bound is of any interest only for $n \geq 9$.

$\mathbb{T}^{\rtimes}$-*Remark.* In section 15.2 we introduce the notion of $\mathbb{T}^{\rtimes}$-*stabilized scalar curvature,* $Sc^{\rtimes}(X)$, improve the inequalities **E** and **F** and will see, for example, that

$$curv^{\perp}\big(\mathbb{T}^n \hookrightarrow B^{n+1}(1)\big) > 2.5 \text{ for } n \geq 7.$$

Codimension two Remark. The inequality **E** applied to the unit tangent bundles of immersed n-tori with codimensions 2,[54] shows (see $[1+2c]$-Example in section 3)

$$curv^{\perp}\big(\mathbb{T}^{n+1} \hookrightarrow B^{n+2}(1)\big) \leq 1 + 2curv^{\perp}\big(\mathbb{T}^n \hookrightarrow B^{n+2}\big)$$

and

$$curv^{\perp}\big(\mathbb{T}^n \hookrightarrow B^{n+2}\big) \geq \frac{1}{2}curv^{\perp}\big(\mathbb{T}^{n+1} \hookrightarrow B^{n+2}(1)\big) - \frac{1}{2} \geq \frac{n+2}{2\pi} - 1.$$

This has any merit only for $n \geq 11$, where $\frac{n+2}{2\pi} - 1 > 1$, and it becomes better than Petrunin's inequality only for $n \geq 15$, where $\frac{n+2}{2\pi} - 1 > \sqrt{\frac{3n}{n+2}}$.

(The improvement with the $\mathbb{T}^{\rtimes}$-remark doesn't significantly change the picture.)

Conjectures. (a) Immersed n-tori in the unit $(n+k)$-ball satisfy

$$curv^{\perp}\big(\mathbb{T}^n \hookrightarrow B^{n+k}(1)\big) \geq \frac{n}{k}.$$

(b) All immersions of all n-manifolds $X \overset{f_0}{\hookrightarrow} B^N(1)$ a regularly homotopic to immersions $f_1 : X \to\hookrightarrow B^N(1)$ where $curv^{\perp}(f_1 : X) \leq Cn$ for someuniversal constant C, (probably, $C \leq 100$).)

These , by no means (not even conjecturally) optimal, inequalities are motivated only by their simple forms.

15.D. Immersions with curvatures $\sim n^{\alpha}$. It not impossible (but unlikely) that all immersion of n-tori to unit balls satisfy

$$curv^{\perp}\big(\mathbb{T}^n \hookrightarrow B^{n+k}(1)\big) \geq \frac{cn^{\alpha}}{k}$$

for some small $c > 0$, $\alpha > 1$, e.g. $c = 0.001$ and $\alpha = \frac{3}{2}$, where the exponent $\alpha = \frac{3}{2}$ is maximal possible.

Indeed, n-tori embed to $B^{n+n}(1)$ with curvatures $n^{\frac{1}{2}}$ and also there exit codimension one embedding of n-tori with curvatures about $n^{\frac{3}{2}}$,

$$curv^{\perp}\big(\mathbb{T}^n \subset B^{n+1}(1)\big) < 6n^{\frac{3}{2}}.$$

[54]If the Euler class of such an immersion is non-zero one needs a mild generalisation of **E**.

In fact, arguing as in section 4.A one construct $X_m = S^{n_1} \times ... \times S^{n_m} \subset B^{n_1+...n_m+1}(1)$ by induction on m as boundaries of ρ_m-neighbourhoods of

$$X_{m-1} = S^{n_1} \times ... \times S^{n_{m-1}} \subset B^{n_1+...n_{m-1}+1}(1-\rho_m) \subset B^{n_1+...n_m+1}(1),$$

where the curvatures of these embeddings grow *exponentially* with m, roughly as 2^{m-1}.

Thus one embeds X_m to the ball $B^{n_1+...n_m+1}(1)$ with the curvature growing *polynomially* in $n = dim(X_m)$ (rather than in m):

$$curv^{\perp}\big(X_m \subset B^{n+1}(1)\big) \leq const_{\mu} n^{\frac{\mu+2}{\mu+1}}, \; n = dim(X_m) = n_1 + ...n_m, \; \mu = \min_i n_i.$$

For all we know, if all n_i are equal to a single n_o, then all immersions of $(S^{n_o})^m$ immersions to the unit (mn_o+1)-ball satisfy

$$curv^{\perp}\big((S^{n_o})^m \hookrightarrow B^{mn_o+1}(1)\big) \leq const_{n_o}(mn_o)^{\frac{\mu+2}{\mu+1}}.$$

15.1 On Three Proofs of $\frac{2\pi}{n}$-Inequalities

All three proofs apply to manifolds V, where their boundaries are decomposed into two disjoint parts $\partial V = \partial_- \sqcup \partial_+$, and show that

$$dist(\partial_-, \partial_+) > 2\pi\sqrt{\frac{n}{\sigma(n+1)}} \text{ for } \sigma = \inf_{x\in X} Sc(X,x).$$

under certain topological assumptions on V specific to each proof.

1. The first proof applies to suitably *enlargeable* manifolds[55] V, e.g. to $V = X \times [-1,1]$, where X admits a metric with $sect.curv \leq 0$.

This proceeds by induction on n with minimal hypersurfaces with boundaries as in §12 from GL, where the original Schoen-Yau argument was augmented with Fischer-Colbrie&Schoen warped product symmetrization idea.

If $dim(V) > 7$, the proof encounters a technical difficulty where minimal hypersurfaces may have singularities, but this can be resolved modulo the partial regularity theorem 4.6 from [SY 2017].

2. The second proof whenever applies, delivers a hypersurface (μ-*bubble*) $X \subset V$ which separates ∂_-. from ∂_+ and which admits a metric with positive scalar curvature. This shows, in particular that in the following three cases,

V can't be diffeomorphic to $X \times [-1,1]$, where X admits no metric with $Sc > 0$,

(i) X is a *spin manifold* , e.g. as in the above **A** and **B**.

(ii) X is SYS as in [SY 1979] or a manifold as in [GH 2024].

(iii) X is an aspherical manifold of dimension| ≤ 5 or a closely related manifold (see [Cho-Li 2020], [Gro 2021])

(These (i), (ii) and (iii) cover all *known* classes of manifolds, except for dimension 4, which admit no metrics with $Sc > 0$.)

This second proof also encounter the singularity problem for $dim(V) > 7$, where it is more serious than in the first proof, since the Schoen-Yau partial regularity theorem is not sufficient in this case.

[55]See [GL 1973], [Gro 2021] and references therein.

However if $dim(V) = 8$ then a required desingularisation follows by a version of Nathan Smale argument and if $n = 9, 10$, then the desingularisation from [Cho-Ma-Sch 2023] most probably apply in the present case.

3. The third proof, which relies on the generalized Callias-Dirac operators technique (see Cecc-Zeid 2023], [Guo-Xie-Yu 2022]), needs V to be a spin manifold.

This proof applies, in particular, to V diffeomorphic to $X \times [-1, 1]$, where X admits no metric with $Sc > 0$, ad where non-existence of such a metric follows via the index theorem for a generalized Dirac operator, as for instance, for X from the above **A** and **B**. 1mm

As far as the curvature of immersion is concerned, this is most useful for the Hitchin's spheres Σ^n for $n = 8l + 1, 8l + 2$ and which admit immersions to $\mathbb{R}^{n+1}$ by Hirsch theorem[56] and all immersions Σ^n to the unit $(n+1)$ ball satisfy the same inequality as tori

$$curv^{\perp}\big(\Sigma^n \hookrightarrow B^{n+1}(1)\big) \geq \frac{n+1}{\pi} - 1$$

and, by a similar argument,

$$curv^{\perp}\big(\Sigma^n \hookrightarrow B^{n+2}(1)\big) \geq \frac{n+2}{\pi} - 2.$$

These inequalities can be improved for small n the same way as in the above (b) for tori, but unlike conjecture **G** for tori, there is *no (known) reason to expect* that immersions of Σ^n to the unit balls B^{n+k}, $k \geq 3$, satisfy $curv^{\perp} \geq const_k n$.)

Question. Do *all* Milnor's spheres Σ^n, including those, which carry metrics with $Sc > 0$, develop large normal curvatures when immersed to the balls $B^{n+1}(1)$?

15.2 $\mathbb{T}^{\rtimes}$-Stabilized Scalar Curvature.

Given a compact Riemannian manifold X, let

$$Sc^{\rtimes}(X) = 4\lambda_1^{\rtimes}(X),$$

where $\lambda_1^{\rtimes}(X)$ is the lowest eigenvalue of the operator $-\Delta + \frac{1}{4}Sc$ on X with the Dirichlet (vanishing on the boundary) condition.[57]

It is easy to see that $Sc^{\rtimes}$ is additive for Riemannian products

$$Sc^{\rtimes}(X_1 \times \underline{X}) = Sc^{\rtimes}(X) + Sc^{\rtimes}(\underline{X}).$$

and, more relevantly,

$Sc^{\rtimes}(X)$ is *decreasing* under equidimensional locally isometric immersions:

if X immerses to Y then $Sc^{\rtimes}(X) \geq Sc^{\rtimes}(X)$.

About $-\Delta + \beta \cdot Sc$. The two above relations remain valid for the first eigenvalues of the operators

$$f(x) \mapsto -\Delta f(x) + \beta \cdot Sc(X, x) \cdot f(x)$$

[56]Lichnerowicz's manifolds, which have non-zero $\hat{A}$-genus admit no Euclidean immersions with codimenension one and two.

[57]See [Gr 2024] for justification of this definition/notation and for the proofs of the properties of this $Sc^{\rtimes}$-curvature used in this paper.

for all $\beta \geq 0$, but $\beta = 1/4$ is essential for the $\frac{2\pi}{\sqrt{Sc^{\rtimes}}}$-inequality below.

Besides 1/4, a significant value is $\beta = \frac{1}{4}\frac{n-2}{n-1}$, where positivity of the operator $-\Delta_X + \beta \cdot \frac{1}{4}\frac{n-2}{n-1}Sc(X)$ for $n \geq 3$ on X implies that X admits a metric with positive scalar curvature (as in the proof of the Petrunin's inequality in section 14.2).

Since $\frac{1}{4}\frac{n-2}{n-1} < \frac{1}{4}$ the inequality $Sc^{\rtimes} > 0$ also implies the existence of a metric with positive scalar curvature on X.

This shows that the conditions *∄PSC and ∄PSC*$^{\rtimes}$ *are equivalent.*

But unlike how it is with the effects of the positive signs of $Sc(X)$ and of $Sc^{\rtimes}(X)$ on the *topology* of X, the $Sc(X)$ and $S^{\rtimes}(X)$ plays different roles in the geometry of X.

Let V be a Riemannian manifold homeomorphic to the product $X \times [-1,+1]$, where X is a *basic ∄PSC n-manifold*, i.e. where the underlying reason for non-existence of a metric with $Sc > 0$ on X is *of the same kind as what is presented in section 16.1.*[58] For instance X is diffeomorphic to the product of the *torus by Hitchin's sphere.*

$\frac{2\pi}{\sqrt{Sc^{\rtimes}}}$-Inequality.[59] Let V be a Riemannian manifold homeomorphic to the product $X \times [-1,+1]$, where X is a *basic ∄PSC n-manifold*, i.e. where the underlying reason for non-existence of a metric with $Sc > 0$ on X is *of the same kind as what is presented in section 16.1.*[60] For instance X is diffeomorphic to the product of the *torus by Hitchin's sphere.*

Then the distance between the two boundary components of V is *bounded as follows:*

$$dist(X \times \{-1\}, X \times \{+1\}) \leq 2\pi\sqrt{\frac{n}{Sc^{\rtimes}(V)(n+1)}}.$$

Examples of Evaluation of $Sc^{\rtimes}$. The rectangular solids satisfy

$$Sc^{\rtimes}\left(\bigtimes_1^n [-a_i, b_i]\right) = 4\sum_1^n \lambda_1[a_i, b_i] = \sum_1^n \frac{4\pi^2}{(b_i - a_i)^2},$$

the unit hemispheres satisfy:

$$Sc^{\rtimes}(S^n_+) = n(n-1) + 4n = n(n+3),$$

the unit balls satisfy

$$Sc^{\rtimes}(B^n) = 4j_\nu^2,$$

for the first zero of the Bessel function J_ν, $\nu = \frac{n}{2} - 1$, where $j_{-1/2} = \frac{\pi}{2}$, $j_0 = 2.4042...$, $j_{1/2} = \pi$ and if $\nu > 1/2$, then

$$\nu + \frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}} < j_\nu < \nu + \frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}} + \frac{3}{20}\frac{2^{\frac{2}{3}}a^2}{\nu^{\frac{1}{2}}}$$

where $a = \left(\frac{9\pi}{8}\right)^{\frac{2}{3}}(1+\varepsilon) \approx 2.32$ with $\varepsilon < 0.13\left(\frac{8}{2.847\pi}\right)^2 < 0.1$.

[58] Conjecturally, all ∄PSC manifolds will do, at least for $n \neq 4$

[59] See [Gro 2024] for more about it.

[60] Conjecturally, all ∄PSC manifolds will do, at least for $n \neq 4$

Corollary. Let X be a basic $\nexists$PSC$^{\rtimes}$ manifold of dimension $n-1$, e.g. $X = \mathbb{T}^{n-1}$, and $f : X \to B^n(r)$ be a smooth immersion.

Then the focal radii of immersions $X \hookrightarrow B^n(r)$ satisfy:

$$[foc.rad]_{j_\nu} \qquad rad^{\perp}\big(X \hookrightarrow B^n(r)\big) \leq \frac{\pi r}{2j_\nu}\sqrt{\frac{n}{n+1}}$$

and

$$[curv^{\perp}]_{j_\nu} \qquad curv^{\perp}\big(X \hookrightarrow B^n(r)\big) \geq \left(\frac{2j_\nu}{\pi r}\sqrt{\frac{n+1}{n}}\right) - r$$

where

$$\frac{2j_\nu}{\pi r} \geq \frac{n - 1/2 + 3.68(n/2-1)^{1/3}}{\pi r}$$

This implies, in particular, the *low curvature bounds from the* $\mathbb{T}^{\rtimes}$*-remark in section 15.*

Also this can be used along with the following.

Mean Curvature/Ricci $4j_\nu^2$–Inequality. Let Y be a compact connected Riemannian n-manifold with a non-empty boundary, such that *the Ricci curvature of Y is nonnegative*, e.g. Y is a bounded Euclidean domain, and *the mean curvature of the boundary of W is bounded from below* by that of the unit ball,

$$mean.curv(\partial Y) \geq n - 1 = mean.curv(\partial B^n).$$

Then

$$Sc^{\rtimes}(Y) \geq Sc^{\rtimes}(B^n) = 4j_\nu^2.$$

Thus, the above inequalities

$[foc.rad]_{j_\nu}$ and $[curv \perp]_{j_\nu}$ remain valid for immersions $X \hookrightarrow Y_r$ for *all compact connected Riemannian n-manifolds Y_r with non-empty boundaries, such that $Ricci(Y_r) \geq 0$ and $mean.curv(\partial Y_t) \geq \frac{n-1}{r}$.*

Remark/Question. Let $V \subset \mathbb{R}^n$ be a bounded domain with two boundary components, let $d(V)$ be the distance between these componetns and let $\lambda_1(V)$ the first eigenvalue of the Dirchlet problem in V.

The above shows that

topology of V may impose a non-trivial bound on the product $d^2(V)\lambda_1(V)$.

What are other cases of a similar role of the topology of a $V \subset \mathbb{R}^n$ on metric invariants of V?

15.3 Curvatures of Regular Homotopies of Immersions

Due to the Atiyah-Singer index theorem for families of Dirac operators, the index theoretic obstructions to $Sc > 0$ apply to families of metrics with $Sc > 0$, which imply the following (see [Hit 1974])

16.3.A. The spheres S^{n-1}, $n = 8k+1, 8k+2$, $k = 1, 2, \ldots$. These admit (Smale/Milnor)

$$\text{diffeomorphisms } \mu : S^{n-1} \to S^{n-1},$$

such that the usual spherical metric g_o ($sect.curv(g_o) = 1$) and the induced metric $g_o^* = \mu^*(g_o)$ (also $sect.curv(g_o^*) = 1$) *can't be joined by a C^2-continuous homotopy g_t, such that $Sc(g_t) > 0$.*

(The diffeomorphism μ establishes an *isometry* of (S^{n-1}, g_o^*) with the usual sphere (S^{n-1}, g_o), where Milnor's theorem doesn't allow a homotopy g_t between g_o and g_o^*, such that the metrics g_t have *constant sectional* curvatures.)

16.3.B. $O(\sqrt{n})$**-Curvature Corollary.** Let $f_o : S^{n-1} \to S^n(1)$, be the standard equatorial embedding of the sphere and let $f_t : S^{n-1} \to S^n(1)$, $t \in [0,1]$, be a C^2-continuous regular homotopy, (a family of C^2-immersions[61]) between f_o and $f_o^* = f_o \circ \mu : S^{n-1} \to S^n(1)$. Then *there exists* $t_0 \in [0,1]$, such that the normal curvature of the immersion f_{t_0} satisfies :

$$curv^\perp\big(S^{n-1} \overset{f_{t_0}}{\hookrightarrow} S^n(1)\big) \geq \sqrt{n-2}.$$

Indeed, if $curv^\perp\big(S^{n-1} \overset{f_{t_0}}{\hookrightarrow} S^n(1)\big) < \sqrt{n-2}$. for all t then, by Petrunin's Gauss formula from section14.1, the f_t-induced metrics g_t on S^{n-1} would have $Sc > 0$ in contradiction with 16.3.A.

16.3.C. $O(n)$**-Curvature Conjectural Corollary.** Let $f_o : S^{n-1} \to B^n(1) \subset \mathbb{R}^n$ be the standard embedding of the sphere and let $f_t : S^{n-1} \to B^n(1)$, $t \in [0,1]$, be a C^2-continuous regular homotopy, (a family of C^2-immersions[62]) between f_o and $f_o^* = f_o \circ \mu : S^{n-1} \to B^n(1)$. Then *there exists* $t_0 \in [0,1]$, such that the normal curvature of the immersion f_{t_0} satisfies :

$$curv^\perp\big(S^{n-1} \overset{f_{t_0}}{\hookrightarrow} B^n(1)\big) \geq j_\nu/\pi > \frac{n+1}{\pi} - 1.$$

To show this one needs an index theorem for families of Callias operators on Riemannian bands.

Milnor's diffeomorphsms seem very different in this regard from the following;

16.3.D. Let $S^n \subset_- B^{n+1}(1)$ be the embedding obtained from the standard one $S^n \subset B^{n+1}(1)$ by an orientation reversing transformation from $O(n+1)$.

If $n = 2, 6 \mod 8$, then the two can be joined by a regular homotopy of immersions $S^n \hookrightarrow B^{+1}(1)$. ("Turning a sphere inside out" .)

Conjecture A regular homotopy between these two embeddings can be achieved for all $n = 2, 6 \mod 8$ with immersions f_t, where $curv^\perp(f_t(S^n)) \leq C$, where C is a universal constant (probably, $C \leq 100$).

16.3.E. Higher Homotopy Remark. There is a body of results on higher homotopy groups of the space $\mathcal{G}_{Sc>0}(S^n)$ of metrics g with $Sc(g) > 0$ on S^n, but it is unclear what to do with (the homotopy structure of) the map from the space of immersions $S^n \to B^{n+k}(1)$ (and/or $S^n \to S^{n+k}(1)$) with sufficiently small curvatures to $\mathcal{G}_{Sc>0}(S^n)$.

Not only Hitchin's spheres but all $\nexists$PSC manifolds X of dimension $n \geq 5$ contain hypersurfaces $H \subset X$, which support pairs of Riemannian metrics g_0 and g_1, such that $Sc(g_i) > 0$, $i = 0, 1$, and where these metrics *can't be joined by a C^2-continuous homotopies g_t, such that $Sc(g_t) > 0$, $0 \leq t \leq 1$.*

[61] Such a family does exist by the Smale immersion theorem.
[62] Such a family does exist by the Smale immersion theorem.

To see that, let $\psi : X \to \mathbb{R}$ be a Morse function and let $Z = \psi^{-1}(r_0) \subset X$, for some $r_0 \in \mathbb{R}$ be a level of ψ, such that all critical point $x \in X$ of ψ with indices $\leq m$ lie below Z, i.e. $\psi(x)(x) < r_0$.

Then Z serves as the common boundary of the regions $X_0 \subset X$ and $X_1 \subset X$, where

$$X_0 = \{x \in X\}_{\psi(x) \leq r_0} \text{ and } X_1 = \{x \in X\}_{psi(x) \geq r_0}.$$

Since X_0 represents a regular neighbourhood of a (ψ-cellular) m-skeleton of X the manifold X_0 carries a natural Riemannian metric g_0 with $Sc(g_0) > 0$, provided $n-m \geq 3$ and since X_1 represents a regular neighbourhood of a $n-m-1$-skeleton of X there is another "natural"metric g_1 on Z with $Sc(g_1) > 0$ for $m \leq 2$.

Also on knows that if g_0 and g_1 lie in the same connected component of $\mathcal{G}_{Sc>0}(Z)$, then X admits a metric with $Sc > 0$.

Similarly, if g_0 and g_1 lie in the same connected component of $\mathcal{G}_{Sc^{\rtimes}>0}(Z)$, then X admits a metric with $Sc^{\rtimes} > 0$.

16.3.F. Higher Homotopy Problem. Is there a development of this construction in the spirit of *3.5.D. Higher Homotopy Remark*, e.g. something about the fundamental group of the space $\mathcal{G}_{Sc^{\rtimes}>0}(Z')$ for some hypersurface $Z' \subset Z$?

16.3.G. Toral Example/Question. Let $X = \mathbb{T}^n$ and $2 \leq m \leq n-3$. Then, one can show that Z admits an immersion $f_0 : Z \to B^n(1)$ with

$$curv^{\perp}\big(Z \overset{f_0}{\hookrightarrow} S^n(1)\big) \leq c_m.$$

It follows that if $n >> m$, then the induced metric g_{f_0}. on Z has $Sc > 0$; moreover, one can find an f_0 such that g_{f_0} is homotopic to g_0 in $\mathcal{G}_{Sc>0}(Z)$.

When does Z also admits a similar immersion f_1 to S^n with a sufficiently small curvature and a homotopy between g_{f_1} and g_1?

When do manifolds like Z admit pairs of regularly homotopic immersion $f_0, f_1 : Z \hookrightarrow B^n(1)$ with curvatures $\leq c$, yet not regularly homotopic by immersions with curvatures $\leq C$ for some costants c and $C >> c$?

16 Overtwisted Immersions.

Riemannian

Let $Y = (Y, g = g_Y)$ be a Riemannian manifolds, such as a bounded Euclidean domain, e.g. the unit ball $B^N(1) \subset \mathbb{R}^N$.

An *overtwisted* immersion from X to Y is a C^1 continuous family of smooth immersions $f_t : X \to Y$, $0 \leq t < \infty$, such that the curvature remains bounded

$$curv^{\perp}(f_t(X) \leq C < \infty,$$

such that the induced Riemannian metrics $g_t = f_t^*(g_Y)$ "tends to infinity" which (to allow non-compact X) is understood as $g_{t+\delta} \geq\geq 2g_t$ h all t and $\delta > 1$.

Here are two motivating examples of families of immersions from the circle to the disc of radius 3 in the (x,y) plane

$$f_t : S^1 \to B^2(3), \text{ with } curv^{\perp}(f_t(X) \leq 1,$$

(1) *Winding on a Circle.* This f_t is a family of immersions from S^1 to the annulus $A^2(1,3)$ between the unit circle $S^1(1) \subset B^2(3)$ and $S^1(3) = \partial B^2(3)$, which are compositions of two maps that are:

• embeddings ϕ_t from S^1 to the band of width 2, i.e. to $\mathbb{R}\times[-1,1]$, where the image of ϕ_t is the 1-encircling (boundary of the 1-neighbourhood) of the straight segment of length t in the central line of the band, $[0,t]\subset\mathbb{R}\times 0\subset\mathbb{R}\times[-1,1]$;

• the covering map $\mathbb{R}\times[-1,1]\to A^2(1,3)=S^1(1)\times[-1,1]$.

(2) $\bigcirc\!\bigcirc$-*Construction.* Another family is obtained by repetitively using (compare with 1.C) regular homotopy from the 1-encircling $f_0(S^1)$ of the interval $[-2,2]$ in the x-line to an immersion f_1 with the image in the union of three unit circles with the centres at $-2,0,2$ on the x-line,

$$f_2 : S^1 \to \bigcirc\!\bigcirc\!\bigcirc \subset B^2(3),$$

where the two parallel horizontal bars (of lengths = 4) in the (convex curve) $f_0(S^1)\subset B^2(2)$ are moved to the two arcs of the central unit circle, where the upper bar is pushed down to the lower arc and the lower bar is pushed up to the arc on the top of this circle.

(This is achieved by moving the the third disc $\bigcirc_3$ (positioned on the right) to the position of $\bigcirc_1$ by "rolling" $\bigcirc_3$ along $\bigcirc_2$.)

Remark. The resulting immersion f_1 from S^1 to $\bigcirc\!\bigcirc\!\bigcirc$ can be regularly homotoped with $curv^\perp\leq 1$ inside $B^2(3)$ to $f_2 : S^1\to\bigcirc\!\bigcirc$, which is contained in the smaller disc $B^2(2)\subset B^2(3)$.

Conjecture. There exist no regular homotopy with curvature ≤ 1 from f_1 to f_2 within a disc or radius $r<3$.

A version of the $\bigcirc\!\bigcirc$-construction can be adapted to families of maps, and also to 1-dimensional foliations. Here is a potential application.

16.A. Parametric $1D$-Approximation Conjecture.[63] Let τ be a smooth non-vanishing vector field on a manifold X Then there exists a smooth map from X to the disc $B^2(4+\varepsilon)$ for a given $\varepsilon>0$, such that the orbits of τ are sent to smooth (immersed) curves with curvatures $curv^\perp\leq 1$ in the disc.
[64]

Example. There exists a smooth map $f: S^{2n+1}\to B^2(4+\varepsilon)$, for all $n=1,2,...$ and $\varepsilon>0$, such that the f-images of the Hopf circles are smooth immersed circles with curvatures $curv^\perp\leq 1$.

The above (1) and (2) generalize to immersions of n-manifolds for $n>1$, yet only in a limited way.

16.B. n-Dimensional Overtwisting Conjecture. Smooth immersions of compact n-manifolds to the unit balls, $f_0: X\to B^N(1)$, must admit *overtwisted* regular homotopies $f_t : X\to B^N(1)$, i.e. where the induced metrics g_t in X tend to infinity, and such that the curvatures of f_t are bounded as follows:

$$curv^\perp(f_t(X))\leq C_n curv^\perp(f_0(X)) \text{ for } t\leq 1 \text{ and } curv^\perp f_t(X)\leq C_n \text{ for } t\geq 1,$$

where, ideally, $C_n\leq 100n$.

[63] This is announced in [Gro 2023]. The proof I had in mind is technical, I don't intend writing it and would be only happy if somebody else does it.

[64] Probably, the minimal possible radius of the receiving disc is $3+\varepsilon$, where "extremal members" in families of immersions $S^1\to B^2(3-\varepsilon)$ with curvatures ≤ 1 must be associated with certain patterns comprised of unit circles in $B^2(3-\varepsilon)$ tangent to the unit circle cantered at zero. and where such a pattern associated with a regular isotopy from the unit circle to an immersion with the image 8 is comprised of three mutually tangent unit circles inside $B^2(1_2/\sqrt{3}$ (which seems to imply that $\Xi_1>1+2/\sqrt{3}\approx 2.1547...$).

Overtwisting immersion f_t (with no control of the curvature for the initial values of t) of special n-manifolds X can be obtained with "winding X on an n-torus".

The existence of these. *overtwisted* $f_t : X \to B^N(1)$*, such that*

$$curv^\perp(f_t(X)) \leq 1000n^{3/2} \text{ for } t > 1,$$

is not hard to show in two cases. [65]

(i) X *is an orientable* n*-manifold, which admits an immersion to* $\mathbb{R}^{n+1}$

(ii) X *is an* n*-manifold, which admits an immersion to* $\mathbb{R}^{N+1}$.

16.C. Curvature Stable Flexibility Conjecture. Let Y be a compact Riemannian manifold, e.g the unit ball $B^N(1)$ or the unit n-sphere S^N

Loosely speaking, the conjecture claims the existence of a constant $C(Y) < \infty$, such that the all homotopy theoretic invariants of immersions f and subspaces $Im_C(X,Y) \subset Im_C(X,Y) = Im_\infty C(X,Y)$ of immersions $f : X \hookrightarrow Y$ with curvatures $curv^\perp(f(X)) \leq C$ do not depend on C for all smooth manifolds X and $C \geq C(Y)$.

One should be aware of possible existence of "rigid immersion" $f_\bullet : X \to Y$ with large (large) $curv^\perp(f) = C$, where no small deformation of such an $f_\bullet$ decreases the curvature. (I suspect these exist, except for $n = 1$).

To be safe, we conjecture that every compact subset $K \subset Im_(C_1)(X,Y)$ can be brought to $Im_{C_2}(X,Y)$ by a homotopy of K in $Im_{C_3}(X,Y)$ for all $C_1 \geq C_2 \geq C(Y)$ and $C_3 \leq C_1(1 + C(Y))$.

Critical Curvature Problem. What is the set $\mathcal{C}^n_{crit} = \mathcal{C}^n_{crit}(Y) \subset \mathbb{R}$ of "critical values" C_{crit}, e.g. where the "homotopy content" of the subspace $Im_C(X,Y)$ increases for some n-manifold X at the point $C = C_{crit}$.

For instance a number $C_\circ$ is critical, if there exits a manifold $X_\circ$ and an immersion $f_\circ : X_\circ \to Y$, such that $curv^\perp(f_\circ(X_\circ) = C_\circ$ and there is *no immersion* f regularly homotopic to $f_\circ$, such that $curv(f(X_\circ)) < C_\circ$.

What is the topology of the set $\mathcal{C}^n_{crit}(Y)$ e.g. for $Y = B^N(1)$, $Y = S^N$ and $n = 2$?

Is this set finite? discrete?

On Immersions between Manifolds with Boundaries

One of the problems in proving a topological Smale-Hirsch type h-principle for overtwisted immersions is typical non-extendability of immersions from $X = X_0$ with controlled curvature to such immersions from the (wide) band $X \times [0,1] \subset X_0 = \{0\} \times X$.

On the other hand, much of what we know and what we don't know about closed manifolds applies to the curvatures of immersions between pairs of manifolds, especially to immersions $f : (X, \partial X) \hookrightarrow (Y, \partial Y)$, (with $f(X)$ normal to ∂Y?), where the mean curvature of $\partial X \subset X$ may play a similar role to that of the scalar curvature of X.

(A promising X is the complement of a small neighbourhood of the 2-skeleton of the n-torus.)

Final Question. Are immersions with $curv^\perp \leq const$ rigid or mainly flexible? Is this "hard" or "soft"mathematcs?

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[65]See section 4.4. in [Gro 2022]. There is also an attempt in this paper to prove a version of the above n-dimensional overtwisting conjecture, but there is an error at Step 2 in 4.3.A of the "proof".

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