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SIGN AND GEOMETRIC MEANING OF CURVATURE

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SUNTO. — Questa monografia è una versione estesa di una mia Conferenza tenuta in Milano nell'ambito delle Lezioni Leonardesche. E' un tentativo di rivelare ai non iniziati il meccanismo di sviluppo della geometria Riemanniana seguendo le tracce di relativamente poche idee dalle fondamenta al tetto dell'edificio.

CONTENTS.

§ 0.	The second fundamental form and convexity in the Euclidean space	Pag. 10
§ ½.	Generalized convexity	» 18
§ 1.	Recollection on length, distance and Riemannian metric	» 29
§ 2.	Equidistant deformation and the sectional curvature $K(V)$	» 38
§ 2½.	Influence of $K(V)$ on small balls in V	» 46
§ 3.	Manifolds with positive sectional curvature	» 49
§ 3½.	Distance function and Alexandrov-Toponogov theorem	» 59
§ 3¾.	Singular metric spaces with $K \geq 0$	» 61
§ 3¾.	The sphere theorem and equidistant deformation of immersed hypersurfaces	» 63
§ 4.	Negative sectional curvature	» 70
§ 5.	Ricci curvature	» 75
§ 6.	Positive scalar curvature	» 87
§ 6½.	Spinors and the Dirac operator	» 95
§ 7.	The curvature operator and related invariants	» 101
§ 7½.	Harmonic maps of surfaces and the complexified curvature $K_{\mathbb{C}}$	» 106
§ 7¾.	Harmonic maps into manifolds with $K_{\mathbb{C}} \leq 0$	» 117
§ 7¾.	Metric classes defined by infinitesimal convex cones	» 120

The curvature tensor of a Riemannian manifold is a little monster of (multi)linear algebra whose full geometric meaning remains obscure. However, one can define using the curvature several significant classes of manifolds and then these can be studied in the spirit of the old-fashioned *synthetic geometry* with no appeal to the world of infinitesimals where curvature tensors reside. A similar interplay between infinitesimal quantities and visual features of geometric objects appears in all corners of geometry and analysis. The simplest example is provided by the equivalence of the two definitions of a *monotone* function

$$\frac{df}{dt} \geq 0 \iff f(t_1) \leq f(t_2) \text{ for } t_1 \leq t_2.$$

Then the infinitesimals of the second order bring along a geometrically more interesting phenomenon of *convexity*.

$$\frac{d^2 f}{dt^2} \geq 0 \iff f\left(\frac{1}{2}(t_1 + t_2)\right) \leq \frac{1}{2}(f(t_1) + f(t_2)).$$

Our next example lies at the very verge of the Riemannian domain so we look at it in a greater detail.

§ 0. THE SECOND FUNDAMENTAL FORM AND CONVEXITY IN THE EUCLIDEAN SPACE.

The basic infinitesimal invariant of a smooth hypersurface $W \subset \mathbb{R}^n$ («hyper» means $\text{codim } W = n - \dim W = 1$) is the *second fundamental form* $\overset{\text{def}}{\Pi} = \Pi^W$ which is the field of *quadratic* forms Π_w on the tangents spaces $T_w(W) \subset T_w(\mathbb{R}^n) = \mathbb{R}^n$ defined as follows.

AFFINE DEFINITION OF Π . Move w to the origine of $\mathbb{R}^n$ by a parallel translation of W in $\mathbb{R}^n$ and compose the resulting embedding $W \subset \mathbb{R}^n$ with the linear quotient map $\mathbb{R}^n \rightarrow \mathbb{R}^n / T_w(W) \overset{\text{def}}{=} N_w$. Identify the (one dimensional linear) space N_w with $\mathbb{R}$ and thus obtain a function, say $p = p_w : W \rightarrow \mathbb{R}$, whose differential Dp vanishes at w (by the definition of $T_w(W)$). Then define the *second differential* $D^2 p$ that is a quadratic form on $T_w(W)$, such that for every pair of vector fields ∂_1 and ∂_2 on W the (second Lie) derivative of p at w satisfies

$$\partial_1(\partial_2 p)(w) = (D^2 p)(\partial_1(w), \partial_2(w)).$$

(The existence of such $D^2 p$ follows from $Dp = 0$ by a simple computation). This construction applies to all $w \in W$ and gives us our Π for $\Pi_w = D^2 p$, thought of as a quadratic form on $T(W)$ with values in the normal bundle $N = T_w(\mathbb{R}^n)/T(W)$ over W , where $T_w(\mathbb{R}^n)$ denotes the restriction $T(\mathbb{R}^n)|_W$.

Here is the familiar picture for $n = 1$.

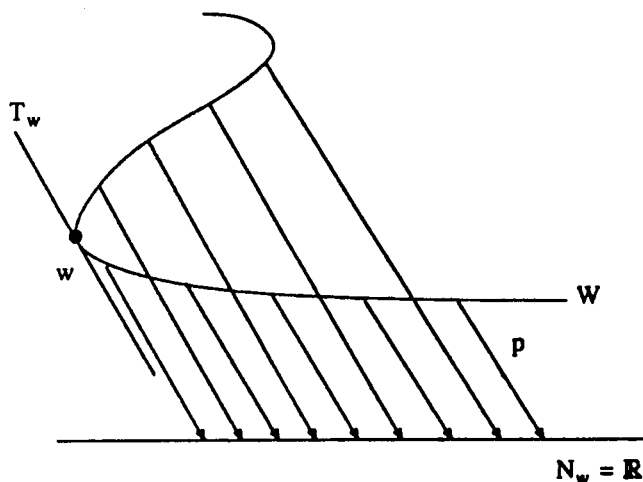


Fig. 1.

Geometrically speaking, Π_w measures the second order infinitesimal deviation of W from the affine subspace $T_w(W) \subset \mathbb{R}^n$. In particular, if W is connected, then the vanishing of Π on W is equivalent to W lying in a hyperplane, as everybody knows.

A more interesting relation between Π and the (affine) geometry of W reads

The form Π is positive semi-definite if and only if W is convex.

To make it precise we have to choose a *coorientation* of W that is a way to distinguish between two components into which W locally divides $\mathbb{R}^n$. This is customary done with a transversal (e.g. normal) vector field v along W . Such a preferred field, once chosen, is called

interior looking and the part of $\mathbb{R}^n$ where v looks into it is called the *interior* of W .

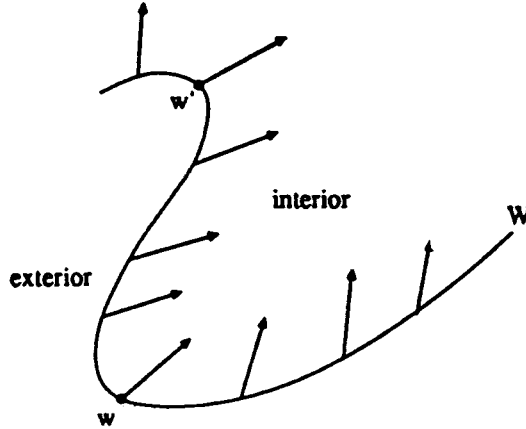


Fig. 2.

Notice that such a field also defines an orientation of the normal fibers N_w and so one can speak of the sign of the forms Π_w with values in N_w . Now we invoke the following

AFFINE DEFINITION OF CONVEXITY. W is called convex at w if it is contained in the interior half-space $T_w^+ \subset \mathbb{R}^n$ bounded by the hyperplane $T_w(W) \subset \mathbb{R}^n$.

For example, the curve W in Fig. 2 is convex at w but not at w' . Yet it becomes locally convex at w' if we change the coorientation by inverting the sign of v .

In the language of the projection $p : W \rightarrow N_w = \mathbb{R}$ the convexity claim is $p \geq 0$ which well agrees with the positivity of $\Pi_w = D^2 p$. In fact, positive definiteness of Π_w obviously implies *local convexity* (i.e. the convexity of small neighbourhood $U \subset W$ of w) of W . But it is slightly harder to derive local convexity of W from positive semidefiniteness of Π in a *neighbourhood* of w (rather than at w alone). Also, the proof of the global convexity of W is not quite trivial. Recall that the global convexity, (i.e. the convexity

at all $w \in W$) follows from $\Pi \geq 0$, provided W is a *closed* connected hypersurface, where « closed » means compact without boundary.

The affine definition of Π given above is quite general. Namely, it applies to all dimensions and codimensions (but we need $\text{codim } W = 1$ to speak of convexity), it makes sense for arbitrary smooth maps $f: W \rightarrow \mathbb{R}^m$ (not only embeddings) and it generalizes to non-Euclidean ambient spaces endowed with *affine connections*. However, the affine nature of this definition makes it poorly adapted to the needs of the Riemannian geometry where the major object of study is the *distance function* associated to the Riemannian structure. With this in mind we turn to our second definition of Π which is based on the following important notion.

EQUIDISTANT DEFORMATION. Let W be a cooriented hypersurface in $\mathbb{R}^n$ and denote by $\delta(x)$, $x \in \mathbb{R}^n$ the signed distance function to W . That is $\delta(x) = \text{dist}(x, W)$ for the exterior points x and $\delta(x) = -\text{dist}(x, W)$ in the interior. Notice that in general the distinction between interior and exterior points makes sense only locally near W and then $\delta(x)$ is defined only in some small neighbourhood of W . Also recall that

$$\text{dist}(x, W) \stackrel{\text{def}}{=} \inf_{w \in W} \text{dist}(x, w)$$

for the Euclidean distance

$$\text{dist}(x, w) = \|x - w\| = \langle x - w, x - w \rangle^{1/2}.$$

Then we look at the levels of δ , that are

$$W_\varepsilon = \delta^{-1}(\varepsilon) = \{x \in \mathbb{R}^n \mid \delta(x) = \varepsilon\},$$

and call them ε -*equidistant hypersurfaces* or ε -*equidistant deformations* of $W = W_0$.

It is easy to show that for small ε the manifolds W_ε are smooth if W_0 is smooth, but W_ε may become singular for large ε . In fact we shall see in a minute that the *inward* (i.e. $\varepsilon < 0$) equidistant deformation necessarily produces singularities for every *convex* initial hypersurface W_0 . (See fig. 3 below). For example such a deformation of the round sphere $W_0 = S^{n-1}(r) \subset \mathbb{R}^n$ of radius r

brings W_0 to the center of the sphere for $\varepsilon = -r$. (Here $W = S^{n-1}(r + \varepsilon)$ for all $\varepsilon \geq -r$).

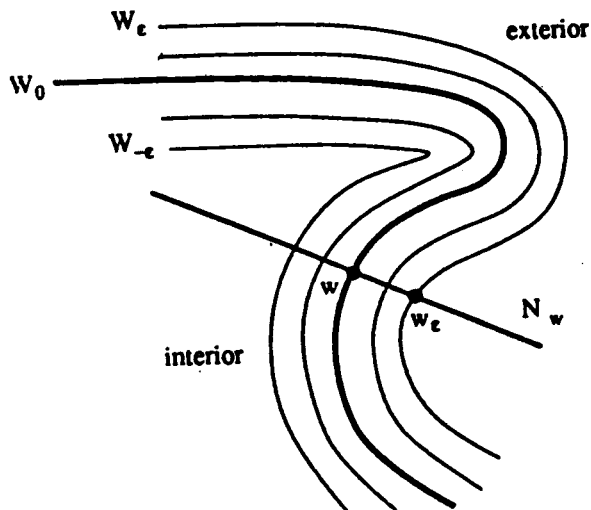


Fig. 3.

Next we consider the lines N_w in $\mathbb{R}^n$ normal to W at the points $w \in W$. It is easy to show that every such line meets each W_ε , for small ε , at a single point denoted w_ε or $(w, \varepsilon) \in W_\varepsilon$ and the resulting map $d_\varepsilon: W \rightarrow W_\varepsilon$ for $d_\varepsilon(w) = w_\varepsilon$ is smooth. (In fact, d_ε is a diffeomorphism; moreover, N_w is normal to W_ε at w_ε , as elementary differential geometry tells us). Now we are going to define the second quadratic form as the rate of change of the lengths of curves $C \subset W_0$ as we pass from W_0 to infinitesimally close hypersurface W_ε . We recall that the length of C is determined by (integration of) the length of the tangent vectors of C which is given, in turn, by the *first fundamental form* g on W that is just the restriction of the Euclidean scalar product (which is a quadratic form on $\mathbb{R}^n$) to the tangent spaces $T_w(W) \subset T_w(\mathbb{R}^n) = \mathbb{R}^n$, $w \in W$. In other words g is the *Riemannian metric* on W induced from the standard Riemannian metric on $\mathbb{R}^{n+1}$. Geometrically speaking, « induced » amounts to the relation

$$g\text{-length}(C) = \text{The Euclidean length}(C)$$

for all smooth curves C in W .

We denote by g_ε the first fundamental form of W_ε , we pull it back to W by the differential of the map d_ε and we denote the pulled back form g_ε^* on $W = W_0$. Then we set

$$(*) \quad \Pi^W = \frac{1}{2} \frac{d}{d\varepsilon} g_{\varepsilon=0}^*$$

EXAMPLE. Let W be the unit sphere $S^{n-1}(1) \subset \mathbb{R}^n$ (e.g. the circle in the plane). Then g_ε^* comes from the concentric sphere $W_\varepsilon = S^{n-1}(1 + \varepsilon)$ and we clearly see that $g_\varepsilon^* = (1 + \varepsilon)^2 g_0$. Thus $\Pi^W = g_0$ for $V = S^{n-1}(1)$, as everybody knows from the kindergarten.

It is not hard to show (by an elementary infinitesimal computation) that the above « equidistant » definition of Π is equivalent to the affine definition given earlier. In fact, the equidistant definition extends to all codimensions and still remains equivalent to the affine definition, (see, e.g., Appendix 1 in [Mi-S]).

EQUIDISTANT DEFORMATION OF A CONVEX HYPERSURFACE. If the hypersurface $W = W_0$ is convex then W_ε is convex for all $\varepsilon \in \mathbb{R}$, even where W_ε becomes singular and one needs a definition of convexity applicable to non-smooth hypersurfaces. First we indicate an infinitesimal proof of the convexity of W_ε based on the classical *tube formula* which tell us how Π^{W_ε} develops with ε for variable ε . To write down this formula we pass from the form Π to the associated operator A defined on $T(V)$ by the usual relation

$$\Pi(\tau_1, \tau_2) = g(A\tau_1, \tau_2) = \langle A\tau_1, \tau_2 \rangle_{\mathbb{R}^{n+1}}.$$

Notice that A is a symmetric operator (on each tangent space $T_w(W)$) and it is sometimes called the *shape operator*. Then the tube formula for the operators A_ε^* on $T(W)$, which are the Dd_ε -pull-backs of the operators A_ε on W_ε corresponding to Π^{W_ε} , reads

$$(**) \quad \frac{dA_\varepsilon^*}{d\varepsilon} = - (A_\varepsilon^*)^2$$

for the ordinary square of the linear operator A_ε^* . This formula actually says that (the differential Dd_ε of) d_ε maps the principal axes of the form Π^W (that are the eigenvectors of A) to those of

Π^{W_ε} and the *principal curvatures* $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \dots, \lambda_{n-1}(\varepsilon)$ of W_ε (that are the eigenvalues of A_ε on W_ε corresponding to Π^{W_ε}) satisfy

$$(+)\quad \lambda_i^{-1}(\varepsilon) = \lambda_i^{-1}(0) + \varepsilon.$$

This agrees with what we know for the sphere $S^{n-1}(r)$ for W , where $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = r^{-1}$ and $W_\varepsilon = S^{n-1}(r + \varepsilon)$, also (+) agrees with (**) as $\lambda_i(\varepsilon) = (c_i + \varepsilon)^{-1}$, for $c_i = \lambda_i^{-1}(0)$ according to (+) and then the derivative is $-(c_i + \varepsilon)^{-2}$. (See Appendix 1 in [Mi-S] for a proof of the tube formula adapted to the present notations). Now it is clear that if $\Pi^{W_0} \geq 0$ then Π^{W_ε} remains positive semi-definite for all $\varepsilon \geq 0$ and also for negative $\varepsilon \geq -\max_{i=1, \dots, n-1} \lambda_i^{-1}(0)$. In fact, whenever ε becomes equal to $-\lambda_i^{-1}(0)$ at a point w , then the map $d_\varepsilon : W \rightarrow \mathbb{R}^n$ (which moves w to the ε -end of the normal $[0, \varepsilon]$ -segment to W at w) fails to be regular at w in the sense that the differential of this map (which is smooth for all ε) becomes *non-injective* on $T_w(W)$ and so the image $d_\varepsilon(W)$ (which is not the same as W_ε for large ε where there is no regularity) may acquire a singularity at $d_\varepsilon(w) \in \mathbb{R}^n$.

Now let us look at W_ε from a global point of view where $W = W_0$ is a closed convex hypersurface in $\mathbb{R}^n$. The above discussion shows that W_ε is smooth and convex for all $\varepsilon \geq 0$ and so the induced metric on W_ε is monotone increasing in ε (where one compares two Riemannian metric by bringing them to the hypersurface W_0 by the diffeomorphism $d_\varepsilon : W \rightarrow W_\varepsilon$). This implies the following global consequence of the convexity (defined infinitesimally by $\Pi^W \geq 0$).

For each exterior (i.e. outside the compact region bounded by W) point $x \in \mathbb{R}^n$ there exists a unique point $w = p(x) \in W$, such that the segment $[x, w] \subset \mathbb{R}^n$ is normal to W at w . Moreover, the resulting map $p : \text{Exterior}(W) \rightarrow W$ is distance decreasing.

We shall see later on (in § 2) that this property is characteristic for the ambient manifolds (replacing $\mathbb{R}^n \supset W$) of *non-positive sectional curvature*.

Now, let us look at the internal deformation W_ε where $\varepsilon < 0$. As we mentioned earlier, such W_ε inevitably become singular at

some moment $\varepsilon < 0$. For example, if $W = S^{n-1}(r)$ then the (only) singular moment is $\varepsilon = -r$ and then W_ε becomes empty (though the normal map d_ε sends W on the concentric sphere $S^{n-1}(r + \varepsilon)$ for $\varepsilon < -r$). But for non-round W the singular region occupies an entire interval in ε before W_ε disappears, see Fig. 4 below.

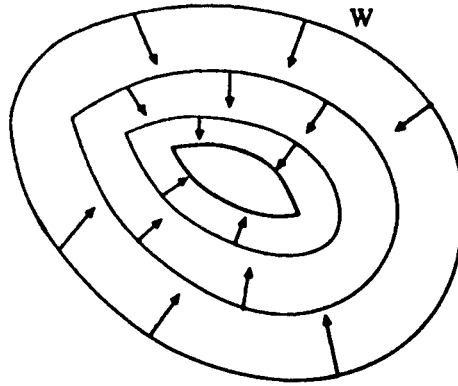


Fig. 4.

The presence of singularities makes it harder to prove the convexity of W_ε by infinitesimal means but this is rather obvious geometrically as one easily sees that the interior $\text{Int } W_\varepsilon$ is a convex set in the ordinary sense. Namely if x_1 and x_2 are two points in $\text{Int } W_\varepsilon$, then the segment $[x_1, x_2]$ also lies in $\text{Int } W_\varepsilon$. In fact $\text{Int } W_\varepsilon$ consists of the points $x \in \mathbb{R}^n$ satisfying $\text{dist}(x, \text{Ext } W) \geq \varepsilon$ and so the inclusion $[x_1, x_2] \subset \text{Int } W_\varepsilon$ is equivalent to $U_\varepsilon([x_1, x_2]) \subset \text{Int } W_\varepsilon$, where U_ε denotes the ε -neighbourhood that is the set of the points within distance $\leq \varepsilon$ from the segment $[x_1, x_2]$. Now, this $U_\varepsilon([x_1, x_2])$ obviously equals the *convex hull* of the union of the ε balls $B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$, which must lie in $\text{Int } W$ as this interior is *assumed* convex in the framework of our present convexity discussion.

If we still insist on an infinitesimal proof, we may define singular convex hypersurfaces W as appropriate limits of those W' whose interiors are finite intersections of regions bounded by smooth convex hypersurfaces. Then we can apply the tube formula to the smooth convex pieces of W' (which can be chosen with an upper bound on the principal curvatures in order to avoid premature singularities) and prove the convexity of W_ε by a simple approximation

argument as W' converges to W . The advantage of such approach is the applicability to non-euclidean ambient manifolds $V \supset W$. In fact, the convexity of the *interior* equidistant manifolds W_ε (i.e. for $\varepsilon \leq 0$) is characteristic for the manifolds V of *non-negative sectional curvature* (see § 3).

Let us draw the moral of the story. The second fundamental form Π is an easily computable tensorial object has several meaningful interpretations on the infinitesimal level. Furthermore, the class of *convex* hypersurfaces, defined by the (infinitesimal) condition $\Pi \geq 0$, has a global geometric interpretation and can be studied by means of synthetic geometry. In fact, the geometric approach naturally brings in *singular* convex hypersurfaces but their global geometry does not harbour any surprises as they can be approximated by smooth convex hypersurfaces.

§ 1/2. GENERALIZED CONVEXITY.

The above discussion leads to the following question.

What are other geometrically significant classes of hypersurfaces (and submanifolds of higher codimension) distinguished by some properties of Π ?

One interesting notion generalizing convexity is *positive mean curvature*,

$$\text{MeanCurv } W \stackrel{\text{def}}{=} \text{trace } \Pi^W = \sum \lambda_i \geq 0,$$

where λ_i denote the principal curvatures of W . This is equivalent in terms of W_ε to the *monotonicity* of the *volume element* of W_ε rather than of the induced metric. Geometrically this monotonicity says that the $(n-1)$ -dimensional volume of every domain $U \subset W = W_0$ increases as we pass to W_ε with $\varepsilon \geq 0$. More precisely the domains $U_\varepsilon = d_\varepsilon(U) \subset W_\varepsilon$ satisfy the relation

$$\frac{d \text{Vol } U_\varepsilon}{d\varepsilon} \geq 0 \text{ at } \varepsilon = 0.$$

The positivity of the mean curvature of the boundary $W = \partial V$ of a domain $V \subset \mathbb{R}^n$ implies the following property of the signed

distance function

$$\delta(v) = -\operatorname{dist}(v, W) = -\inf_{w \in W} \operatorname{dist}(v, w).$$

The function $\delta(v)$ is subharmonic,

$$\Delta\delta(v) \geq 0$$

for all $v \in V$.

Notice, that the function δ is not everywhere smooth in V and at the singular points the sign of the Laplace operator Δ must be understood in an appropriately generalized sense.

The inequality $\Delta\delta \geq 0$ at the smooth points of δ can be easily derived by applying the tube formula to the levels of δ which are just our equidistant hypersurfaces W_ϵ , and at the singular points one needs an extra approximation argument. (We shall come back to the positive mean curvature in the more general framework of manifolds $V \supset W$ with $\operatorname{Ricci} V \geq 0$ where the equidistant deformation of hypersurfaces provides the major tool for the study of such V (see § 5).

k -CONVEXITY. A cooriented hypersurface W in $\mathbb{R}^n$ is called *k -convex* for some integer $k = 1, 2, \dots, n-1, = \dim W$, if among its $n-1$ principal curvatures λ_i at least k are ≥ 0 . Then W is called *strictly k -convex* if k among λ_i are > 0 . For example, $(n-1)$ convexity is the same as the ordinary convexity.

Notice that k convexity is invariant under *projective transformation* of $\mathbb{R}^n$ which allows us to extend the notion of k -convexity to the sphere S^n and the projective space P^n which are locally projectively equivalent to $\mathbb{R}^n$. Then we observe that k -convexity is stable under small inward equidistant deformations of W in $\mathbb{R}^n$, as follows from the tube formula. (This is also true for large deformation with an appropriate generalization of k -convexity to non-smooth hypersurfaces). Furthermore, the inward equidistant deformation performed in S^n with respect to the *spherical metric* also preserves k -convexity since S^n has (constant) positive curvature where the generalized tube formula (see (**)) in § 2) leads to the desired conclusion (compare the discussion in § 2 following (**)). Moreover, since the curvature of S^n is *strictly positive*, an arbitrarily small equidistant deformation in S^n makes every k -convex W *strictly*

k -convex. As both notions are projectively invariant, we conclude that *every k -convex hypersurface in $\mathbb{R}^n$ admits a strictly k -convex approximation.*

So an elementary Riemannian geometry of positive curvature leads to a purely Euclidean conclusion.

More interesting global properties of *closed* k -convex hypersurfaces can be obtained with elementary Morse theory of linear functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ restricted to W . If the critical points of such f are *non-degenerate*, which is the case for *generic* linear functions f , then all critical points of f on W have indices either $\geq k$ or $\leq n - 1 - k$. Therefore W admits a cell decomposition with no cell of dimension ℓ in the interval $n - 1 - k < \ell < k$.

EXAMPLES. (a) If $k = n - 1$, the only possible cells are of dimension 0 or $n - 1$ and the above reduces to the standard property of locally convex closed hypersurfaces W in $\mathbb{R}^{n-1}$: these are *homeomorphic to disjoint unions of spheres* (we do not assume W is connected).

(b) Let $k = n - 2$. Now the above cell restriction becomes non-vacuous starting from $n = 5$. It says that the cells have dimensions 0, 1, $n - 2$ and $n - 1$. An obvious consequence of that for $n \geq 5$ is the freedom of the fundamental group and the vanishing of $H_i(W)$ for $i \neq 0, 1, 2, n - 2, n - 1$.

In general, the Morse theory of W tells us something non-trivial only if $n \geq 2k + 1$. However, there are non-trivial restrictions for all $k \geq 1$ on the domain $V \subset \mathbb{R}^n$ bounded by W .

If a bounded domain V in $\mathbb{R}^n$ has smooth k -convex boundary, then V admits a homotopy retraction onto an ℓ -dimension subpolyhedron in V for $\ell = n - 1 - k$.

This is immediate with the Morse theory applied to f on V which is a compact manifold with boundary.

It is not hard to see that the converse is also true. If V can be built in $\mathbb{R}^n$ by sequentially attaching handles of indices $\leq \ell$, then it is diffeomorphic to a domain with $(n - 1 - \ell)$ -convex boundary. For example, every small ε -neighbourhood of a smooth submanifold $V_0 \subset \mathbb{R}^n$ of codimension $k + 1$ obviously has k -convex boundary.

Now, we want to use the Morse theory to say something interesting about the *geometry* of V . First we observe that the intersection of a k -convex domain V (i.e. ∂V is k -convex) with an affine subspace $X \subset \mathbb{R}^n$ of codimension d is $(k-d)$ -convex in X (i.e. has a $(k-d)$ -convex boundary, now of dimension $n-1-d$) in-so-far as the intersection $V \cap X$ has *smooth* boundary in X . Then we apply the Morse theory to linear functions on the intersections of V with linear subspaces and obtain by induction on d (the case $d=1$ follows by the above Morse theory) the following

LEFSCHETZ THEOREM. *The homology homomorphism*

$$H_\ell(V \cap X) \rightarrow H_\ell(V)$$

is injectiv for $\ell = n-1-k$ and $\text{codim } X = d \leq n-k$.

EXAMPLE. If $k=n-1$ and $d=n-1$ the above says, in effect, that the intersection of every connected component of V with a line is connected. In other words, a connected domain with locally convex boundary is convex in the ordinary sense.

It is easy to see that the Lefschetz property is characteristic for k -convexity.

If a compact domain $V \subset \mathbb{R}^n$ with smooth boundary has $H_\ell(V \cap X) \rightarrow H_\ell(V)$ injective for all affine $(\ell+1)$ -dimensional subspaces X in $\mathbb{R}^n$ then (the boundary of) V is k -convex for $k=n-\ell-1$.

Now, one can accept the above injectivity as the *definition* of k -convexity without any smoothness assumption on $W = \partial V$. The first (obvious) theorem of the resulting theory reads

If V_1 and V_2 are k -convex in $\mathbb{R}^n$ then the intersection $V_1 \cap V_2$ is also k -convex.

Notice that the homological definition of $(n-1)$ -convexity allows *disconnected* domains in $\mathbb{R}^n$ with locally convex boundaries. The connectedness condition generalizes for $k < n-1$ by the requirement that $H_\ell(V \cap X) = 0$ for $\ell = n-1-k$ and for all $(\ell+1)$ -dimensional subspaces $X \subset \mathbb{R}^n$. For example, in the next-to-convex case of $k=n-2$ this requires the intersection of V with every plane $X \subset \mathbb{R}^n$ to be simply connected.

Finally we indicate yet another k -convexity condition by the analogy with the classical convex case, *for every point x in $\mathbb{R}^n$ outside V there is a k -dimensional affine subspace passing through x and missing V .*

This is a truly global property of $V \subset \mathbb{R}^n$ which is much stronger than the above k -convexity and which becomes indispensable if we want to reconstruct V by its linear projections to $(n - k)$ -dimensional subspaces in $\mathbb{R}^n$.

IMMERSED k -CONVEX HYPERSURFACES. Here our W in $\mathbb{R}^n$ is allowed to have *self-intersection*. This means, W is a smooth $(n - 1)$ -dimensional manifold which comes along with an *immersion* $W \rightarrow \mathbb{R}^n$ that is a locally diffeomorphic map. If W is oriented as an abstract manifold, then the immersed $W \subset \mathbb{R}^n$ becomes cooriented if we fix once and for ever some orientation of $\mathbb{R}^n$. In this case we can define the second fundamental form and the notion of k -convexity of W .

One sees in fig. 5 below a locally convex immersed closed curve in $\mathbb{R}^2$. Notice that the *image* of this immersion is singular at the double points and in no sense convex

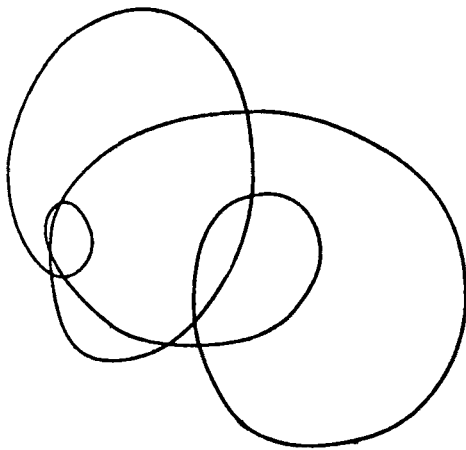


Fig. 5.

A classical convexity theorem claims that every locally convex closed connected hypersurface in $\mathbb{R}^n$ is *embedded* (i.e. has no double

point) for $n \geq 3$ and thus bound a convex domain in $\mathbb{R}^n$. The latter statement generalizes to the k -convex case as follows.

Let $W \subset \mathbb{R}^n$ be a closed immersed k -convex hypersurface for $k > \frac{n}{2}$. Then W bounds an immersed manifold V in $\mathbb{R}^n$ of dimension n (i.e. W bounds V as an abstract manifold and the immersion of $W = \partial V$ to $\mathbb{R}^n$ extends to an immersion $V \rightarrow \mathbb{R}^n$).

The construction of V is achieved by following the levels of a linear functions on W that are the intersections $W \cap X_t$ for a family of parallel hyperplanes $X_t \subset \mathbb{R}^n$. These intersections are $(k-1)$ -convex for non-critical t where X_t is transversal to W and the intersection $W_t = W \cap X_t$ is a smooth immersed hypersurface in $X_t = \mathbb{R}^{n-1}$. As we move t in a non-critical interval this hypersurface moves by a regular homotopy (i.e. remaining immersed) but the self-intersection pattern of W_t may change with t . However, the inequality $k < \frac{n}{2}$ rules out the interior head-on collision of two pieces of W_t as indicated in Fig. 6 below

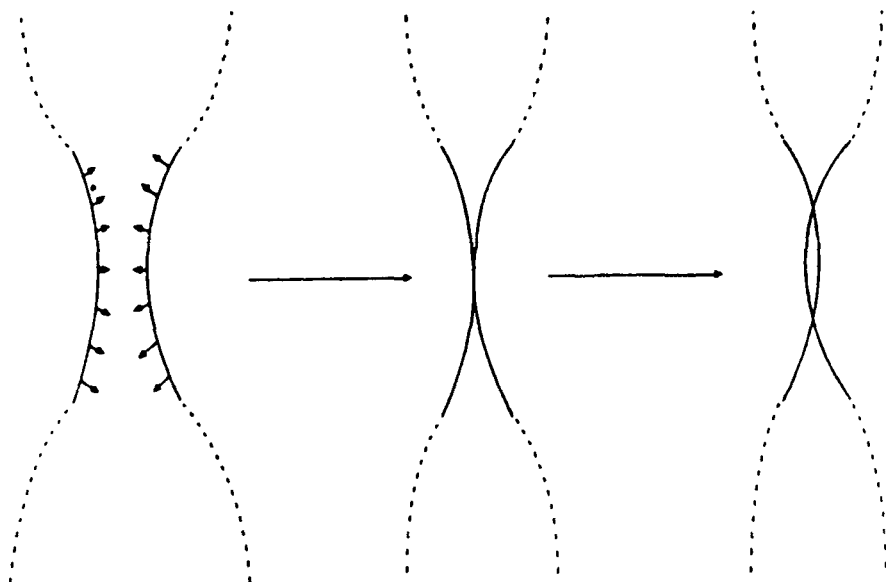


Fig. 6.

(The vector field on the initial position of W_t marks the coorientation). It is easy to see that if W_{t_0} bounds some immersed manifold W_{t_0} in $X_{t_0} = \mathbb{R}^{n-1}$, then this also so for W_{t_1} for $t_1 > t_0$ if in the course of the regular homotopy $W_{t_0} \rightarrow W_{t_1}$ the above head-on collision does not occur. Then the manifolds V_t filling in all W_t add up to the required V filling in W .

REMARK. The circle in Fig. 5 gives us a conterexample for $k=1$ and $n=2$ and it is easy to produce non-fillable W in $\mathbb{R}^n$ for all n and $k \leq \frac{n-1}{2}$. But the case $k = \frac{n}{2}$ for even $n \geq 4$ is less obvious.

PSEUDO-CONVEXITY. If one is content to restrict the symmetry group preserving k -convexity one comes up with a vast amount of generalizations among which the most important is *pseudo-convexity* of hypersurfaces $W \subset \mathbb{C}^n$. The complex structure in $\mathbb{C}^n$ distinguishes certain affine subspaces in $\mathbb{R}^{2n} = \mathbb{C}^n$, namely those which not only $\mathbb{R}$ -affine but also $\mathbb{C}$ -affine in $\mathbb{C}^n$. In particular, the distinguished planes in $\mathbb{R}^{2n} = \mathbb{C}^n$ are (called) $\mathbb{C}$ -lines in $\mathbb{C}^n$. With this terminology, W is called *pseudoconvex*, if for each point $w \in W$ and every $\mathbb{C}$ -line X in $\mathbb{C}^n$ tangent to W at w the restriction of the second fundamental form Π_w^W to $X = \mathbb{R}^2$ has the eigenvalues λ_1 and λ_2 satisfying $\lambda_1 + \lambda_2 \geq 0$. In other words, W is *mean curvature convex along all $\mathbb{C}$ -directions*. Similarly one defines k -pseudo-convexity by insisting on the above inequality at each w for the $\mathbb{C}$ -lines inside some k -dimensional $\mathbb{C}$ -affine subspace tangent to W at w .

We suggst to the reader to state and prove the Lefschetz and the filling theorems in this case.

The beautiful feature of pseudo-convexity and k -pseudo-convexity is the invariance under the (local as well as global) biholomorphic transformations of $\mathbb{C}^n$. (The proof is not hard). This allows an extension of these notions to arbitrary complex manifolds V where the pseudo-convexity plays a major role in the analysis and geometry on V . For example, there is a remarkable theorem of Grauert which claims that every compact connected complex manifolds V with a non-empty strictly pseudoconvex boundary admits a non-constant holomorphic function. Moreover, there is a proper holomorphic map f of the interior of V into some $\mathbb{C}^N$, such that f is injective on the complement to some compact complex submanifold $V_0 \subset V$ of positive codimension.

Finally, we suggest to the reader to work out the notion of k -convexity in the quaternion space $\mathbb{H}^n$ and then extend it further to the mean curvatur convexity with respect to a given set of (distinguished) subspaces in $\mathbb{R}^n$. Then the reader may state and prove the Lefschatz and the filling theorems.

HYPERSURFACES OF TYPE (k_+, k_-) . This refers to the condition which requires that W has at each point w exactly k_+ strictly positive and k_- strictly negative principal curvatures. We also assume Π^W is *nowhere singular* on W and so $k_+ = k_- = n - 1$. Notice that the non-singularity of Π^W is equivalent (by a trivial argument) to *regularity* of the *Gauss map*. Recall that the Gauss map v sends a *co-oriented* hypersurface W to the unit sphere $S^{n-1} \subset \mathbb{R}^n$ by assigning to each $w \in W$ the outward looking unit normal vector $v(w)$ at w brought to the origin of $\mathbb{R}^n$. In the non-cooriented case the Gauss map goes to the projective space P^{n-1} by sending w to the line in $\mathbb{R}^{n+1}$ through the origin parallel to $v(w)$. Such a map is smooth if W is smooth and the above mentioned regularity of v means that the differential $Dv : T(W) \rightarrow T(S)$ is injective on the tangent spaces $T_w(W)$ for all $w \in W$, or, equivalently, that v is *locally diffeomorphic*.

For example, for surfaces W in $\mathbb{R}^3$ there are two possibilities. The first, is, where Π is *definite*, positive or negative (one may switch from positive to negative by changing the coorientation) and so W is locally convex or concave. The second possibility is that of indefinite Π where W is a *saddle surface*.

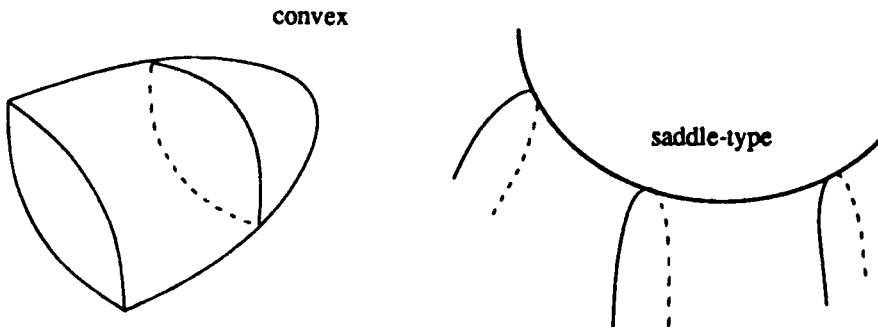


Fig. 7.

Recall that $w \in W$ is called a *saddle point* if W is neither convex nor concave at w , that is the tangent space $T_w(W) \subset \mathbb{R}^n$ intersects the interior and the exterior (parts of the complement) of W arbitrarily close to w . Equivalently, w is contained in the (Euclidean) convex hull of the boundary of every sufficiently small neighbourhood $U \subset W$ of w . Then W is called of *saddle type* if all $w \in W$ are saddle points.

If one want to reap global consequences of the type condition on Π^W one must make some assumptions on the behavior of W at infinity. Here it is worth-while noticing that *every closed* hypersurface always contains at least one convexity/concavity point. For example, W is (obviously) convex at each maximum point of the distance function $\text{dist}(x_0, w)$ on W for every fixed $x_0 \in \mathbb{R}^n$. Thus the saddle type makes V non-closed and one may expect interesting geometry if W has *no apparent boundary* in $\mathbb{R}^{n+1}$ of this kind or another. Here are three conditions one usually imposes on W to rule out such a boundary.

(1) W is *properly* embedded (or immersed if a self-intersection is allowed) into $\mathbb{R}^n$. That is the inclusion map $W \subset \mathbb{R}^n$ is proper: the intersections of the compact subsets in $\mathbb{R}^n$ with W are compact in W . In other words, if a sequence of points $w_i \in W$ goes to infinity in W then it also goes to infinity in $\mathbb{R}^n$ (and so no subsequence may create in the limit a boundary point of W in $\mathbb{R}^n$).

(2) W is *quasi-proper* in $\mathbb{R}^n$. This means that the intersection of W with every compact subset in $\mathbb{R}^n$ is a union of *disjoint compact* subsets in W . That is every connected curve in W going to infinity in W must be *unbounded* in the ambient $\mathbb{R}^n$. Clearly, $\text{proper} \implies \text{quasi-proper}$.

(3) W is *complete*. This refers to the completeness of the induced Riemannian metric. Equivalently every connected curve in W going to infinity must have infinite length in $\mathbb{R}^n$ (and hence in the induced metric on W). This is weaker than quasi-properness.

These three conditions serve their purpose of ruling out boundary (or limit) points of W in $\mathbb{R}^n$ if W itself has no boundary as a topological manifold.

Here are some examples clarifying the meaning of the above definitions.

(a) Let $W_0 \subset \mathbb{R}^n$ be a *closed* submanifold and $W \rightarrow W_0 \subset \mathbb{R}^n$ be an infinite covering map. One may take, for example $W_0 = S^1 \subset \mathbb{R}^2$ and $W = \mathbb{R}$ which covers the circle S^1 . Then this W is complete but neither proper nor quasi-proper in $\mathbb{R}^2$. (Of course an infinite covering map $W \rightarrow W_0$ never give us an *embedding* $W \rightarrow \mathbb{R}^n$, but one can sometimes make an embedding from such a map by an arbitrarily small perturbation which does not affect the properties (1), (2) or (3). This is clearly possible, for example, for $\mathbb{R} \rightarrow S^1 \subset \mathbb{R}^2$).

(b) The graph of the function $\sin \frac{1}{x}$ over $]0, \infty[$ is complete but not quasi-proper in $\mathbb{R}^2$; the graph of $\frac{1}{x} \sin \frac{1}{x}$ over $]0, \infty[$ is quasi-proper though non-proper; but the graph of $x^2 \sin \frac{1}{x}$ over the semi axes $]0, \infty[$ is not even complete.

Now let $W \subset \mathbb{R}^n$ be a submanifold without boundary which satisfies one of the above conditions (1), (2), (3) and has the form Π^W of a given (constant!) type (k_+, k_-) . Then one expects that the global geometry (and topology) of W is rather special. Yet, one can not answer the following simple looking questions.

Is there a bound on certain Betti numbers of W ? What is the structure of the Gauss map $v: W \rightarrow S^{n-1}$? Can this map have $|\text{Jac } v| = |\text{Discr } \Pi| \geq c > 0$? (This is impossible for $n = 3$ by a difficult theorem of Efimov, see [Miln]). Suppose v is a diffeomorphism of W onto an open subset $U \subset S^{n-1}$. Does this U has bounded topology? Can one classify the subsets U appearing this way? (Yes, for $n = 2$ according to Verner, see the discussion on p.p. 188 and 283 in [Gro]).

COMPACTIFICATION OF W . The constant type condition on Π^W is not only an affine invariant but it is also invariant under *projective transformations* of $\mathbb{R}^n$. Therefore, one may speak of hypersurfaces of constant type (k_+, k_-) in the projective space P^n and also in the sphere S^n . Here such a hypersurface may be *closed* and then one

asks what the geometry and topology of such closed $W \subset S^n$ can conceivably be. The simplest examples are provided by codimension one orbits of isometry groups acting on S^n , where not only the type of Π^W but also the *principal curvatures of W in S^n are constant*. The hypersurfaces with constant principal curvatures (recall, these are the eigenvalues of Π) are called *isoparametric* and, amazingly, not all of them are homogeneous for large n (see [F-K-M]). Now, one can not expect the (topological) classification of closed hypersurfaces $W \subset S^n$ of a given constant type (k_+, k_-) to be too simple, but one still believes that such W have « bounded » topology and geometry, e.g. the Betti numbers of W must be bounded in terms of the dimension n alone.

Finally we observe that every closed hypersurface $W \subset P^n$ gives us (properly embedded) hypersurface W' in $\mathbb{R}^n = P^n - P^{n-1}$, that is $W' = W - P^n$. Then we ask if this « compact origin » of W' of constant type imposes extra topological restriction on W' .

We conclude this section by an attempt to formulate a general problem on the relation between Π^W and the global geometry of W . First we notice that Π^W is completely characterized (up to rigid motions) at each point $w \in W$ by the principal curvatures $\lambda_1(w), \dots, \lambda_{n-1}(w)$ which we organize in the increasing order, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1}$. Then we have a non-ambiguous (and hence, continuous) map $\lambda: W \rightarrow \mathbb{R}^{n-1}$ for $\lambda(w) = (\lambda_1(w), \dots, \lambda_{n-1}(w))$ which encodes the infinitesimal information hidden in Π^W at all points $w \in W$. Now, for example, every subset $A \subset \mathbb{R}^{n-1}$ defines a class of hypersurfaces W in $\mathbb{R}^n$ if one requires the image $\lambda(W)$ to be contained in A . (This includes k -convexity, $\text{MeanCurv} \geq 0$, and constant type as special cases). Another important invariant besides the geometric image $\lambda(W)$ is the push-forward $\alpha_* = \lambda_*(dw)$ of the Riemannian measure dw on W . Then every class $\mathcal{M}$ of measures on $\mathbb{R}^{n-1}$ defines a class of W in $\mathbb{R}^n$ with $\lambda_*(dw)$ belonging to $\mathcal{M}$. With these A and $\mathcal{M}$ the « local-to-global » problem sounds as follows. What is the (geometric and topological) « shape » of W in the class defined by a given A or $\mathcal{M}$? We want an answer in terms of A or $\mathcal{M}$ and we may expect it only for exceptionally nice A and $\mathcal{M}$. Unfortunately we never know which problem is nice and which is ugly unless we solve it.

§ 1. RECOLLECTION ON LENGTH, DISTANCE AND RIEMANNIAN METRIC.

A Riemannian structure on a smooth manifold V is given by a positive definitive quadratic form g on the tangent bundle $T(V)$. Such g assigns the *norm* (or *length*) to each tangent vector $\tau \in T(V)$ by

$$||\tau||_g = (g(\tau, \tau))^{\frac{1}{2}}$$

and then one has the g -length of every C^1 -smooth curve in V , that is a map $c : [0, 1] \rightarrow V$, defined by integrating the norm of the vectors $c'(t) = (Dc) \frac{\partial}{\partial t} \in T_v(V)$, $v = c(t)$,

$$\text{length}(c) = \int_0^1 ||c'(t)||_g dt.$$

Geometrically speaking, the norm $||\tau||_g$ pulled back to $[0, 1]$ by the differential of c defines a *measure density* on $[0, 1]$ whose total mass is the length of c . Thus the length is invariant under reparametrization of $[0, 1]$.

There is nothing sacred in the quadratic nature of $|| \cdot ||_g$. One could start with an arbitrary continuous family of (non-Euclidean) *norms* $|| \cdot ||$ on the tangent spaces $T_v(V)$, $v \in V$, and then define the length of curves by integrating $||c'||$. Here we only mention that a norm on $T(V)$ is called a *Finsler metric* on V and manifold carrying these are called *Finsler manifolds*.

Let us concentrate our attention for the moment on the function $c \mapsto \text{length}(c)$ defined on the space of map $[0, 1] \rightarrow V$ with some Riemannian or Finsler norm on $T(V)$. Such a function satisfying a few obvious properties (such as invariance under reparametrization and additivity for subdivisions of curves into smaller pieces) is an interesting geometric object in its own right which is called a *length structure* on V . Granted such a structure, we define the associated metric on V in the usual way by looking at all curves between given points v_1 and v_2 in V and by setting $\text{dist}(v_1, v_2)$ equal the infimum of the lengths of these curves. Clearly, this is indeed a

metric in the Riemannian and Finsler cases (but it may be not so for less regular length structures where, for example, every curve between v_1 and v_2 may happen to have infinite length, e.g. if we allow a *generalized norm* on $T(V)$ which is infinite away from some subbundle $S \subset T(V)$).

Metrics arising from length structures are called *length metrics* and they have the following property which almost characterizes them.

TRIANGLE ε -EQUALITY. For every two points v_1 and v_2 , every $\varepsilon > 0$ and every positive $\delta \leq \text{dist}(v_1, v_2)$ there exists a point $v \in V$, such that

$$\text{dist}(v, v_1) \leq \delta + \varepsilon$$

and

$$\text{dist}(v, v_2) \leq \text{dist}(v_1, v_2) - \delta.$$

In other words the triangle inequality

$$\text{dist}(v_1, v_2) \leq \text{dist}(v, v_1) + \text{dist}(v, v_2)$$

becomes nearly the equality with an appropriate choice of v . In fact, one can make an actual equality if there exists a *shortest* curve c between v_1 and v_2 for which $\text{length}(c) = d = \text{dist}(v_1, v_2)$. Such a curve c in V with the induced metric clearly is isometric to the segment $[0, d] \subset \mathbb{R}$ and it is customarily called a *minimizing geodesic segment* between v_1 and v_2 and, accordingly, denoted (even if such segment is non-unique) by $[v_1, v_2] \subset V$. Then for every $\delta \in [0, d]$ the corresponding $v \in [v_1, v_2] \leftrightarrow [0, d]$ with $\text{dist}(v, v_1) = \delta$ satisfies the triangle equality

$$\text{dist}(v, v_1) + \text{dist}(v, v_2) = \text{dist}(v_1, v_2).$$

Notice that if V is a compact (possibly with a boundary) *Riemannian* (or Finsler) manifold then the minimizing segment does exist for all v_1 and v_2 in V , as everybody knows, and this is also true in the non-compact case if V is *complete* as a metric space.

LOCALITY OF LENGTH METRICS. Every length metric on V is uniquely determined by its restrictions to the elements of an arbitrary cover

of V by open subsets U_i . That is if two such metrics coincide on every U_i then they equal on V . In fact for an arbitrary metric d on V one can define d^+ as the supremum of the metrics d' for which there exists an open covering of V by U_i (depending on d'), such that $d' \leq d$ on each U_i . Then the triangle ε -equality shows that $d^+ = d$ for the length metrics d . But in general $d^+ > d$. For example, if we start with the Euclidean metric d on a submanifold $V \subset \mathbb{R}^m$, then d^+ corresponds to the *induced Riemannian structure* on V defined with the *Euclidean* length of curves (but these must be taken in V). Thus

$$\text{dist}_V(v_1, v_2) = d^+(v_1, v_2) > d(v_1, v_2) = \text{dist}_{\mathbb{R}^m}(v_1, v_2),$$

unless V (or at least the closure of V in $\mathbb{R}^m$) contains the straight segment between v_1 and v_2 in $\mathbb{R}^m$.

The locality of the length metrics, and in particular of the Riemannian metric is the major (non-psychological) reason why one adheres to the *local-to-global principle* in the Riemannian geometry.

NON-EFFECTIVENESS OF THE LENGTH DEFINITION OF A METRIC. Even if a Riemannian metric on V is written down quite explicitly one may have hard time in evaluating the corresponding distance between two given points. For example, if V is (diffeomorphic to) a domain in $\mathbb{R}^n$, $n = \dim V$ then every Riemannian structure is given by $\frac{n(n+1)}{2}$ functions on V that are the components of g in the standard basis,

$$g_{ij} = g(\partial_i, \partial_j), i, j = 1, \dots, n$$

for the vector fields $\partial_i = \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$. But even for very simple functions (e.g. polynomials) g_{ij} we can not see very well what happens in the course of minimization of the g -length of curves.

Another example, where the logical nature of the problem is especially clear, is where V equals the universal covering of a compact manifold V_0 . The length structure lifts easily from V_0 to V by just assigning to each curve in V the length of its image in V_0 . Yet it does not tell us much about the corresponding *metric* in V .

For example, there is no way to tell by looking at V_0 if the *diameter* of V (i.e. $\sup_{v_1, v_2 \in V} \text{dist}(v_1, v_2)$) is finite or not, as this is equivalent to finding out whether the fundamental group $\pi_1(V_0)$ is finite or not. But the latter problem is well known to be *undecidable* and so the diameter of V can not be effectively computed in terms of V_0 .

Our last example is where V is a Lie group and g is a *left invariant* Riemannian structure. Such a g is uniquely determined by what happens on a single tangent space, say at $T_e(V)$ for the neutral group element $e \in V$. Thus g may be given by prescribing a quadratic form on the linear space $T_e(V)$. Yet one has very poor knowledge (especially for non-nilpotent solvable Lie groups) on the metric structure of these (V, g) .

The above mentioned difficulties make quite valuable any kind of metric information one can obtain by looking at effectively computable *infinitesimal invariants* of V . A great deal of these is harboured by the *Riemannian curvature tensor* of g which is expressed by direct (but messy) algebraic formulae involving g and the first and second derivatives of the components g_{ij} of g in given local coordinates. (These formulae appears explicitly later in this §). For example, there is a particular combination of these derivatives, called the *sectional curvature* $K(V)$ (see § 2), whose *strict positivity* $K(V) \geq \varepsilon > 0$ implies V is compact, whenever it is complete as a metric space. This gives an (effective!) partial solution to the above diameter problem for coverings $V \rightarrow V_0$. Conversely, if the sectional curvature of V_0 (and hence of V) is everywhere *negative* then V is non-compact and its diameter is infinite (see §§ 2 and 4). Unfortunately, most manifolds have sectional curvature of variable sign and the above criterion does not apply. But the constant sign of the sectional curvature does occur for certain interesting examples including some *homogeneous* Riemannian manifolds (yet solvable groups mentioned above do not fall into the constant sign category).

RECAPTURING g FROM THE METRIC. There is the following simple way to reconstruct g from the corresponding distance function. For a given point v we define the function $\varrho(v')$ on V by $\varrho(v') = (\text{dist}(v, v'))^2$. We observe that ϱ is smooth at $v' = v$ and the differential

Dg vanishes at v . Then the second differential D^2g is a well defined quadratic form on $T_v(V)$ which equals (by an easy argument) to g on T_v .

Thus we established the equivalence of the three basic viewpoints on a Riemannian structure: infinitesimal, for g a quadratic form on $T(V)$; path-theoretic, for the length function on curves and the metric (or distance function) point of view. Often one does not distinguish between the three structures and apply the expression « Riemannian metric » to all of them. It should be noted that although the three structures are formally equivalent they represent objects from quite different worlds. For example, g is a tensor, specifically a quadratic form, whose positivity may be sometimes forfeited. But then nothing remains of the metric structure attached to g (except for some residual terminology such as « Lorentz metric »). On the other hand one may have a (non-Riemannian) metric space with rather nasty singularities where the infinitesimal approach becomes hard to pursue. So it seems we are quite lucky in having so diverse notions and ideas merging together in the Riemannian stream.

RIEMANNIAN VOLUME. Each Riemannian manifold V of dimension n carries a canonical measure, which is (uniquely) characterized by the following two axioms.

MONOTONICITY. If there exists a distance decreasing surjective map between two n -dimensional manifolds, say $f : V_1 \rightarrow V_2$, then

$$\text{Vol } V_2 \leq \text{Vol } V_1$$

where « Vol » denotes the total volume (or mass) of the measure on the manifold in question.

NORMALIZATION. The unit cube in $\mathbb{R}^n$ has volume = 1.

Notice that the above definition makes sense on the level of the length structure and of the Riemannian norm on $T(V)$ as well as for the distance function. In fact, the following three conditions on f are obviously equivalent,

- (i) f is distance decreasing,
- (ii) f decreases the length of curves,
- (iii) the differential of f decreases the length of the tangent vectors (here we must additionally assume f differentiable).

The existence and uniqueness of the Riemannian measure for the continuous structures g on V immediately follows from the obvious *infinitesimal approximation* of g at each point $v_0 \in V$ by a Euclidean metric g_0 . Namely, if we take some local coordinates $u_1 \dots u_n$ in V near v_0 , then g defines a Euclidean metric g_0 on the coordinate neighbourhood U by

$$g_0(\partial_i(u), \partial_i(u)) = g(\partial_i(v_0), \partial_i(v_0)),$$

where $\partial_i = \frac{\partial}{\partial u_i}$ and u run over U while $v_0 \in U$ is fixed. It is clear that (U, g_0) is isometric to $\mathbb{R}^n$ and that g_0 *approximates* g at v_0 with zero order. That is for every $\varepsilon > 0$ there exists a neighbourhood $U_\varepsilon \subset U$ of v_0 , such that

$$(1 - \varepsilon) g_0 \leq g \leq (1 + \varepsilon) g_0 \text{ on } U_\varepsilon.$$

It follows, by monotonicity, that the Riemannian g -volume of U_ε is ε -close (in an obvious sense) to the g_0 -volume (which is Euclidean and may be assumed known) and then the uniqueness of Vol_g follows with $\varepsilon \rightarrow 0$. The existence is also seen in this framework but it is more convenient to use a purely infinitesimal definition. Namely, (the discriminant of) g defines a norm on the top exterior power of $T(V)$ which gives us a measure (density) on V . In the down-to-earth terms one has the notion of $|\text{Jacobian}|$ for every C^1 -map $f: V_1 \rightarrow V_2$ which is computed at each $v \in V$ with the (Euclidean!) metrics $g_1(v)$ on $T_v(V)$ and $g_2(w)$ on $T_w(V_2)$, $w = f(v) \in V_2$, as $|\text{Det}|$ of the differential $Df: T_v(V_1) \rightarrow T_w(V_2)$. That is

$$|\text{Jac } f| = |\text{Det } f| = (\text{Det } DD^*)^{\frac{1}{2}},$$

where $D = Df$ and D^* is the adjoint of D with respect to $g_1(v)$ and $g_2(w)$. Then one defines the Riemannian volume of every small neigh-

neighbourhood $U \subset V$ by taking a diffeomorphism $f : U' \rightarrow U$ for some $U' \subset \mathbb{R}^n$ and by setting

$$\text{Vol}_g U = \int_{U'} |\text{Jac } f| \, du'$$

for the Euclidean volume element du' .

Notice that the Riemannian structure restricts to submanifolds $W \subset V$ of dimension $k < n = \dim V$ and then we have the Riemannian volume Vol_k on W . In particular, for $k=1$ we come up again with the length of curves, that is $\text{Vol}_1 W$ for $\dim W=1$.

FIRST ORDER INFINITESIMAL APPROXIMATION OF g BY g_0 . Since g is Euclidean at v_0 with zero order one might think that *non-flatness* of g (i.e. the deviation from being locally Euclidean) can be measured by the first derivatives of $g_{ij} = g(\partial_i, \partial_j)$ in some local coordinates. Then, surprisingly, this does not work because there always exist particularly nice local coordinates $u_1, \dots, u_n$ around v , called *geodesic coordinates*, such that

$$\partial_k g_{ij}(v_0) = 0 \text{ for all } i, j, k = 1, \dots, n,$$

where we need the metric g to be C^1 -smooth at v_0 . In fact, a little thought explains how this could happen. When we change coordinate systems which are n -tuples of functions $u_i : V \rightarrow \mathbb{R}$, we observe that the change of $\partial_k g_{ij}(v_0)$ is determined by the *second* derivatives of u_i at v_0 . Altogether, there are $\frac{n^2(n+1)}{2}$ of these derivatives (for n functions u_i) which are (somewhat miraculously) the same in number as the *first* derivatives of $\frac{n(n+1)}{2}$ functions g_{ij} at v_0 . Then it is easy to believe (and not hard to prove) that one can adjust the second derivatives of u_i such that $\partial_k g_{ij}$ become zero. In fact, one can prove that these second derivatives are uniquely determined by the condition $\partial_k g_{ij}(v_0) = 0$. Namely, if u_i and u'_i are geodesic coordinates, such that $\partial_{u_i} u'_j = \delta_j^i$, then the second derivatives of u'_j with respect to u_i vanish at v_0 .

The Euclidean metric g_0 on U constructed as earlier, but now with *geodesic* coordinates u_i , approximates g at v_0 with the *first order*,

$$-\varepsilon^2 g_0 \leq g - g_0 \leq \varepsilon^2 g_0,$$

where ε is a smooth function on V vanishing at v_0 . The name usually applied to g_0 is *the osculating metric at v_0* .

One may try to proceed further and eliminate the second derivatives of g at v_0 by manipulating the third derivatives of u_i . But now we have only $\frac{n^2(n+1)(n+2)}{6}$ of the third derivatives of u_i to play against $\frac{n^2(n+1)^2}{4}$ second derivatives of g_{ij} . The difference between the two numbers is $\frac{n^2(n^2-1)}{12}$ which tells us how many parameters must measure the second order non-flatness of g at v_0 . In fact the following linear combinations $R_{ijk\ell}$ of $\partial_k \partial_\ell g_{ij}(v)$ are constant under the changes of u_i which keep fixed the first and the second derivatives of geodesic coordinates u_i at v , as a straightforward (an exceedingly boring) computation shows

$$R_{ijk\ell} = \frac{1}{2} (\partial_j \partial_\ell g_{ik} + \partial_i \partial_k g_{j\ell} - \partial_i \partial_\ell g_{jk} - \partial_j \partial_k g_{i\ell}),$$

where $\frac{1}{2}$ is the matter of convention. It is clear that

$$R_{ijk\ell} = -R_{ij\ell k} = R_{k\ell ij} = -R_{\ell k ij}$$

and that

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0,$$

which is called the (first) *Bianchi identity*. Then one can easily see that the number of linearly independent $R_{ijk\ell}$ is exactly $\frac{n^2(n^2-1)}{12}$ and that $R_{ijk\ell}$ transform as tensors should under the changes of local coordinates (now, with no restriction on the first and second derivatives of these coordinates). Thus we obtain a tensor on V called the *curvature tensor* $R = \{R_{ijk\ell}\}$ of g which measures non-flatness of g in the following sense.

A manifold (V, g) has zero curvature if and only if each point $v \in V$ admits a neighbourhood U which is isometric to some open subset $U' \subset \mathbb{R}^n$, for $n = \dim V$.

The curvature R at v is a 4-linear tensor on the tangent space $T_v(V)$ which is given a Euclidean structure by $g|_{T_v(V)}$. One can form a vast amount of numerical characteristic of $R|_{T(V)}$ which are invariant under g -orthogonal transformations of $(T(V), g)$ and thus give us *scalar invariants* of g , that are real valued functions on V built up in an invariant way at each $v \in V$ from the second derivatives of g at v in geodesic coordinates. For example, one can

take the g -norm $\|R\|_g$ that is $\left(\sum_{i,j,k,\ell} R_{i,j,k,\ell}^2\right)^{\frac{1}{2}}$ which gives us an overall

scalar measure of non-flatness of V . Now we can say that V is ε -flat if $\|R\|_g \leq \varepsilon$ on V and then try to study the global geometry of ε -flat manifolds for a given $\varepsilon > 0$ (see, e.g. [Bu-Kar]). But we are concerned with more subtle scalar invariants which are not automatically positive and whose sign conveys a non-trivial geometric information about V . As a matter of comparison we may look again at the second fundamental form Π^V of $V \subset \mathbb{R}^{n+1}$ whose norm $\|\Pi^V\|$ measures non-flatness of V in $\mathbb{R}^{n+1}$ but where the *sign of the eigenvalues* λ_i of Π^V (which are scalars invariant under rigid motions of V in $\mathbb{R}^{n+1}$) tells us much more than the sheer size of Π . (Notice

that $\sum_{i=1}^n \lambda_i^2 = \|\Pi\|^2$).

In the Riemannian case there are many similar scalar invariants (which correspond to functions on the space of curvature tensors on $T(V)$ invariant under fiberwise orthogonal transformations) but only few of them have found a meaningful geometric interpretation so far. Most studied among these are the *sectional curvature* $K(V)$, which is, in fact, a function on the Grassmann bundle of the tangent 2-planes in V , the *Ricci curvature*, which is a quadratic form on V and the *scalar curvature* which is a function on V . These curvatures are defined and studied in the following §§ 2-6.

§ 2. EQUIDISTANT DEFORMATIONS AND THE SECTIONAL CURVATURE $K(V)$.

For every hypersurface W in a Riemannian manifold $V = (V, g)$ (recall that « hyper » signifies $\dim W = n - 1$ for $n = \dim V$) we define the equidistant hypersurfaces W_ε as the levels of the signed Riemannian distance function $\text{dist}_g(v, W)$ exactly as we did it for hypersurfaces in $\mathbb{R}^n$ in § 0. We can also define the map $d : W = W_0 \rightarrow V$, which moves W_0 to W_ε for small ε using *geodesic ε -segments* normal to W instead of straight segments in $\mathbb{R}^{n+1}$ (see § 0). In order to have good theory of geodesics in V we assume g is C^2 -smooth (in some coordinate system). Then one knows since the work of Riemann that for every unit tangent vector $\tau \in T_v(V)$ there exists a unique *geodesic issuing from v in the direction τ* . Here the word « geodesic » means a *locally isometric* map of $\mathbb{R}$ or of a connected subset of $\mathbb{R}$ into V , where « locally isometric » refers to the distance function in V . Namely, every sufficiently small subsegment of a geodesic must be a *minimizing segment* in V that is of the length equal the distance between its ends in V . (For example, the geodesics in the unit sphere $S^n \subset \mathbb{R}^{n+1}$ are the great circles or rather lines and segments going around these circles with the unit speed. They are minimizing in-so-far as their length does not exceed π).

If V is a complete (e.g. compact) manifold without boundary, then it is classically known (probably, already to Riemann) then one has a *geodesic ray* $\gamma : \mathbb{R}_+ \rightarrow V$ with $\gamma(0) = v$ and $\gamma'(0) = \tau$ for all $v \in V$ and unit $\tau \in T_v(V)$. If V has a boundary, the ray may hit the boundary at some finite moment. Similarly, if V is non-complete the ray may reach the « infinity » of V in finite time (as actually happens to straight rays issuing from a point in a bounded domain $V \subset \mathbb{R}^n$). However, for every *interior* point $v \in V$ and some $\varepsilon > 0$ depending on v there always exists an *ε -segment* $\gamma : [0, \varepsilon] \rightarrow V$ with $\gamma(0) = v$, $\gamma'(0) = \tau$ (for a given unit $\tau \in T_v(V)$ which is a minimizing geodesic segment between $v = \gamma(0)$ and $v' = \gamma(\varepsilon) \in V$. Thus the local geometry of geodesics in V is very much the same as that in $\mathbb{R}^n$ (where the geodesics are straight lines, rays and segments).

Now we define d_ε with an exterior unit normal field ν to W by sending each $w \in W$ to the ε -end of the geodesic segment issuing from w in the direction $\nu(w)$. (If $\varepsilon < 0$ we use the segment

$\gamma: [-\varepsilon, 0] \rightarrow V$ with $\gamma(0) = w$ and $\gamma'(0) = v$. If V is complete we thus obtain a map (called the *normal exponential map*) $d: W \times \mathbb{R} \rightarrow V$ for $d(w, \varepsilon) = d_\varepsilon(w)$ in the above notation. It is characterized by the geodesic property of d on the lines $w \times \mathbb{R}$ and by the initial conditions

$$d(w, 0) = w \text{ and } \frac{\partial}{\partial \varepsilon} d(w, 0) = v(w)$$

for all $w \in W$. (In the non-complete case we have such a map on some neighbourhood $U \subset W \times \mathbb{R}$ of $W \times 0 \subset W \times \mathbb{R}$, provided W lies in the interior of V).

THE SECOND FUNDAMENTAL FORM OF W IN V . Using d_ε one defines the second fundamental form Π^W which measures how much W is curved inside V (exactly as in the Euclidean case, see § 0), by

$$(+)\quad \Pi^W = \frac{1}{2} \frac{d}{d\varepsilon} g_{\varepsilon=0}^*,$$

where g_ε^* is the metric on W induced from g by the map $d_\varepsilon: W \rightarrow V$.

There is another way to define Π^W by using geodesic coordinates $u_1, \dots, u_n$ at a point $w \in W \subset V$ where we want to evaluate Π^W . Namely, these coordinates identify the coordinate neighbourhood U with a domain $U' \subset \mathbb{R}^n$ (with the Euclidean coordinates $u_1, \dots, u_n$), such that $w \leftrightarrow 0$ and $W \cap U$ becomes a hypersurface $W' \subset U' \subset \mathbb{R}^n$ passing through the origin. Then we *define* the form Π^W of $W \subset V$ at w as that of W' in $\mathbb{R}^n$ at 0,

$$\Pi^W | T_w(W) = \Pi^{W'} | T_0(W')$$

where the tangent spaces $T_w(W)$ and $T_0(W')$ are identified by the differential of the implied diffeomorphism $U \leftrightarrow U'$ (sending $W \cap U$ to W' and $T_w(W)$ onto $T_0(W')$). A little thought shows this definition to be independent of the coordinates u_i and with a minor extra effort one can see this form is the same as defined by the above (+).

An offshot of the second definition is the existence, for every tangent hyperplane $S \subset T_v(V)$, of a smooth hypersurface $W \subset V$ passing through v , tangent to S (i.e. having $T_v(W) = S$) and being *geodesic at v* which means, by definition, $\Pi^W | T_v(W) = 0$. For example, one may take W corresponding to the Euclidean *hyper-*

plane $W' \subset \mathbb{R}^n$ in geodesic coordinates, such that W' is tangent to S at $T_0(\mathbb{R}^n) = T_v(V)$.

Now, as $\frac{d}{d\varepsilon} g_{\varepsilon=0}^* = 0$ at v for the above W and the corresponding equidistant W_ε , we want to look at the second derivative. This is better done (for the reason which will become clear later on) with the operators A_ε^* on $T(W)$ defined as the pull-backs of the *shape* operators A_ε on W_ε under the differentials of the maps $d_\varepsilon: W \rightarrow W_\varepsilon$ (compare § 0; here as in the Euclidean case d_ε is a diffeomorphism of W on W_ε for small ε and A_ε is defined by $\Pi^{W_\varepsilon}(\tau_1, \tau_2) = g(A_\varepsilon \tau_1, \tau_2) = \langle A_\varepsilon \tau_1, \tau_2 \rangle_v$). Namely we set

$$(\quad) \quad B_S = \frac{d}{d\varepsilon} A_{\varepsilon=0}^* | S.$$

This B_S is a symmetric operator on $S = T_v(W)$ which depends only on S and on g but not on a choice of W with $T_v(W) = S$, as a simple infinitesimal computation shows. Also notice that B_S does not depend on the choice of coorientation of S (and thus of W) since the *second* derivative is invariant under the sign change of the variable.

The operators B_S on the tangent hyperplanes $S \subset T(V)$ carry the same infinitesimal information as the curvature tensor and there are simple algebraic formulae expressing one in term of the other. On the other hand, one can define with B_S the *sectional curvature* $K(\sigma)$ for all tangent 2-planes $\sigma \subset T_v(V)$ as follows. Take an arbitrary hyperplane $S \subset T_v(V)$ meeting σ at a line $\ell = S \cap \sigma \subset T_v$ and being normal to σ (i.e. normal to the line $\ell^\perp \subset \sigma$ normal to ℓ). Then we take a unit vector $\tau \in \ell$ and define

$$(*) \quad K(\sigma) = -g(B_S(\tau), \tau).$$

Again one should go through some simple algebraic computation to see why the result does not depend on the choice of S and τ . As for the minus sign, this comes about because we want the round sphere in $\mathbb{R}^{n+1}$ to have *positive* curvature. In fact, let V equal the unit sphere $S^n = S^n(1) \subset \mathbb{R}^{n+1}$ with the induced Riemannian metric g and $W \subset V = S^n$ be an equatorial hypersphere. Then W clearly is geodesic at all points $w \in W$ and so is suitable for computing B_S for

$S = T(W)$. The ε -equidistant concentric spheres $W_\varepsilon \subset V$ clearly are *smaller* than W . More precisely, the metrics of these spheres W_ε brought to $W = W_0$ are given by the following well known (and obvious) formula

$$g_\varepsilon^* = (\cos^2 \varepsilon) g_0.$$

Therefore the second derivative of g_ε^* measured by B is negative (definite) and the sectional curvature K is positive in accordance with the usual convention. Also notice that the diminishing of g_ε^* for growing $|\varepsilon|$ agrees with the behavior of the second form II^{W_ε} : the smaller of the two balls in S^n bounded by W_ε is convex and the larger ball is concave. Thus the interior equidistant deformation of W (for a given coorientation) makes W convex and the exterior deformations W_ε are concave.

Now, let us compute K for $W = S^n(1)$. First,

$$\frac{d}{d\varepsilon} g_\varepsilon^* = -2 \sin \varepsilon \cos \varepsilon g_0,$$

and so

$$A_\varepsilon^* = -tg \varepsilon Id,$$

(formally, A_ε^* equals $\frac{d}{d\varepsilon} g_\varepsilon^*$ divided by g_ε^*). Then

$$\frac{d}{d\varepsilon} A_\varepsilon^* = (-1 - tg \varepsilon) Id,$$

for the identity operator Id on $S = T(W)$. Thus $B_s = -Id$ and $K(\sigma) = 1$ for all tangent planes σ in $W = S^n(1)$.

Notice that K scales quadratically under the scalar change of the metric. For example, the sphere $S^n(R)$ of radius R has $K = R^{-2}$. (This can also be seen directly for $g_\varepsilon^* = \cos^2(\varepsilon R^{-1})$ in this case). In general, we denote by RV the manifold with the new distance defined by

$$\text{dist}_{\text{new}} = R \text{dist}_{\text{old}},$$

which correspond to $g_{\text{new}} = R^2 g_{\text{old}}$. Then the scaling formula for K reads

$$K(RV) = R^{-2} K(V).$$

SECTIONAL CURVATURE FOR SURFACES. If $\dim V = 2$ then the curvature tensor reduces to a function on V (as the number of components $\frac{n^2(n^2 - 1)}{12}$ becomes 1 for $n = 2$, see § 1) which adequately is expressed by the sectional curvature $K(v) = K(\sigma = T_v(V))$ for the points $v \in V$. The famous Gauss' formula expresses K for surfaces in $\mathbb{R}^3$.

TEOREMA EGREGIUM. *The sectional curvature $K(v)$ of $V \subset \mathbb{R}^3$ equals the Jacobian of the Gauss map $V \rightarrow S^2$ at v , or equivalently to the product of the principal curvatures (eigenvalues of Π^v or of the shape operator A) at v .*

Of course, the proof is trivial by the standards of the modern infinitesimal calculus. Yet, the major consequence of the theorem looks as remarkable as it appeared 200 years ago: the Jacobian of the Gauss map does not change if we *bend* V in $\mathbb{R}^3$, that is if we apply a deformation preserving the length of the curves in V . For example, when we start bending an initially flat sheet of paper it does not remain flat in $\mathbb{R}^3$ but its intrinsic geometry does not change and so the Jacobian of the Gauss maps remains zero.

Another corollary of Gauss theorem reads

Convex (and concave) surfaces have $K \geq 0$ while saddle surfaces have $K \leq 0$.

Notice that the first claim extends to convex hypersurfaces $V^n \subset \mathbb{R}^{n+1}$ for all n : they have $K \geq 0$ by Gauss' formula extended to the higher dimension. On the other hand, *saddle surfaces* $V^2 \subset \mathbb{R}^n$ have $K \leq 0$ for all n , where « saddle » means the following.

CONVEX HULL PROPERTY. Each point $v \in V$ is contained in the Euclidean convex hull of the boundary of every sufficiently small neighbourhood $U \subset V$ of v . (Compare saddle surfaces in § 1/2).

The proof of the $K \leq 0$ statement follows from a higher codimensional version of Gauss' formula. (This formula applies to all $V^n \subset \mathbb{R}^{n+k}$ but it leads to no nice geometric interpretation for $K(V^n) \leq 0$ if $n \geq 3$).

SURFACES $\Sigma \subset V$. The sectional curvature K of V can be computed at every tangent plane $\sigma \subset T_v(V)$ with an appropriate surface Σ in V . Namely one takes Σ , such that $T_v(\Sigma) = \sigma$ and such that Σ is

geodesic at v . The geodesic condition is equivalent to the existence of geodesic coordinates $u_1, \dots, u_n$ at v with respect to which Σ becomes a Euclidean plane in $\mathbb{R}^n$ (with the Euclidean coordinates $u_1, \dots, u_n$, compare § 1). Then, by yet another generalization of Gauss' formula, $K(\sigma)$ in V equals $K(v)$ in Σ for the metric in Σ induced from V .

SECTIONAL CURVATURE OF HYPERSURFACES. Consider a hypersurface $W \subset V$ and let us formulate Gauss' *teorema egregium* which relates $K_W(\sigma)$ to $K_V(\sigma)$ for the planes $\sigma \in T_w(W)$, $w \in W$. For this we need the second fundamental form Π^W restricted to σ , where σ is given the (Euclidean) metric inherited from g on $T_w(V)$. Now, every quadratic form on $\mathbb{R}^2 = (\sigma, g|_\sigma)$ is characterized by its eigenvalues (which are the eigenvalues of the corresponding symmetric operator A on $\mathbb{R}^2$) and the product of these eigenvalues for the form Π^W on σ is denoted $\text{Dis}(\sigma)$. With this the Gauss formula reads

$$K_W(\sigma) = K_V(\sigma) + \text{Dis}(\sigma).$$

Here as earlier the proof is algorithmic but the corollaries are quite nice. For example if W is *convex* (see next § for a discussion on convexity in V) and so Π^W is definite, then we conclude

$$K_W \geq K_V.$$

In particular if V has positive sectional curvature then so does W .

TUBE FORMULA. The tube formula for hypersurfaces in $\mathbb{R}^n$ (see (**) in § 0) generalizes to hypersurfaces W in an arbitrary Riemannian manifold V by

$$(**) \quad \frac{d}{d\varepsilon} A_\varepsilon^* = - (A_\varepsilon^*)^2 + B,$$

where B is the operator on the (tangent spaces S of the) tangent bundle $T(W_\varepsilon)$ defined earlier in this § by $(++)$. Notice that $(**)$ for $\varepsilon = 0$ reduces to $(++)$ for geodesic submanifolds W (i.e. where $\Pi^W = 0$). As usual we do not provide the proof as we do not attempt to present the infinitesimal computational formalism of the Riemannian geometry. Yet, we want to point out here the following important feature of $(**)$. The term B measures the curvature of V and does not depend on W . In fact, we have B on every tangent hyperplane $S \subset T(V)$ and B in $(**)$ is obtained by restricting to the hyperplanes $T_w(W_\varepsilon) \subset T_w(V)$. On the other hand the operators A_ε

measure the relative curvature of $W = W_0$ in V and, for $\varepsilon \neq 0$, of the equidistant hypersurfaces W_ε .

Using (**) we can give our first geometric characterization of manifolds with $K \geq 0$ and $K \leq 0$ in term of equidistant hypersurfaces W_ε .

LOCAL CONVEXITY CRITERION. *If $K(V) \geq 0$ then the inward equidistant deformations W_ε of every convex hypersurface $W \subset V$ remain convex, and if $K(V) \leq 0$ then the outward deformation is convex. Conversely, if the inward equidistant deformation preserves convexity of all convex hypersurfaces in V then $K(V) \geq 0$ and if this happens for the outward deformation then $K(V) \leq 0$.*

In this statement we speak of cooriented hypersurfaces and convexity is defined by $\Pi^W \geq 0$. The equidistant deformation in question is only considered for small ε so that the normal geodesic map $d_\varepsilon : W \rightarrow V$ is a diffeomorphism of W onto W_ε (as is needed for our version of tube formula). Then the claims

$$K(V) \geq 0 \implies \text{inward deformation preserves convexity}$$

and

$$K(V) \leq 0 \implies \text{outward deformation preserves convexity}$$

become obvious as

$$K \geq 0 \iff B \leq 0$$

and

$$K \leq 0 \iff B \geq 0.$$

To prove the opposite implication, from preservation of convexity to the sign of $K(V)$, one needs sufficiently many convex hypersurfaces W in V whose equidistant deformations W_ε are non-convex whenever the curvature sign is wrong. Such a W must have quite small second fundamental form (and hence, small $\|A\|$) to be sensitive to the B -term in (**). This can be easily arranged by using hypersurfaces W corresponding to (pieces of) Euclidean spheres of large radius in geodesic coordinates. (We suggest the reader to actually produce these W and conclude the proof following the above hint.).

The above convexity criterion makes sense for quite general metric spaces (e.g. for Finsler manifolds), where one can define convexity but where our infinitesimal definition of the curvature

does not work. On the other hand, the usefulness of the (infinitesimally defined) conditions $K \geq 0$ and $K \leq 0$ in the Riemannian framework owns very much to the possibility of several different geometric interpretations which by no means follow one from another for non-Riemannian manifolds. For example, one does not know how to extend the implication

$$K(V) \geq 0 \implies K(W) \geq 0$$

to *convex* hypersurfaces W in *non-Riemannian* spaces V . Even for non-smooth convex hypersurfaces in $\mathbb{R}^n$ the only simple proof of $K(W) \geq 0$ uses an approximation by smooth hypersurfaces followed by the application of Gauss' teorema egregium.

We conclude this § by relating the sectional curvature to the curvature tensor defined in § 1.

SECTIONAL CURVATURE AND THE CURVATURE OPERATOR. The sectional curvature at each point $v \in V$ is a function on the Grassmann manifold $Gr_2 \mathbb{R}^n = Gr_2 T_v(V)$ of the planes in $\mathbb{R}^n = (T_v(V), g_v)$. To understand the nature of this function we use the standard (Plücker) embedding of $Gr_2 \mathbb{R}^n$ into the unit sphere of the exterior power $\Lambda^2 \mathbb{R}^n$ which assigns to each plane $\sigma \in Gr_2 \mathbb{R}^n$ the bivector $\beta = x_1 \wedge x_2$ for an orthogonal basis (x_1, x_2) in $\sigma \subset \mathbb{R}^n$. This β does not depend on the choice of x_1, x_2 (here we need σ oriented and the basis respecting this orientation) and the norm $||\beta||$ (naturally defined with the Euclidean norm in $\mathbb{R}^n$) equals one. Now a simple algebra shows that the sectional curvature function $\sigma \mapsto K(\sigma)$ on $Gr_2 \mathbb{R}^n \subset \Lambda^2 \mathbb{R}^n$ is quadratic: there exists a (necessarily unique) quadratic form Q on $\Lambda^2 \mathbb{R}^n$, such that $K(\sigma) = Q(\sigma, \sigma)$ for all $\sigma \in Gr_2 \mathbb{R}^n$. Following the established tradition one often uses instead of Q the corresponding symmetric operator R defined by $\langle R\alpha, \beta \rangle = Q(\alpha, \beta)$ for the scalar product on $\Lambda^2 T(V)$ induced by g on $T(V)$. This is called the *curvature operator* $R: \Lambda^2 T_v(V) \rightarrow \Lambda^2 T_v(V)$.

Notice that $d = \dim \Lambda^2 \mathbb{R}^n = \frac{n(n-1)}{2}$ and so quadratic forms Q on $\Lambda^2 \mathbb{R}^n$ constitute the space of dimension

$$\frac{d(d+1)}{2} = \frac{n(n-1)(n+1)(n-2)}{8}.$$

This is more than the number of the independent indices in the curvature tensor (which is $\frac{n^2(n^2-1)}{12}$, see § 1) and in fact, the

form Q satisfies some symmetry relation, called the *Bianchi identity*, which reduces the dimension to $\frac{n^2(n^2-1)}{12}$. Then the form Q (and the curvature operator R) can be identified with curvature tensor of (V, g) .

§ 21½. INFLUENCE OF $K(V)$ ON SMALL BALLS IN V .

We want to give here another geometric criterion for the sign of $K(V)$, now in terms of the size of small balls in V . Namely we shall show that small concentric balls grow *slower* in V with $K(V) \geq 0$ than the balls in $\mathbb{R}^n$. On the contrary, if $K(V) \leq 0$, then the balls in V grow *faster* with the growth of the radius than it happens in $\mathbb{R}^n$. Here is the precise statement.

MONOTONICITY CRITERION. *If $K(V) \geq 0$ then for every point $v \in V$ there exists a number $\delta_0 > 0$ such that every two concentric balls $B(v, \delta)$ and $B(v, \lambda\delta)$ with $\delta \leq \lambda\delta \leq \delta_0$ satisfy*

$$(0) \quad B(v, \lambda\delta) \leq \lambda B(v, \delta)$$

which is understood according to the following

DEFINITION. The inequality

$$B \leq \lambda B'$$

for two metric spaces B and B' signifies that there exists a bijective (sometimes « surjective » is enough) map $f : B' \rightarrow B$, such that

$$\text{dist}_B(f(a), f(b)) \leq \lambda \text{dist}_{B'}(a, b)$$

for all a and b in B' .

The inequality (0) is characteristic for $K \geq 0$. If it holds for all small balls around v then the sectional curvatures at v are ≥ 0 . Similarly, the negative curvature $K \leq 0$ is characterized by the inverse ball inequality

$$B(v, \lambda\delta) \geq \lambda B(v, \delta)$$

for $0 < \delta \leq \lambda\delta \leq \delta_0(v)$.

IDEA OF THE PROOF. One knows that every point $a \in V$ lying sufficiently close to v can be joined with v by a *unique* geodesic segment $[v, a] \in V$. Then for every $a \in B(v, \delta)$ we define $b = f(a) \in B(v, \lambda\delta)$

as the b -end of the geodesic segments $[v, b]$ which extends $[v, a]$ and has

$$\text{length } [v, b] = \lambda \text{ length } [v, a].$$

The resulting map $f : B(v, \delta) \rightarrow B(v, \lambda\delta)$ preserves the geodesics issuing from v and also it respects the spheres around v : the sphere $S(v, \alpha)$ of radius α , for every $\alpha \leq \delta$, goes to $S(v, \beta)$ for $\beta = \lambda\alpha$. This f expands exactly by λ in the radial direction and we must show that it expands the spheres $S(v, \alpha)$ no more than that. Now the spheres $S(v, \varepsilon)$ form an equidistant family to which the tube formula (**) applies. This shows for $K \geq 0$ that $S(v, \varepsilon)$ grow *slower* with ε than the corresponding spheres in $\mathbb{R}^n$ (where $K=0$ and there is no negative B -term in the tube formula), while initially, for « infinitely small » ε the spheres $S(v, \varepsilon)$ are (asymptotically) Euclidean. In other words, the (eigenvalues of the) shape operator on the sphere $S(v, \varepsilon)$ are smaller in V than in $\mathbb{R}^n$ and so the spheres do grow slower in V . This implies the required λ -inequality on f comparing $B(v, \delta)$ and $B(v, \lambda\delta)$, for $K \geq 0$ and the case $K \leq 0$ follows by a similar argument.

If $\dim V = 2$, then the converse statement giving the sign of $K(V)$ in terms of the balls follows from what we have just proved as near each point v where $K(v) \neq 0$ the curvature is either positive or negative (because there is a single 2-plane σ at v). Then this extends to $n \geq 2$ by looking at the growth of the small balls $B(v, \beta)$ intersected with a geodesic surface Σ at v tangent to the plane $\sigma \in T_v(V)$ where we study the (sign of the) curvature $K(\sigma)$. The details here are not hard to fill in and this is suggested to the reader.

Notice, that the inequality (0) for the balls in V can be used as a definition of $K \geq 0$ for an arbitrary metric space V but the corresponding theory has not been truly developed. For example, one does not know when this definition agrees with that using convex hypersurfaces.

Another remark is that the above argument gives us besides a comparison between concentric balls in V also a comparison of the small δ -balls $B(v, \delta) \subset V$, with the Euclidean ball $B'(\delta) \subset \mathbb{R}^n$. Namely,

$$K(V) \geq 0 \iff B(v, \delta) \leq B'(\delta)$$

and

$$K(V) \leq 0 \iff B(v, \delta) \geq B'(\delta).$$

Again this can be used as a definition of $K \geq \leq 0$ but for non-Riemannian V this is quite different from the above definition using the inequality (0) for concentric balls in V .

EXAMPLE. Let V be a finite dimensional *Banach space* that is an n -dimensional linear space with a norm $\| \cdot \|$ and the corresponding $\text{dist}(v_1, v_2) = \|v_1 - v_2\|$. This V , like $\mathbb{R}^n$, admits a similarity transformation at each point $v \in V$ by

$$v' \rightarrow v + \lambda(v' - v)$$

for variable v' and each $\lambda \in \mathbb{R}_+$. This establishes the *metric equality*

$$B(v, \lambda\delta) = \lambda B(v, \delta)$$

for all balls and thus suggests the *vanishing* of the curvature $K(V)$. On the other hand if a ball $B = B(v, \delta)$ in such a V is comparable with the Euclidean δ -ball B' by either of the two inequalities $B \geq B'$ or $B \leq B'$, then necessarily $B = B'$ and V is isometric to $\mathbb{R}^n$. (This is a simple exercise for the reader).

One may ask at this point what is the deep reason which makes various geometric definitions of the sign of the curvature coincide for the Riemannian manifolds. First of all, by their very definition, Riemannian manifolds are infinitesimally Euclidean and so their basic geometry is similar to that of $\mathbb{R}^n$. Furthermore, as we assume the Riemannian structure g smooth, we tremendously restrict the infinitesimal geometry at each point $v \in V$. For example, all infinitesimal information of the second order (which is reflected in the curvature) is defined by *finitely many* parameters at each point of V (that are the values of the first and second derivatives of g_{ij}), and so there are plenty of algebraic relations between these parameters. When integrated, these infinitesimal relations acquire a geometric meaning such, for example, as the equivalence of different geometric definitions of (the sign of) the curvature. On the other hand the infinitesimal geometry, say, of a Finsler manifold at a given point involves *infinitely many* parameters as these are needed to specify a general (Banach) norm at every tangent space. However, there exist some non-Riemannian spaces with finite dimensional infinitesimal geometry. Among them most known are those called *sub-Riemannian* or *Carnot-Caratheodory* spaces but their geometry has not been studied as deeply as in the Riemannian case (compare [Str]).

§ 3. MANIFOLDS WITH POSITIVE SECTIONAL CURVATURE.

As we already know the curvature condition $K(V) \geq 0$ is characterized by preservation of convexity of small inward equidistant deformations $W_\varepsilon \subset V$ of convex hypersurfaces W in V . Now we want to establish the convexity of W_ε for all negative ε (« negative » corresponds to « inward » with our conventions, see § 0) and we need first of all a definition of convexity suitable for non-smooth hypersurfaces. We start with the following basic notion of

CONVEX BOUNDARY. Let V' be a Riemannian manifold with boundary called W' . We say that W' is (geodesically) convex of in the interior $\text{Int } V' = V' - W'$, every two points can be joined by a minimizing segment provided such a segment exists for the two points in question in the ambient space $V' \supset \text{Int } V'$. (The latter condition is satisfied for all *complete*, in particular *compact* manifolds V'). In other words, non-convexity of W' is manifested by the minimizing segments between v_1 and v_2 in $\text{Int } V'$ which meet W' at some point w between v_1 and v_2 . See Fig. 8.

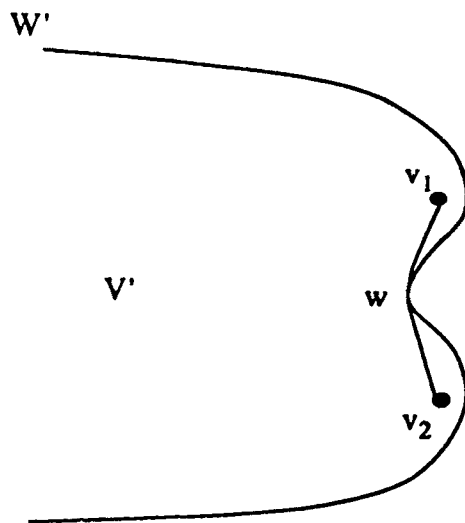


Fig. 8.

Notice that such a minimizing segment $[v_1, v_2]$ in V' typically « bends » at the points w where it meets W' . For example if V' is part of a larger manifold $V \supset V'$ with $\dim V = \dim V'$ then $[v_1, v_2]$

may be (and typically is) non-geodesic at w in V . With this in mind one can see that W' is convex if and only if the second fundamental form $\Pi^{W'}$ is positive semi-definite, provided W' is C^2 -smooth in order to have $\Pi^{W'}$ defined. It follows that the convexity is a *local* property of W and this locality remains valid (for the above reason) for non-smooth W' as well. (Notice that the above argument which appeals to the length minimization inside V' gives us a very quick proof of the classical result on the convexity of *connected locally convex* subsets in $\mathbb{R}^n$. We challenge the reader to find a purely elementary proof of this classical local to global convexity criterion for finite polyhedra V' in $\mathbb{R}^3$).

Now a hypersurface W in V is called *convex* if near each point $w \in W$ it can be made into the (part of the) boundary of a convex domain $V' \subset V$ where the latter convexity refers to the boundary convexity of V' defined above. Again, if W is smooth this is equivalent to $\Pi^W \geq 0$, but now no apparent global convexity of W follows from our local definition, as is seen in Fig. 9 below.

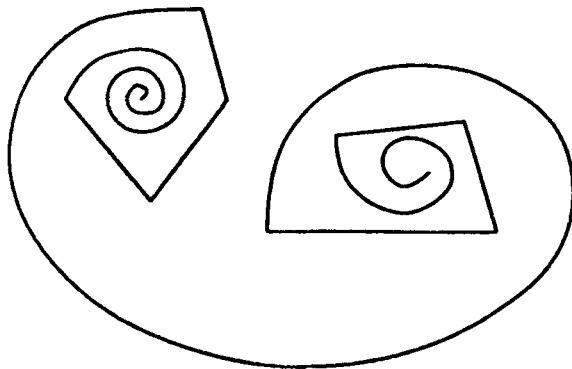


Fig. 9.

We shall also apply the notion of geodesic convexity to subsets $V_0 \subset V$ as follows. V_0 is called *geodesically convex* if for every two points v_1 and v_2 there exists a path between v_1 and v_2 which is length minimizing among all paths between v_1 and v_2 in V_0 and which is also geodesic in the ambient manifold V . Notice, that convex hypersurfaces W are not convex in this sense but what is bounded by W may be convex. On the other hand every connected *totally geodesic* submanifold $V_0 \subset V$ is geodesically convex. (Recall that V_0 is called

totally geodesic if every geodesic in V which is tangent to V_0 at a point is necessarily contained in V_0). In fact, one can think of every k -dimensional convex V_0 , for $k \leq n = \dim V$, as a convex domain inside a totally geodesic submanifold of dimension k in V .

INWARD DEFORMATION OF THE BOUNDARY. Let V be a compact manifold with boundary $\partial V = W$ and set

$$V_\varepsilon^- = \{v \in V \mid \text{dist}(v, W) \geq \varepsilon\}.$$

If V_ε^- happens to be a manifold with boundary then

$$\partial V_\varepsilon^- = W_{-\varepsilon} = \{v \in V \mid \text{dist}(v, W) = \varepsilon\},$$

(where the minus sign at ε is due to our coorientation convention, see § 0). If W is smooth then also $W_{-\varepsilon}$ is smooth for small ε but as ε grows $W_{-\varepsilon}$ may develop singularities. There are two slightly different reasons for the appearance of singularities. First, two different parts of W may meet inside V as they move inward, see Fig. 10 and 11 below.

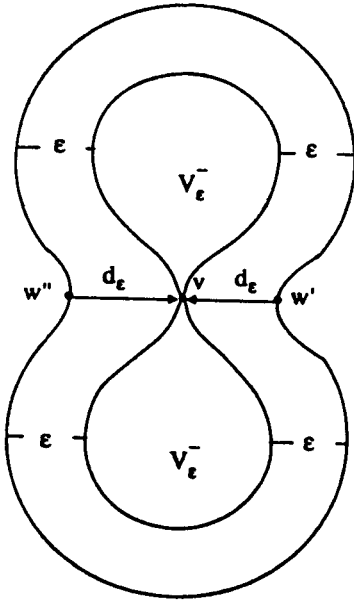


Fig. 10.

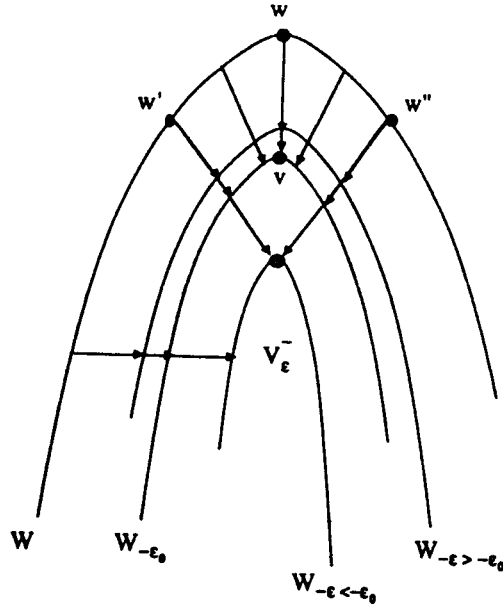


Fig. 11.

In other words a point $v \in V_\varepsilon^-$ becomes singular if there are two distinct points w' and w'' in W for which

$$\varepsilon = \text{dist}(v, w') = \text{dist}(v, w'') = \text{dist}(v, W).$$

Notice, that this v is the *double-point* of the normal geodesic map d_ε and $d_\varepsilon(w') = d_\varepsilon(w'') = v$.

The second reason for the singularity is the meeting in V of two « infinitely close » points of W . This means v is the *focal* point for some point $w \in W$ with $\text{dist}(v, w) = \varepsilon$, where « focal » signifies that the normal geodesic map $d_\varepsilon : W \rightarrow V$ is non-regular at w , i.e. the differential of d_ε is non injective at w . (Recall that d_ε moves each w to the ε -end of the geodesic ε -segment normal to W at w). Notice that the first moment ε_0 where a focal point appears is characterized by the blow-up of the second fundamental form of W mapped to V by d_ε .

$$||\Pi|| \rightarrow \infty \text{ for } \varepsilon \rightarrow \varepsilon_0.$$

This is clearly seen, for example, in the inward deformation of the sphere of radius ε_0 in $\mathbb{R}^n$.

Now the reader may appreciate the elegance of the following basic theorem by Gromoll and Meyer (see [Ch-Eb]).

CONVEX CONTRACTION FOR $K \geq 0$. *Let V be a compact connected manifold with convex boundary and non-negative sectional curvature. Then the subsets $V_\varepsilon^- \subset V$ are convex for all $\varepsilon \geq 0$. (We assume V is connected to satisfy our current definition of convexity).*

IDEA OF THE PROOF. Assume for the moment that $W = \partial V$ is smooth. Then V_ε^- remains smooth, and hence convex, in-so-far as the normal geodesic map $d_\varepsilon : W \rightarrow V$ is a smooth embedding. Furthermore, if $d_\varepsilon(W)$ develops a self-intersection without focal points, then V_ε^- becomes locally represented as an intersection of smooth convex subsets and so again it is convex. Then it is easy to believe in the convexity at the focal points as well as these are just « infinitesimal » double points (vanishing of the differential of a map at a tangent vector $\tau \in T(W)$ brings together the « infinitely closed points » corresponding to the « two ends » of τ).

To make the above rigorous, one may use a piecewise smooth approximation (compare § 0) of convex hypersurfaces (and subsets) as in Fig. 12 below.

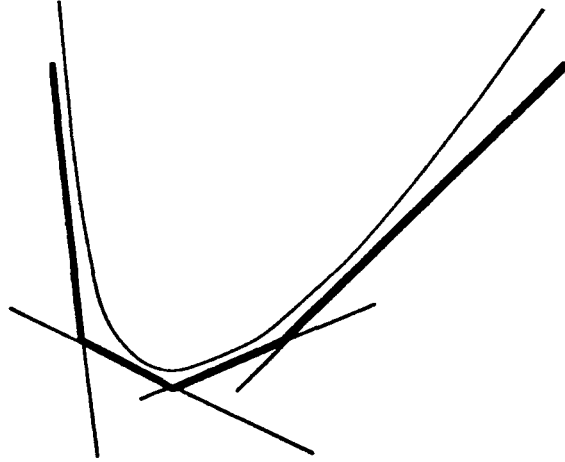


Fig. 12.

We require that each piece is convex and has the second fundamental form Π bounded by $||\Pi|| \leq c$ for some fixed constant, e.g. $c = 1$. Then the small inward ε -deformation of this piece-wise smooth hypersurface is again convex and piecewise smooth, where the deformed pieces may, unfortunately, have $||\Pi||$ slightly greater than c . This increase of $||\Pi|| \rightarrow \infty$ which corresponds to the appearance of a focal point. But this can be prevented since the deformed hypersurface can be arbitrarily close approximated again by another piecewise smooth convex hypersurface having $||\Pi|| \leq c$ for all pieces. Thus by sequentially applying small equidistant deformations followed by approximations

$$W \xrightarrow{\text{def}} W_\varepsilon \xrightarrow{\text{appr}} (W'_\varepsilon \xrightarrow{\text{def}} W'_\varepsilon)_\varepsilon \rightarrow \dots$$

we manage to keep in the category of piecewise smooth convex hypersurfaces for large inward deformations.

To conclude the proof, we must somehow produce small convex pieces out of which we construct the approximating hypersurfaces.

This is done at each point v by using geodesic coordinates at v which relate (small pieces near v of) strictly convex hypersurfaces W in V (« strictly » means $\Pi^W > 0$) with those in $\mathbb{R}^n$ (where the Euclidean coordinates correspond to the geodesic coordinates in V , see § 1). Thus the approximation of a *strictly* convex W reduces (locally and then globally) to the corresponding Euclidean problem where the approximation is quite easy, but the non-strict case is somewhat more delicate.

Notice that the notion of strict convexity of W extends to non-smooth points $w \ni W$ by requiring the existence of a *smooth strictly convex* hypersurface $W' \rightarrow w$ (i.e. $\Pi^{W'}(w) > 0$) whose « interior region » locally contains W as in Fig. 13 below.

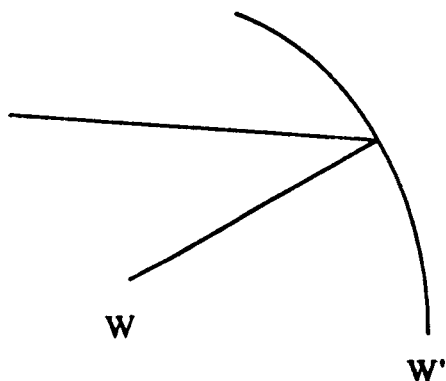


Fig. 13.

One might think that there is little point in fussing about non-strict convexity as a small perturbation could make every convex hypersurface W strictly convex. In fact, this works if V has *strictly positive* curvature ($K(\sigma) > 0$ for all $\sigma \in T(V)$) where a small inward equidistant deformation leads to strict convexity. Similarly, if $K < 0$, one obtains strict convexity with the outward deformation. Also in $V = \mathbb{R}^n$ convex hypersurfaces can be approximated by strictly convex ones (see § 1½). But if we look at a product manifold such as $V = V_0 \times \mathbb{R}^k$, where V_0 is a closed manifold with $\dim V_0 > 0$ and take $W = V_0 \times S^{k-1}$ for the round (and strictly convex!) sphere S^{k-1} in $\mathbb{R}^k$, we shall see that this W is convex but not strictly convex, nor can it be approximated by anything strictly convex.

The geometry of the equidistant hypersurfaces is very simple in the above product example. Namely, $W_{-\varepsilon} = V_0 \times S^{k-1}(\varrho - \varepsilon)$ where ϱ is the radius of the original sphere $S^{k-1} = S^{k-1}(\varrho) \subset \mathbb{R}^k$. Furthermore the region $V_{\varepsilon}^- \subset V$ bounded by $W_{-\varepsilon}$ equals $V_0 \times B^{k-1}(\varrho - \varepsilon)$ for the balls $B^{k-1}(\varrho - \varepsilon)$ bounded by the spheres in $\mathbb{R}^k$. Thus V_{ε}^- and $W_{-\varepsilon}$ disappear at the moment $\varepsilon = \varrho$ and at the last moment V_{ε}^- and $W_{-\varepsilon}$ equal $V_0 \times 0$ in $V = V_0 \times \mathbb{R}^k$.

A similar picture is observed for all manifolds V with $K(V) \geq 0$. As we deform the boundary $W = \partial V$ inward there is the first moment ϱ

$$\varrho = \text{inrad } V \stackrel{\text{def}}{=} \sup_{v \in V} \text{dist}(v, W),$$

such that $\dim V_{\varepsilon}^- = n = \dim V$ for $\varepsilon < \varrho$ and $\dim V_{\varepsilon}^- < n$. If $K(V) > 0$ or if $W = \partial V$ is strictly convex, then the only possibility for V_0 is to be a *single point* since no totally geodesic submanifold V_0 of positive dimension in V can be strictly convex at a non-boundary point $v_0 \in V$ with our definition of strict convexity given above for singular points. On the other hand, in the non-strict case, the inward equidistant deformation may terminate with a subset $V_{\varepsilon_1}^- \subset V$ of positive dimension which is, as we know, convex. This $V_{\varepsilon_1}^-$ is itself a compact manifold with or without boundary. If $V_{\varepsilon_1}^-$ has a boundary, call it $W^1 = \partial V_{\varepsilon_1}^-$, one can shrink $V_{\varepsilon_1}^-$ further with the inward deformation $W_{-\varepsilon}^1$ of W^1 in $V_{\varepsilon_1}^-$. If the process stops at a closed (i.e. without boundary) manifold we are through; if not we go to yet lower dimensional manifolds

$$(V_{\varepsilon_1}^-)_{\varepsilon_2}^-, ((V_{\varepsilon_1}^-)_{\varepsilon_2}^-)_{\varepsilon_3}^-, \dots$$

unless we do arrive at a closed totally geodesic submanifold $V_0 \subset V$ without boundary, called the *soul* of V . Then it is not very hard to show that V is *homeomorphic to a bundle of balls over V_0* . For example, V is homeomorphic to the n -ball in the strict case i.e. where either $K > 0$ or $W = \partial V$ is strictly convex. (The strict case is due to Gromoll-Meyer and the general one to Cheeger-Gromoll (see [Ch-Eb])). This indicates that manifolds V with $K \geq 0$ tend to have a rather simple topology and whenever this topology approaches the critical level of complexity compatible with $K \geq 0$, then the geometry of V becomes very special. For example, if the above V with $K(V) \geq 0$

and convex boundary has non-trivial homology in dimension k , then V contains a closed totally geodesic submanifold (the above soul) of dimension $\geq k$. It is worth noticing that the existence of a totally geodesic submanifold of dimension k for $2 \leq k \leq n-1$, $n = \dim V$, is an exception rather than a rule: there is no such submanifold for a *generic* Riemannian metric g on V .

The above discussion also shows that the homotopy classification of manifolds V with $K(V) \geq 0$ admitting a convex boundary reduces to that for closed manifolds. (Notice that a soul $V_0 \subset V$ has $K(V_0) \geq 0$ as it is totally geodesic in V). This result extends to *non-compact complete* manifolds V without boundary: every such V fibers over its soul which is a closed totally geodesic submanifold $V_0 \subset V$ and the fibers are homeomorphic to some $\mathbb{R}^k$. (This is shown by constructing an exhaustion of V by compact convex domains with convex boundaries, see [Ch-Eb]).

Then one may ask what is a possible homotopy type of a *closed* manifold V with $K(V) \geq 0$.

On the positive side, one knows that every compact homogeneous space, $V = G/H$ for a compact Lie group G , admits a metric with $K \geq 0$. In fact, every bi-invariant metric g on G has $K(g) \geq 0$ according to the following formula (see [Ch-Eb]) which expresses the value of K at the span $\sigma = x \wedge y$ of two orthonormal vectors x and y at the tangent space T_e of G at the identity,

$$K(\sigma) = \frac{1}{4} \|[x, y]\|^2,$$

where $[\cdot, \cdot]$ is the bracket in the Lie algebra $L(G) = T_e$. Then g descends to the metric $\bar{g}$ on V defined by the following condition: the differential of the projection $G \rightarrow V$ *isometrically* sends the *horizontal* subbundle of $(T(G), g)$ to $(T(V), \bar{g})$ (where the horizontal subbundle consists of the vector g -normal to the fibers of the projection which are also the orbits of H in G). One knows, that the curvature of $\bar{g}$ satisfies $K(\bar{x} \wedge \bar{y}) + \frac{3}{4} \|[x, y]_{\text{vert}}\|^2$, where x and y are orthonormal horizontal vectors in T_e and $\bar{x}, \bar{y}$ are their images in $T(V)$, (see [Ch-Eb]), and so $K(\bar{g}) \geq 0$.

Among homogeneous manifolds with $K \geq 0$ the most remarkable are *compact symmetric spaces* V , where for each point $v \in V$ there is an isometric involution $I: V \rightarrow V$ fixing v and having the differential $DI = -Id|_{T_v(V)}$. In fact one may think that the symmetric examples provide the major motivation for the study of $K \geq 0$.

There are some non-homogeneous manifolds with $K \geq 0$ but they do not influence much further our intuition. For example, one believes that the topologically « largest » n -dimensional manifold with $K \geq 0$ is the n -torus T^n (which admits a metric with $K = 0$ as $T^n = \mathbb{R}^n / \mathbb{Z}^n$). One knows in this regard that, indeed, the fundamental group $\pi_1(V)$ for $K(V) \geq 0$ cannot be much greater than $\mathbb{Z}^n$ as it is commensurable with $\mathbb{Z}^n$, (this is already true for $\text{Ricci} \geq 0$, see § 5) and one also knows that the Betti numbers $b_i(V)$ are bounded by universal constants $b_{i,n}$. Yet one is unable to bound $b_i(V)$ by $b_i(T^n) = \binom{n}{i}$. (See [Che] about it).

The above bound on $\pi_1(V)$ becomes radically better if we assume $K(V)$ is *strictly* positive (i.e. $K(\sigma) > 0$ for all $\sigma \in T(V)$). Namely $\pi_1(V)$ is *finite* in this case by the following classical

BONNET THEOREM. *If $K(V) \geq \kappa^2$ then the diameter of V is bounded by*

$$\text{Diam } V \leq \pi / \kappa,$$

where

$$\text{Diam } V \stackrel{\text{def}}{=} \sup_{v_1, v_2 \in V} \text{dist}(v_1, v_2).$$

IDEA OF THE PROOF. Take a minimizing segment between two points in V , say $[v_0, v_1]$ between v_0 and v_1 and look at the spheres $S(\varepsilon)$ of radius ε around v_0 near the points $v \in [v_0, v_1]$. (See Fig. 14 below).

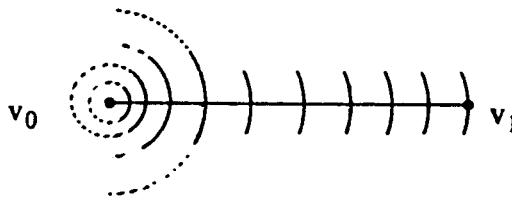


Fig. 14.

The minimizing property of $[v_0, v_1]$ implies that every sphere $S(\varepsilon)$ is smooth at the point $v \in [v_0, v_1]$ with $\text{dist}(v_0, v) = \varepsilon$ for all $\varepsilon < \text{dist}(v_0, v_1)$. (This is a simple general fact which is true without any curvature condition). On the other hand a simple analysis of the tube formula (**) in § 2 shows that A_ε^* must blow up for some finite negative ε . Namely if we start with some A_0^* , then A_ε^* becomes infinite for some ε in the interval $[-\pi/x, 0]$. Thus the length of $[v_0, v_1]$ cannot exceed $-\pi/x$ and the theorem follows.

The critical case for the Bonnet theorem which clarifies the picture is that of V equal to the round sphere $S^n(\varrho) \subset \mathbb{R}^{n+1}$ which has constant curvature $\kappa^2 = \varrho^{-2}$. Here the ball $B(v_0, \varepsilon)$ of radius ε around $v_0 \in S^n$ is convex in S^n until the moment $\varepsilon = \pi\varrho/2$ and for larger ε the boundary sphere $S_{(\varepsilon)}^{n-1} = W_{-\varepsilon}$ of this ball becomes concave. As $\varepsilon \rightarrow \pi\varrho$ the curvature of $W_{-\varepsilon}$ (measured by $A_{-\varepsilon}^*$) blows up to infinity while the complementary region $S^n - B(v_0, \varepsilon)$ becomes « infinitely convex » for $\varepsilon \rightarrow \pi\varrho$ and blows out of existence for $\varepsilon > \pi\varrho$. Now the tube formula shows that all this happens even faster for $K(V) \geq \kappa^2$. Namely the spheres $S(\varepsilon)$ are more concave in V than in $S^n(\varrho)$ and the complement $V - B(v_0, \varepsilon)$ is more convex. In particular this complement must become empty for $\varepsilon > \pi\varrho$ as is claimed by the Bonnet theorem.

Now, the finiteness of $\pi_1(V)$ follows from the Bonnet theorem applied to the universal covering $\tilde{V} \rightarrow V$ which has finite diameter and therefore is compact.

It is also worth looking at the case where K is *non-strictly* positive and π_1 is infinite. For example if π_1 is isomorphic to $\mathbb{Z}^n$, then V (isometrically!) is a *flat torus*, i.e. $V = \mathbb{R}^n/L$ for some lattice in $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^n$ (see [Ch-Eb]).

There is no comparable result of this nature for $b_i(V)$ for $i \geq 2$.

REMARK. The conclusion of Bonnet's theorem remains valid with the following (weaker) assumption on $\text{Ricci}(V)$,

$$\text{Ricci} \geq (n-1)\kappa^2,$$

and the above characterization of flat tori by $\pi_1 = \mathbb{Z}^n$ also remains valid for $\text{Ricci} \geq 0$ (see § 5). On the other hand, the π_1 -corollary to

Bonnet's theorem is sharpened by Singe's theorem (see § 7½) which says that if $n = \dim V$ is *even* and $K > 0$, thtn the fundamental group is either trivial or $\mathbf{Z}_2$, where the latter happens if V is non-orientable.

§ 3½. DISTANCE FUNCTION AND ALEXANDROV-TOPONOGOV THEOREM.

If one looks at a metric space V from a finite combinatorial point of view then one wants to know the properties of the $(N \times N)$ -matrix of the pairwise distances between the points in every subset in V containing N elements. In other words, one may try to characterize V by the set of those metric spaces with N elements which isometrically embed into V . Yet another way to see it is by considering the map of the Cartesian power $V^N = \underbrace{V \times V \times \dots \times V}_N$ into $\mathbb{R}^{N'}$ for $N' = \frac{N(N-1)}{2}$, say $M_N: V^N \rightarrow \mathbb{R}^{N'}$, which relates to each N -tuple of points in V the set of the mutual distances between these points. Then our invariant of V is the image $M_N(V^N) \subset \mathbb{R}^{N'}$. (If there is a natural measure on V as in the Riemannian case one should look at the M_N -push-forward of this measure to $\mathbb{R}^{N'}$). One obvious universal restriction on $M_3(V^3) \subset \mathbb{R}^3$ is expressed by the *triangle inequality*. Then one also knows how to characterize the Euclidean and (Hilbert) spaces in terms of M_N (express the scalar products a_{ij} between the vectors $x_0 - x_i$ in $\mathbb{R}^n$, $i=1, \dots, N-1$, in terms of the squared distances and observe that the matrix a_{ij} is positive semi-definite).

Now we want to state the Alexandrov-Toponogov theorem which characterizes the manifolds V with $K(V) \geq 0$ by the image $M_4(V^4) \subset \mathbb{R}^6$. To abbreviate the formulae we shall write below $|v_1 - v_2|$ for $\text{dist}(v_1, v_2)$. We consider three points v_0, v_1 and v_2 in V and also a point v_3 *between* v_1 and v_2 . This means

$$|v_1 - v_3| + |v_2 - v_3| = |v_1 - v_2|.$$

Then we observe that there exist four points in $\mathbb{R}^2$, v'_0, v'_1, v'_2 and v'_3 between v'_1 and v'_2 , such that

$$|v'_i - v'_j|_{\mathbb{R}^2} = |v_i - v_j|_V \text{ for } i, j = 0, 1, 2$$

and also

$$|v_1 - v_3|_V = |v'_1 - v'_3|_{\mathbb{R}^2}.$$

Then automatically

$$|v'_2 - v'_3|_{\mathbb{R}^2} = |v_2 - v_3|_V$$

and the Euclidean distance $|v'_0 - v'_3|_{\mathbb{R}^2}$ can be expressed by a (well known) formula in terms of the four numbers $|v_0 - v_1|$, $|v_0 - v_2|$, $|v_1 - v_2|$ and $|v_1 - v_3|$. Here is the picture which helps to keep everything in mind.

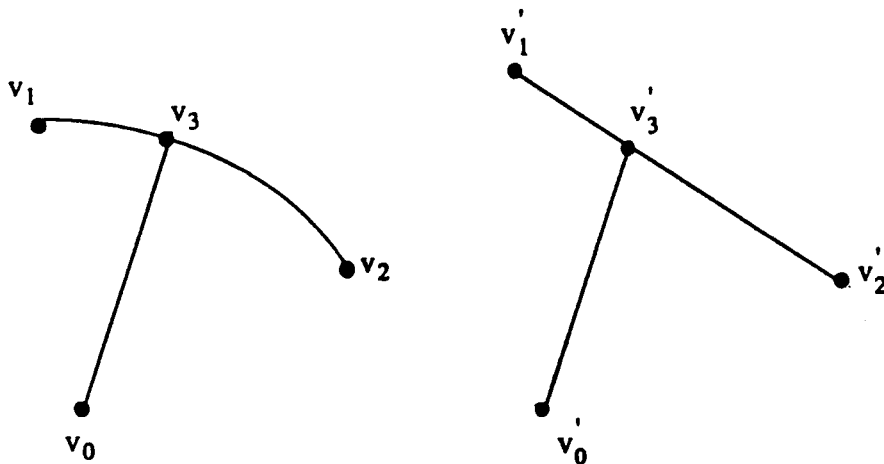


Fig. 15.

THEOREM. If V is complete with $K(V) \geq 0$, then

$$|v_0 - v_3|_V \geq |v'_0 - v'_3|_{\mathbb{R}^2}$$

We call this *AT-inequality* as it was discovered by Alexandrov for $n=2$ and extended by Toponogov to $n \geq 3$.

IDEA OF THE PROOF. One can think of the *AT-inequality* as a kind of concavity relation for the function $d_0(v) = \text{dist}_V(v_0, v)$ on V restricted to the segment $[v_1, v_2]$ as it gives a lower bound on $d_0(v_3)$ in terms of $d_0(v_1)$ and $d_0(v_2)$. More precisely, the theorem says that d_0 is more concave on each segment in V than the Euclidean distance function on the corresponding segment in $\mathbb{R}^2$. Although the function $\text{dist}_V(v_0, \cdot)$ is non-smooth, the concavity type inequalities on geodesic segments follow from the corresponding *local* concavity which, in the smooth

case, can be expressed with the Hessian of $d_0 = \text{dist}(v_0, \cdot)$. Now we invoke the tube formula (**) in § 2 and apply it to the concentric spheres $S(v_0, \varepsilon) \subset V$ for all $\varepsilon > 0$. We can see with this formula (with a little adjustment at non-smooth points of the spheres) that these spheres are less convex (or more concave) than the ε -spheres in $\mathbb{R}^n$. (For example the spheres in the unit sphere $V = S^n(\cdot) \subset \mathbb{R}^{n+1}$ become concave for $\varepsilon > \pi/2$). This « concavity » of the spheres together with the obvious relation $\|\text{grad } d_0(v)\| = 1$ translates into a certain concavity of the function $d_0(v)$ and by inspecting this translation one sees that it gives exactly the desired local version of the AT-inequality on each geodesic segment in V . Q.E.D.

REMARK. It is easy to see that the AT-inequality, when applied to the points v_1, v_2 and v_3 which are infinitely close to v_0 , yields $K \geq 0$ at v_0 . Thus AT is equivalent to $K \geq 0$.

One may wonder if there are further universal metric inequalities related to the curvature, but none besides AT is known today.

However, the inverse AT, namely $|v_0 - v_3|_V \leq |v'_0 - v'_3|_{\mathbb{R}^2}$ is known for complete manifolds V with $K(V) \leq 0$, but here one must additionally assume that V is simply connected. Also, the manifolds V with Ricci $V \geq 0$ satisfy certain metric inequalities (see § 5) but these depend on $\dim V$.

§ 32/3. SINGULAR SPACES WITH $K \geq 0$.

One may try, following Alexandrov, to develop the theory of metric length spaces with $K \geq 0$ using AT as an axiom. Now such a generalized space V of positive curvature may be singular, and in fact, even topologically singular. For example, if we start with a smooth V with $K(V) \geq 0$, acted upon by a finite isometry group Γ , then the AT-inequality for V implies (by an elementary « synthetic » argument) that for V/Γ , which is a singular space if the action of Γ is non-free. The geometry of possible singularities of V can be also seen in convex subsets $V \subset \mathbb{R}^n$ which are regarded (mildly) singular at the boundary points (even if the boundary ∂V is smooth) and also at the singular points of non-smooth convex hypersurfaces $V \subset \mathbb{R}^{n+1}$ with the induced length structure. (One can replace $\mathbb{R}^{n+1}$ by an arbitrary smooth manifold of dimension $n+1$ with $K \geq 0$). An instructive example is the boundary of the convex hull of a generic curve in $\mathbb{R}^{n+1}$.

Another kind of a singular space with $K \geq 0$ is the unit *Euclidean cone* over a manifold S with $K(S) \geq 1$. This cone is singular at the vertex unless S is the round sphere of constant curvature 1, where the cone is just the unit Euclidean ball bounded by the sphere. Notice that one can allow singular points in the above S with $K \geq 1$ and instead of the cone one may take the *suspension* which is the union of two cones over S joined along S .

Finally we observe that the Cartesian product of spaces with $K \geq 0$ has $K \geq 0$ and also that curvature remains positive if we go to a quotient V/G for a compact group G of isometries of V . (We have mentioned above the case of a finite group).

These examples make singular spaces worth of a study and one can extend some known results from smooth manifold to the general case. However one has not developed yet the theory of convex hypersurfaces W in such spaces. For example, one does not know if the *induced (intrinsic) metric* of W has $K \geq 0$. Another question is whether the inward equidistant deformations W_ε are convex (*).

Next one wishes to know the structure of the singularities of V . The known examples indicate that V should be *topologically conical* at each point with *roughly conical geometry*. Recently G. Perelman proved the topological conical property of V , which implies, in particular, local contractibility of V , (see [B-G-P]) but the conical geometry remains conjectural (**). (Notice that these questions are closely related to the geometry lying behind the bound on $b_4(V)$ mentioned earlier).

The final group of questions concerns the structure of the singular loci of spaces with $K \geq 0$. It is known that the singular points must form a rather rare set. Namely, every n -dimensional space V contains an open dense subset which is locally by-Lipschitz homeomorphic to $\mathbb{R}^n$. Moreover, for each $\varepsilon > 0$ there exists an open dense subset $V_\varepsilon \subset V$ which is locally ε -Euclidean in the following sense. For each point $v \in V_\varepsilon$ there exists a flat metric dist_E on some neighbourhood $V' \subset V_\varepsilon$ of v which is ε -bi-Lipschitz to the metric coming from V , that is

$$1 - \varepsilon \leq \text{dist}_V / \text{dist}_E \leq 1 + \varepsilon$$

on U (see [B-G-P]).

(*) « Yes », according to Perelman.

(**) This is now proved by Perelman.

This is still far from what one expects by looking at convex hypersurfaces in $\mathbb{R}^{n+1}$ which are known to be *almost everywhere* C^2 -smooth. The above statement concerns the C^1 -structure (in the case of convex hypersurfaces). It seems that once the C^1 -structure of the singularities is completely understood the C^2 -refinement will follow by mere analysis (*). On the other hand one would need a new geometric idea in order to prove that the n -dimensional Hausdorff measure of the singular set is zero (**). Intuitively, each singular point carries infinite positive curvature while the integral curvature (properly defined) must be universally bounded as one believes but is unable to prove even in the smooth case for $n \geq 3$. (For $n = 2$ such a bound is obtained with the *Gauss-Bonnet theorem* which equates the total curvature of a surface V with $2\pi\chi(V)$. This theorem extends to higher dimensional manifolds but it provides a non-trivial information on the total amount of curvature only for $n = 2$ and $n = 4$, where one may use for $n = 4$ the universal bound on $|\chi(V)|$ which follows from that on the Betti numbers for $K(V) \geq 0$. Compare the discussion following the statement of the sphere theorem in the next §).

§ 3 $\frac{3}{4}$. THE SPHERE THEOREM AND EQUIDISTANT DEFORMATION OF IMMERSED HYPERSURFACES.

The modern period in the global Riemannian geometry starts, according to M. Berger (see [Ber]₁, [Ber]₂), with the work of Rauch in the early fifties who proved among other things that if the sectional curvatures of a closed simply connected Riemannian manifold V are sufficiently close to those of a round sphere, then V is *homeomorphic* to the sphere. (One needs the simply connectedness assumption in order to rule out such manifolds as the real projective space $P^n = S^n/\mathbb{Z}_2$ and the lense spaces $S^{2m-1}/\mathbb{Z}_k$ which have constant positive curvature but not homeomorphic to spheres).

The closeness of the curvature $K = K(V): Gr_2 V \rightarrow \mathbb{R}$ to the (constant) curvature of a sphere is customarily expressed by the inequality.

$$ca < K < a,$$

where $a > 0$ and $0 < c < 1$. Here one thinks of the constant a as

(*) This analysis was started by Otsu and Shioya and it is rather subtle.

(**) This is proven by Otsu and Shioya and also appears in the final version of [B-G-P].

of the curvature of the round sphere of radius $a^{-\frac{1}{2}}$ and c , called the *pinching constant*, measures the allowed amount of non-constancy in $K(V)$. Notice, that by scaling one can reduce the general case to that of $c = 1$ and then the inequality

$$c < K < 1$$

says that the sectional curvatures of V are strictly pinched between those of the unit sphere and the one of radius $c^{-\frac{1}{2}}$.

Rauch conjectured that the best pinching constant in his theorem must be $1/4$. This value is motivated by the fact that the *complex projective space* $\mathbb{C}P^n$, which goes next in roundness after S^n , has the sectional curvatures spread over the closed interval $[1/4, 1]$ for the natural $U(n+1)$ -invariant (Fubini-Study) metric on $\mathbb{C}P^n$. Notice that also the quaternion projective spaces and the projective Cayley plane carry natural homogeneous (even symmetric) metrics with $1/4 \leq K \leq 1$.

The solution of the Rauch problem (achieved in the middle sixties by Berger and Klingenberg) is now known as

THE SPHERE THEOREM. *If a closed simply connected manifold V has*

$$\frac{1}{4} < K(V) < 1,$$

then V is homeomorphic to S^n .

REMARKS

- (a) One still does not know if the above V is *diffeomorphic* to S^n (but this is known for a more narrow pinching).
- (b) If V is not simply connected, the theorem applies to the universal covering of V .
- (c) For the non-strict pinching.

$$\frac{1}{4} \leq K(V) \leq 1,$$

the sphere theorem is complemented by the *rigidity theorem* of Berger which says that if a closed simply connected manifold V with

$\frac{1}{4} \leq K(V) \leq 1$ is non-homeomorphic to S^n then it is necessarily *isometric* to the projective space over complex numbers, quaternions or over Cayley numbers with the standard homogeneous metric.

SKETCH OF THE PROOF OF THE SPHERE THEOREM. If $n = \dim V = 2$, then the result follows from the *Gauss-Bonnet theorem*

$$\int_V K(v) dv = 2\pi\chi(V),$$

for the Euler characteristic $\chi(V)$. Thus the positivity of the curvature K alone (without pinching) implies $\chi(V) > 0$ and hence we can identify V with S^2 as we assume V is simply connected.

REMARK. The Gauss-Bonnet theorem generalizes to all dimensions by

$$\int_V \Omega dv = \chi(V),$$

where $\Omega = \Omega(v)$ is expressible at each v as a certain polynomial in the components of the curvature tensor. One knows that for $\dim V = 4$ the sign conditions $K > 0$ and $K < 0$ both imply $\Omega(v) > 0$. It follows that if the curvature of V does not change sign then $\chi(V) > 0$.

This is not very interesting for $K > 0$, where the universal covering $\tilde{V}$ of V is compact and where $\pi_1(\tilde{V}) = 0 \implies b_1 = b_3 = 0$. So the remaining Betti numbers contributing to $\chi(\tilde{V}) = \chi(V)$ are even: b_0 , b_2 and b_4 . On the other hand the topological conclusion, $\chi(V) > 0$ for closed 4-dimensional manifolds of strictly *negative* curvature cannot be obtained to-day by any other method.

If $\dim V \geq 6$, the sign of Ω is not controlled by the sign of K anymore. Yet Chern conjectures that if $n = 4k$, then $K > 0$ and $K < 0$ imply $\chi > 0$ and for $n = 4k + 2$ the sign of χ equals that of K if K is everywhere strictly positive or strictly negative on V .

Now, we are back to the sphere theorem for $n \geq 3$. We recall that the pinching condition $1/4 < K < 1$ means that the sectional

curvatures of our V are strictly smaller than those ($=1$) of the unit sphere S^n and greater than the curvature of the sphere $2S^n$ of radius 2. Then we take a point $v \in V$ and consider the concentric balls $B(v, r) \subset V$. For a small radius $r > 0$ every such ball has smooth convex boundary. As the ball grows three bad things may happen to it.

(1) The boundary (sphere) may lose convexity and even become everywhere *concave*. For example this happens to $B(v, r) \subset S^n$ for $r > \pi/2$ and it happens in $2S^n$ for $r > \pi$. The condition $K(V) > \frac{1}{4}$ implies, by the tube formula, that $B(v, r)$ necessarily becomes concave for $r \geq \pi$.

We shall see below, that this concavity of the boundary sphere is not a bad thing after all. On the contrary, it turns out very useful as we look at this sphere from outside where it appears convex.

(2) The boundary sphere may develop *double points*. To see how it happens we look at the example of the (flat) cylinder $V = S^1 \times \mathbb{R}$. The universal covering of the cylinder is the Euclidean plane $\mathbb{R}^2$ and the ball $B(v, r)$ in V is the image of a Euclidean 2-ball (disk) B in $\mathbb{R}^2$. As r becomes greater than the half-length of S^1 the map of B to $S^1 \times \mathbb{R}$ becomes non-one-to-one and we can see B wrapping around the cylinder as r grows.

One observes a similar picture in an arbitrary V with the so-called *exponential map* $e : T_v(V) \rightarrow V$ which sends each vector $\tau \in T_v(V)$ to the second end of the geodesic segment in V issuing in the direction of τ and having length $= \|\tau\|$. The ball $B(v, r)$ in V equals the exponential image of the Euclidean r -ball $B \subset T_v(V)$ and the double points of the boundary sphere of $B(v, r)$ are the images of those points in $S = \partial B$ where the map $e|_S$ is non-one-to-one. For example, if $V = S^n$ then the exponential map is one-to-one on the balls $B \subset T_v(V)$ of radii $< \pi$ but the sphere $S \subset T_v(V)$ of radius π is sent by e to the single point in $V = S^n$ which is opposite to v in S^n .

The double points, unquestionably, cause a serious complication of the picture. Yet this will be taken care of in the above concave case.

(3) The geometric image of the ball $B \subset T_v(V)$ wrapping around V under the exponential map $e : T_v(V) \rightarrow V$ is adequate in-so-far as the map e is an *immersion* i.e. is locally one-to-one (and hence locally homeomorphic) on B . A sufficient condition for that is the *regularity* of $e|B$ which means $\text{rank}(De) = n$, where De denotes the differential of e and $n = \dim V$. If $K(V) < 1$ then the tube formula implies that *the map e is regular on the ball $B \subset T_v(V)$ of radius π* . Here one may ignore possible self-intersections of the boundary spheres $S(v, r) = \partial B(v, r)$ by looking at the exponential map e on a narrow sector $A \subset B$ around a given straight segment in $B \subset T_v(V)$ joining the origin (i.e. the center of B) with a point $s \in S = \partial B$. Then the intersections of the concentric spheres in $T_v(V)$ with A go under e into a family of *smooth* mutually equidistant hypersurfaces in V , see Fig. 16 below.

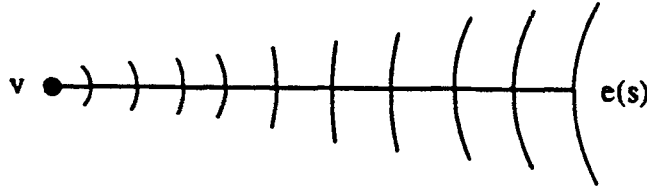


Fig. 16.

Thus, if $\frac{1}{4} < K(V) < 1$, we have an *immersion* of $B \subset T_v(V)$ into V , such that the boundary of this immersed ball is *concave* in V . Now we want to construct another immersion of some ball B' to V which would bound $e(S) = e(\partial B) \subset V$ from the convex side. Thus we would obtain an immersion of the sphere $S^n = B \cup B'$ into V , where the two balls B and B' are glued together over the common boundary S . Notice that such an immersion is a covering map (since S^n is a closed manifold and $n = \dim V$) and so the sphere theorem comes along with an immersed B' in V filling in $e(S) \subset V$ from the convex side. The existence of such B' is insured for $n \geq 3$ by the following

FILLING LEMMA (compare § 1½). *Let V be a complete Riemannian manifold of dimension $n \geq 3$ with $K(V) > 0$ and let $e : S \rightarrow V$ be a (topological) immersion of a closed connected $(n-1)$ -dimensional manifold S into V ; If the immersed hypersurface is locally convex*

in V , then it is diffeomorphic to S^{n-1} . Moreover, there exists a ball B' which bounds this $S^{n-1} = S$ and an immersion $B' \rightarrow V$ extending $e: S \rightarrow V$, such that the immersed ball B' fills in $e(S)$ from the convex side.

REMARK. Notice that the image $e(S) \subset V$ does not have to be convex in any sense. The local convexity means that each point $s \in S$ has a neighbourhood $W \subset S$ on which the map e is one-to-one and whose image is (locally) convex in V . A typical example is provided by the locally convex immersed curves in the plane as in Fig. 17 below. (Compare Fig. 5 in § 1½ and Fig. 9 in § 3).

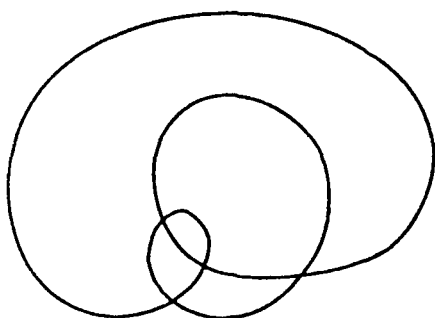


Fig. 17.

Also notice that a closed immersed curve S in $\mathbb{R}^2$ does not bound any immersed disk unless S is embedded (i.e. has no double point). This does not contradict Filling Lemma where we assume $n \geq 3$.

IDEA OF THE PROOF. Given an immersed locally convex hypersurface in V , we may try the inward locally equidistant deformation which applies simultaneously to all small embedded neighbourhoods of $e(W) \subset V$, see Fig. 18 below.

The equidistant deformation sketched in Fig. 18 develops a (cuspidal) singularity at a certain moment ε_0 and cannot be continued beyond ε_0 . However, an elementary argument as in § 1½ shows that no such singularities appear for deformations of locally convex (pieces of) hypersurfaces in $\mathbb{R}^n$ for $n \geq 3$. This conclusion extends to all Riemannian manifolds V with local geodesic coordinates systems at the points $x \in e(S)$ where one is afraid of singularities. Thus one can

continue the equidistant deformation as long as the local convexity is preserved. (In fact one needs here *strict* convexity as it is stable under small perturbations and goes along as one passes to the Euclidean picture in geodesic coordinates). Now, since $K(V) > 0$, the

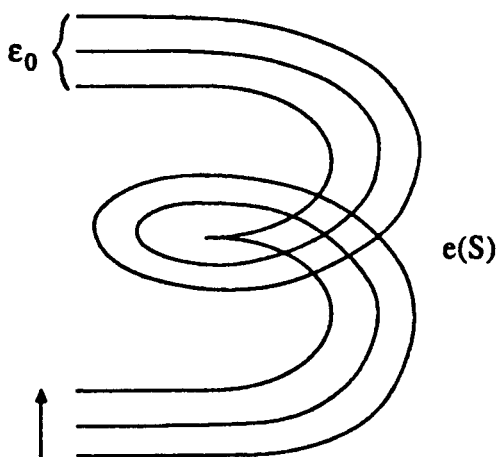


Fig. 18.

convexity only improves in the course of the inward deformation and so S eventually shrinks to a single point $v' \in V$. The totality of the deformed hypersurfaces form a multiple domain in V filling in $e(S)$ which is a manifold B' with $\partial B' = S$ immersed into V , and our locally equidistant hypersurfaces

$$S_\varepsilon = \{b \in B' \mid \text{dist}(b, \partial B') = \varepsilon\}$$

for the Riemannian metric in B' induced by the immersion $B' \rightarrow V$. In this case we know by the Gromoll-Meyer theorem (see § 3) that B' is homeomorphic to the ball B^n (this homeomorphism is easily constructed with the family of convex S_ε shrinking to a point in B') and the proof of Filling Lemma is concluded. (See [Esch] for a detailed argument).

REMARK. Coming back to the theorem, we observe that the simply connectedness of V implies that the (covering) map $S^n = B \cup B' \rightarrow V$ is injective. So the exponential map $e : B \rightarrow V$ is injective after all!

§ 4. NEGATIVE SECTIONAL CURVATURE.

Here $V = (V, g)$ is a complete Riemannian manifold with $K(V) \leq 0$. One can easily derive from the tube formula (see (**)) in § 2) that the condition $K \leq 0$ is equivalent to the preservation of convexity under outward equidistant deformations $W_{\varepsilon \geq 0}$ of convex hypersurfaces W in V , for small ε , see Fig. 19.

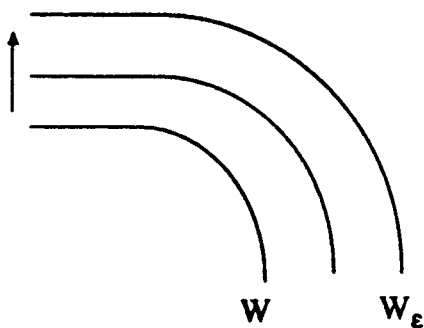


Fig. 19.

This is quite similar to the case $K \geq 0$. But what happens here for large $\varepsilon \geq 0$ is different. Namely, the tube formula shows that the normal geodesic map $d : W \rightarrow V_\varepsilon$ is regular (i.e. an immersion) for all $\varepsilon > 0$ and $d_\varepsilon(W) \hookrightarrow V$ is a locally convex immersed hypersurface. The only problem comes from possible self-intersections of this hypersurface. The simplest case to consider is where W_ε are concentric ε -spheres around a point $v_0 \in V$. In other words we look at the *exponential map* $e : H_{v_0}(V) \rightarrow V$ which locally isometrically sends each straight ray τ in the tangent space $T_{v_0}(V)$ to the geodesic ray in V issuing from v_0 and tangent to τ at v_0 . This e (obviously) maps every Euclidean r -ball $B(0, r) \subset T_{v_0}(V)$ onto the r -ball $B(v_0, r) \subset V$ (this is true for all complete V with no assumption on the curvature) and for $K \leq 0$ the tube formula (applied to the spheres) shows that e is an *immersion*. Moreover, by the tube formula the map e is *infinitesimally distance increasing*, i.e. the metric $\tilde{g}$ on $T_{v_0}(V)$ induced by e from g on V is (non-strictly) greater than

the Euclidean metric g_0 on $T_v(V)$. Since g_0 is complete the (greater) metric $\tilde{g}$ is also complete which then implies by an easy argument that e is a *covering map* and since $T_v(V) = \mathbb{R}^n$ is simply connected this is the *universal covering*. From this one immediately derives the classical

THEOREM OF CARTAN-HADAMARD. *The universal covering of a complete n -dimensional manifold V with $K \leq 0$ is diffeomorphic to $\mathbb{R}^n$. In particular, if V is compact without boundary then the fundamental group $\pi_1(V)$ is infinite.*

The proof of the above theorem shows that if V is *simply connected* then the exponential map $T_v(V) \rightarrow V$ is a bijective diffeomorphism for each $v \in V$. It follows that every two points v and v' in V can be joined by a *unique* geodesic segment in V , which (because of uniqueness) must be *minimizing* in V and that the distance function $\text{dist}(v_1, v_2)$ is smooth for $v_1 \neq v_2$. Then, by applying the tube formula to the ε -neighbourhoods of the diagonal $\Delta = V \subset V \times V$, one can easily prove that dist is a *convex function* on $V \times V$. This means dist is convex on every plane in $V \times V$ which is the Cartesian product of two geodesics in V . In particular, every ball $B(v, \varrho) \subset V$ is convex.

The above discussion indicates a certain duality between manifolds with $K \geq 0$ and *simply connected* manifolds with $K \leq 0$. This duality shows up even better for the monotonicity of the balls

$$B(v, \lambda\varrho) \supseteq \lambda B(v, \varrho)$$

for all $v \in V$, $\varrho \geq 0$ and $\lambda \geq 1$ (compare § 21½) and the Alexandrov-Toponogov inequality

$$|v_0 - v_3|_V \leq |v'_0 - v'_0|_{\mathbb{R}^n}$$

(see Fig. 15 in § 31½), which is a sharpening of the convexity of the function $\text{dist}(v_0, \cdot)$ on V . Notice that both inequalities need $\pi_1 = 0$ as well as $K \leq 0$.

But the duality does not seem to extend much further. In fact the essential features of manifolds V with $K(V) \leq 0$ and $\pi_1(V) = 0$

are seen (if we know where and how to look) *asymptotically* as we go to *infinity* in V , which has no counterpart for $K \geq 0$.

Now, if we turn to closed manifolds V with $K \leq 0$ then they appear as quotient spaces of the universal covering $\tilde{V}$ by the Galois group $\Gamma = \pi_1(V)$ which isometrically acts on $\tilde{V}$. Even if we fix $\tilde{V}$ the variety of different $\Gamma \subset \text{Iso } \tilde{V}$ and the corresponding $V = \tilde{V}/\Gamma$ may be quite astounding. The richest source of examples is the 3-dimensional space $\tilde{V} = H^3$ of *constant negative curvature* which can also be defined as $PSL_2 \mathbb{C}/SO(3)$ with an invariant Riemannian metric. Discrete subgroups $\Gamma \subset PSL_2 \mathbb{C}$ conformally act on the Riemann sphere S^2 and their study under the name of the *Kleinian groups* has been conducted for many years in the framework of complex analysis. A new development emphasizing $K = -1$ was started about 12 years ago by Thurston who has created (or discovered) a magnificent geometric world in dimension 3. There is nothing comparable to this for $K \geq 0$.

Notice that negative curvature accompanies every *non-compact semi-simple group* G . Namely, if we divide G by the *maximal compact* subgroup $H \subset G$ then, by the compactness of H , the space $\tilde{V} = G/H$ admits a G -invariant metric g . Such a metric is *complete* (this is elementary) and by a well known theorem of E. Cartan $K(g) \leq 0$. This curvature is *strictly* negative if and only if $\text{rank}_{\mathbb{R}} G = 1$, and $K(g)$ is *constant* if and only if G is locally isomorphic to $O(n, 1)$. The compact manifolds V covered by $\tilde{V}$ are associated to discrete subgroups $\Gamma \subset G$ which are usually produced by arithmetic constructions.

The above mentioned examples by no means exhaust all compact manifolds with $K \leq 0$. In fact, there is no slightest chance of any meaningful classification of such manifolds (but there may exist a classification of compact manifolds with $K \leq 0$ modulo those with $K < 0$). On the other hand for $K \geq 0$ a rough description of all compact manifolds looks quite feasible. (For example, every convex subset $V \subset \mathbb{R}^n$ is roughly equal to some *solid* $[0, \ell_1] \times [0, \ell_2] \times \dots \times [0, \ell_n]$ and every flat torus roughly is the Riemannian Cartesian product of the circles S_i of certain lengths ℓ_i). The situation here is vaguely similar to the classification of algebraic varieties (e.g. sur-

faces). The varieties of general type look kind of *hyperbolic* (which corresponds to $K \leq 0$) and are unclassifiable. On the contrary, special varieties (such as Fano varieties) sometimes can be classified in every fixed dimension. These correspond to manifolds with $K \geq 0$.

In order to have a better idea of a possible classification for general manifolds with $K \geq 0$ one may look at two examples where the problem was, in principle, solved. The first example is that of *constant curvature 1*, where $V = S^n/\Gamma$ for some finite isometry group Γ freely acting on the unit sphere $S^n \subset \mathbb{R}^{n+1}$. The second class of examples is given by the *flat* Riemannian manifolds ($K=0$) which are $\mathbb{R}^n/\Gamma$ for so-called *crystallographic* groups Γ (isometrically acting) on $\mathbb{R}^n$. In both cases one has a good overall picture of such groups Γ as well as a possibility of classification for every fixed n (but such classification quickly becomes a mess for large n and is not very appealing).

HYPERBOLIC GROUPS. The main topological problem concerning the spaces with $K \leq 0$ is a characterization of the groups Γ which may serve as fundamental groups of such spaces. If we only assume that V with $K \leq 0$ is *complete* then it is unclear if there is any non-obvious restriction on $\Gamma = \pi_1(V)$. (The « obvious » condition is the existence of a free discrete action on a Euclidean space as $\tilde{V}$ is diffeomorphic to $\mathbb{R}^n$). On the other hand, if V is compact, then there are many specific properties of $\Gamma = \pi_1(V)$. For example, Γ *contains a free group on two generators unless V is a flat manifold*. (In the latter case Γ contains a subgroup $\Gamma' = \mathbb{Z}^n$ of finite index). The idea of the freedom theorem goes back to Felix Klein who proved it for groups acting on the hyperbolic space H^3 with $K = -1$. The generalization to the subgroups $\Gamma \subset SL_n$ acting on the manifold $SL_n/SO(n)$ with $K \leq 0$ is a famous result by J. Tits. The extension to the variable *strictly negative* curvature is due to P. Eberlein and in the general case the freedom theorem was recently proven by W. Ballmann on the basis of a deep analysis of the nature of « non-strictness » for $K \leq 0$.

To get some idea on the difference between strictly and non-strictly negative curvature we first recall the old result by Preissmann saying that every *Abelian subgroup* $A \subset \Gamma = \pi_1(V)$ is *free cyclic*,

assuming V is a closed manifold with $K < 0$. Furthermore, if we only assume $K \leq 0$ and insist on the existence of a free Abelian subgroup A in $\pi_1(V)$ of rank $k \geq 2$, then V contains an isometrically and geodesically immersed flat torus $T^k \hookrightarrow V$ and so $K(o) = 0$ on all planes tangent to T^k .

This example shows how the « critical » topology may influence the geometry for $K \leq 0$. This phenomenon (which is somewhat similar to what happens for $K \geq 0$) is seen even better in the following striking result due to Gromoll-Wolf and Lawson-Yau.

SPLITTING THEOREM. *If the fundamental group Γ of a closed manifold V with $K \leq 0$ splits into the direct product by $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_1 and Γ_2 have trivial centers, then V isometrically splits, i.e. $(V, g) = (V_1 \times V_2, g_1 \oplus g_2)$, where $\pi_1(V_i) = \Gamma_i$, $i = 1, 2$.*

Notice that the condition $\text{Center } \Gamma_i = 0$ is essential as is seen in the example of a non-split flat torus. (These tori also suggest a correct generalization to the case $\text{Center} \neq 0$).

Observe that if $\dim V_i > 0$ $i = 1, 2$, then $K(V = V_1 \times V_2)$ vanishes on many 2-planes $\sigma \in T(V)$. Namely, K vanishes on the Cartesian products of geodesics $\gamma_1 \times \gamma_2 \subset V$ for $\gamma_i \subset V_i$, as these products are isometric to $\mathbb{R}^2$ geodesically immersed into V . (This is always true for Riemannian products with no regard for the curvature).

Some results similar to the splitting theorem were more recently established for manifolds V with $K(V) \leq 0$ where there are « sufficiently many » plane σ on which $K(\sigma) = 0$ (see [B-G-S]). This has raised hope for a possible reduction of the general case of $K \leq 0$ to that where $K < 0$. On the other hand, one can axiomatize the essential features of the fundamental groups $\pi_1(V)$ for $K(V) < 0$ and study them independently of differential geometry. This brings to life a new class of groups called *hyperbolic groups* which include the above $\pi_1(V)$ as well as the so-called *small cancellation groups*. The study of the hyperbolic groups appears to be at the moment the main avenue in (strictly) negative curvature. (See [Gh-Ha] for more about it).

§ 5. RICCI CURVATURE.

Let us recall the basic tube formula (see (**) in § 2), which relates the second derivative of the (induced) metric g_ε on the equidistant hypersurface $W_\varepsilon \subset V$ to the sectional curvature $K(V)$ expressed by the symmetric operator $B = B$ on $T(W)$. This formula reads

$$\frac{d}{d\varepsilon} A_\varepsilon^* = -(A_\varepsilon)^2 + B$$

where A_ε^* is the shape operator which is just another face of the second fundamental form Π of W_ε , that is

$$\Pi^{W_\varepsilon} \stackrel{\text{def}}{=} \frac{1}{2} \frac{d}{d\varepsilon} g_\varepsilon.$$

Let us see what happens if we take the traces of the operators A_ε^* and B in the tube formula. First, the trace of A_ε^* is the same as the trace of Π^{W_ε} relative to g_ε , which is called the *mean curvature* $M(W_\varepsilon)$ of W_ε and which equals to the sum of the principal curvatures of W_ε (as follows from the definitions of all these curvatures, see § 0). Then it is clear with the above formula for Π^{W_ε} that $M(W) = \text{Trace}_{g_0} \Pi^{W\delta}$ measure the ε -variation of the Riemannian volume of g_ε , that is

$$(*) \quad \frac{d \text{Vol}_\varepsilon^*}{d\varepsilon} = M(W_\varepsilon) \text{Vol}_\varepsilon^*,$$

where Vol_ε denotes the Riemannian volume density on W_ε (recall that a *density* on W_ε is not a function but rather a $(n-1)$ -form modulo $\pm$ sign). Then Vol_ε^* denotes the pull-back of Vol_ε to $W = W_0$ under the normal geodesic map $d_\varepsilon : W \rightarrow W_\varepsilon$. Now (*) makes sense as the left hand side is a density on W as well as the right hand side being the product of a density by a function.

One can equivalently express (*) by using the background metric g_0 on $W = W_0$ as follows. Let $J(w, \varepsilon)$ denotes the Jacobian of the map d_ε at $w \in W$. Then (*) becomes

$$\frac{d(J(w, \varepsilon))}{d\varepsilon} = J(w, \varepsilon) \text{Trace } A_\varepsilon^*,$$

as

$$\frac{dJ}{d\varepsilon} / J = \frac{d \text{Vol}_\varepsilon}{d\varepsilon} / \text{Vol}_\varepsilon.$$

Yet another way to express (*) is by

$$\frac{d}{d\varepsilon} \log J(w, \varepsilon) = \text{Trace } A_\varepsilon^*,$$

or, if one integrate over W_ε , one gets

$$\frac{d}{d\varepsilon} \text{Vol}(W_\varepsilon) = \int_W J(w, \varepsilon) \text{Trace } A_\varepsilon^* dw.$$

It may be worth noticing at this point that the $(n-1)$ -dimensional volume of the equidistant hypersurface W_ε in the above formula equals the derivative with respect to ε of the n -dimensional volume of the « band » between W_0 and W_ε that is the image of the map

$$W_0 \times [0, \varepsilon] \rightarrow V \text{ by } (w, \varepsilon) \mapsto d_\varepsilon(w).$$

More generally, one may take the ε -neighbourhoods $V_\varepsilon^+ \subset V$ of a fixed subset $V_0 \subset V$. Then the $(n-1)$ -dimensional volume of the boundary $W_\varepsilon = \partial V_\varepsilon^+$ satisfies

$$\text{Vol } W_\varepsilon = \frac{d}{d\varepsilon} \text{Vol } V_\varepsilon^+,$$

(where $V_\varepsilon^+ \stackrel{\text{def}}{=} \{v \in V \mid \text{dist}(v, V_0) \leq \varepsilon\}$). Notice that the above formula remains valid even if the hypersurfaces W_ε are non-smooth. (This is first proven in $\mathbb{R}^n$ and then brought to V with Euclidean metrics is first proven in infinitesimally approximating g at the points $v \in V$, compare § 1).

Now let us turn to $\text{Trace } B$. Recall, that the operator $B = B_S$ was assigned to every co-oriented hyperplane $S \subset T_v(V)$ in each tangent space $T_v(V)$. Every such S is defined by the unit normal vector $\nu = S^\perp$ (unique because of the coorientation) and then $\text{Trace } B$ becomes a function on the unit tangent bundle of V . Then a simple (infinitesimal) algebraic consideration shows that this function is

quadratic on each fiber. That is there exists a (necessarily unique) quadratic form on V , called the *Ricci tensor*, such that

$$\text{Trace } B_s = - \text{Ricci } (v, v)$$

for $v = S^\top$.

With this Ricci we have the following

TRACED TUBE FORMULA.

$$(**) \quad \frac{d}{d\varepsilon} M(W_\varepsilon) = - \text{Trace } A_\varepsilon^2 - \text{Ricci } (v_\varepsilon, v_\varepsilon)$$

where v_ε is the inward (or outward, which makes no difference) normal unit field on W_ε . If we combine this with the above discussion on Vol_ε and the Jacobian of d_ε we have

$$(+) \quad \frac{d^2}{d\varepsilon^2} \log J(w, \varepsilon) = - \text{Trace } A_\varepsilon^2(w) - \text{Ricci}(v_\varepsilon(w), v_\varepsilon(w)).$$

Notice that $\text{Trace } A_\varepsilon^2$ equals the squared norm $\|H^\varepsilon\|_g^2$ measuring the overall curvature of W_ε in V by the sum of principal curvatures.

Now, we may recall the definition of the sectional curvature K (see § 2) and observe the following formula expressing Ricci by K . Let $v = v_1, v_2, \dots, v_n$ be an orthonormal frame in $T_v(V)$ and let $\sigma_2, \dots, \sigma_n$ denote the planes spanned by the pairs of vectors $(v_1, v_2), (v_1, v_3), \dots, (v_1, v_n)$. Then by an easy computation,

$$\text{Ricci } (v, v) = \sum_{i=2}^n K(\sigma_i).$$

Thus the unit round n -sphere S^n has $\text{Ricci} = (n-1)g$ for the spherical metric g . (It is no accident that $\text{Ricci}(S^n) \rightarrow \infty$ for $n \rightarrow \infty$, but an important property of S^n with many implications, see [Mi-S]).

MANIFOLD WITH $\text{Ricci } V \geq 0$. If $\text{Ricci} \geq 0$ (i.e. positive semidefinite), then the traced tube formula shows that the second derivative (va-

riation) of $\log \text{Vol}_\epsilon$ is negative or, equivalently

$$\frac{d^2}{d\epsilon^2} \log J(v, \epsilon) \leq 0.$$

Conversely one can easily see that this inequality is equivalent to $\text{Ricci} \geq 0$.

One can derive a sharper inequality from the traced tube formula (+) by observing that

$$-\text{Trace} A^2 \leq (n-1)^{-1} (\text{Trace} A)^2$$

for $n-1 = \dim W = \dim V - 1$. Then (+) and the preceding formula for $\frac{d}{d\epsilon} J$ yield

$$(++)\quad \frac{d^2 \log J}{d\epsilon^2} \leq -(n-1) \left(\frac{d \log J}{d\epsilon} \right)^2 - \text{Ricci}$$

which becomes

$$(+*)\quad \frac{d^2 \log J}{d\epsilon^2} \leq -(n-1) \left(\frac{d \log J}{d\epsilon} \right)^2$$

for $\text{Ricci} \geq 0$. In terms of the mean curvature M of W_ϵ the equivalent inequalities are

$$(+*)'\quad \frac{d^2 \log \text{Vol}_\epsilon}{d\epsilon^2} = \frac{dM}{d\epsilon} \leq -(n-1)^{-1} M^2 - \text{Ricci}$$

and

$$(+*)''\quad \frac{d^2 \log \text{Vol}_\epsilon}{d\epsilon^2} = \frac{dM}{d\epsilon} \leq -(n-1) M^2 \text{ for } \text{Ricci} \geq 0,$$

where $\frac{d \log \text{Vol}_\epsilon}{d\epsilon}$ is identified with M according to (*).

In order to emphasize the similarity of the above inequalities with those for $K \geq 0$ in §§ 2, 3 we introduce the following terminology.

A cooriented hypersurface $W \subset V$ is called *mean convex* if $M(W) \geq 0$. Then a domain $V_0 \subset V$ with smooth boundary is *mean*

convex if the boundary is mean convex. Clearly, convexity implies mean convexity as the latter requires the positivity of *all* principal curvatures not only of their mean. Also notice, that with our terminology, round spheres in $\mathbb{R}^n$ looked upon as boundaries of balls $B \subset \mathbb{R}^n$ are convex, but the same spheres which bound the complements $\mathbb{R}^n - B$ are regarded *concave*.

Observe that the round ε -sphere $S_\varepsilon \subset \mathbb{R}^n$ has mean curvature $(n-1)\varepsilon^{-1}$ and so the above $(+*)'$ becomes an equality. Thus $(+*)'$ shows that if $\text{Ricci} \geq 0$ then *the inward deformation* (i.e. for $\varepsilon \leq 0$) *makes* $W = W_0$ *mean convex at every* $w \in W$ *faster than it happens to the round sphere in* $\mathbb{R}^n$ *of the same mean curvature as* W *at the point* $w \in W$ *in question. In particular, if* W *is mean convex then so is* W_ε *for* $\varepsilon \leq 0$.

NON-SMOOTH MEAN CONVEXITY. The above discussion was made under the tacit assumption of smoothness of W_ε along with the requirement on $d_\varepsilon : W \rightarrow V$ to send W diffeomorphically onto W_ε . This assumption, as we know, is satisfied for smooth W and small $|\varepsilon|$ but it is usually violated for large $|\varepsilon|$. However, the above mean convexity property for $\text{Ricci} \geq 0$ remains valid for all ε with an appropriate generalizations of mean curvature and mean convexity to non-smooth hypersurfaces (notice that exactly at this point the geometry truly comes into play. The above formulae for small ε would remain a futile infinitesimal exercise if they were not valid globally for all W). The idea of such a generalization comes from the remark that the intersection of two mean convex domains V_1 and V_2 in V should be mean convex, though the boundary of $V_1 \cap V_2$ may be (and usually is) non smooth. Thus one can enlarge the class of mean convex domains with smooth boundaries by taking finite and (with some precaution) infinite intersections. Then one defines mean convex non-smooth hypersurfaces as those which locally are the boundaries of such domains. Alternatively, one can make a definition of mean convexity of W at a given point $w \in W$ with an « ambient » *smooth* mean convex W' touching W at w from outside as in Fig. 13 used earlier in § 3 to define convexity.

From this moment on we assume we know the meaning of the relation $M(W) \geq 0$ for non-smooth W and we take a similar attitude

toward the relation $M(W) \geq \delta$ for all real δ . We also introduce the notions of *strict mean convexity*, denoted $M(W) > 0$, as we did earlier in the convex surrounding of § 3.

Now let us apply the above principle of the fast mean convexity rate to concrete geodesic situations.

PUSHING-IN MEAN CONVEX BOUNDARY. Let V be complete with compact mean convex boundary $W = \partial V$ (V itself may be non-compact but look like $W \times [0, \infty)$). Then we define as earlier

$$V_\varepsilon^- = \{v \in V \mid \text{dist}(v, W) \geq \varepsilon\}$$

and observe that the above discussion for $\text{Ricci} \geq 0$ implies that $V_\varepsilon^- \subset V$ is mean convex for all $\varepsilon > 0$. Then we can estimate the $(n-1)$ -dimensional volume of $W_{-\varepsilon} = \partial V_\varepsilon^-$ since the derivative of $\text{Vol } W_{-\varepsilon}$ equals $-M(W_{-\varepsilon})$ integrated over W_ε with a positive weight. Thus we see that $\text{Vol } W_{-\varepsilon}$ is monotone decreasing in ε . Finally we recall that $\text{Vol } W_{-\varepsilon}$ integrated over ε gives the n -dimensional volume of the « band » $V - V_\varepsilon^-$ which we can now bound by $\varepsilon \text{Vol } W$.

Suppose furthermore that either $W = \partial V$ is strictly mean convex (i.e. $M(W) > 0$) or $\text{Ricci} > 0$. In this case the inequality $(+*)''$ integrated over W_ε shows that $\text{Vol } W_\varepsilon$ becomes zero at some finite moment ε . It easily follows that V is *compact* in this case. In particular, it can not be homeomorphic to $W \times [0, \infty)$. One knows more in this regard. For example, the *splitting theorem* of Cheeger and Gromoll (who generalized the earlier splitting theorem of Toponogov for $K \geq 0$) implies that if a *complete manifold* V with $\text{Ricci} \geq 0$ has *no boundary and has more than one end* (i.e. $V - V_0$ is disconnected for all sufficiently large compact subsets V_0 in V) then V splits into the isometric product by $V = W \times \mathbb{R}$ for some closed manifold W .

MEAN CURVATURE AND THE LAPLACE OPERATOR. Recall that the Laplace operator Δf on smooth functions $f: V \rightarrow \mathbb{R}$ is

$$\Delta f \stackrel{\text{def}}{=} \text{div grad } f,$$

where the vector field $\text{grad } f$ is defined with the Riemannian metric

g as the dual of the differential df , i.e.

$$\langle \text{grad } f, \tau \rangle = df(\tau)$$

for all $\tau \in T(V)$, and the divergence is defined as the *Lie derivative* of the Riemannian volume (density) with respect to the gradient.

Now let $f(v) = -\text{dist}(v, W)$. This function obviously has $\|\text{grad}\| = 1$ and the divergence of the gradient equals the variation (i.e. derivative with respect to ε) of the volume densities Vol_ε of W_ε , since the volume of the region between W_{ε_1} and W_{ε_2} in V equals the integral of $\text{Vol } W_\varepsilon$ over $[\varepsilon_1, \varepsilon_2]$, and the same remains true for all subdomains $U_\varepsilon \subset W_\varepsilon$. Thus we obtain the equality between Δf at v and the mean curvature of W_ε passing through v . This equality,

$$\Delta f(v) = M(M_\varepsilon, v),$$

makes sense so far only where W_ε and f are smooth, but with our earlier discussion we can extend the story to all points $v \in V$. In particular, if $\text{Ricci } V \geq 0$ and $W = \partial V$ is mean convex, then

$$-\Delta \text{dist}(v, w) \geq 0,$$

that is the minus distance to W is a *subharmonic* function on W which may be called *mean convex* in our language. Notice that this function is *convex* for $K \geq 0$ (compare §§ 0 and $\frac{1}{2}$ where all this is done for $V \subset \mathbb{R}^n$).

VOLUME MONOTONICITY FOR BALLS. Consider concentric balls $B(\varepsilon)$ in V around a fixed point $v_0 \in V$ and observe with our tube inequalities for $\text{Ricci } V \geq 0$ that these balls are « less mean convex » than the corresponding balls in $\mathbb{R}^n$, that is

$$M(\partial B(\varepsilon)) \leq (n-1) \varepsilon^{-1} = M(S_\varepsilon^{n-1}),$$

for the Euclidean ε -spheres S_ε^{n-1} . (If we look from the point of view of the complement $V_\varepsilon^- = V - B(\varepsilon)$, then the boundary sphere $\partial B(\varepsilon) = \partial V_\varepsilon^-$ appears *more* mean convex than that in $\mathbb{R}^n$ with our convex-concave convention). Also notice that in term of the function $d_0(v) = \text{dist}(v, v_0)$ the above mean curvature relations becomes $\Delta d_0 \leq ((n-1) \varepsilon^{-1})$. It follows that the volumes of $\partial B(\varepsilon)$ and $B(\varepsilon)$

grow slower than those in $\mathbb{R}^n$. Namely,

$$\text{Vol } \partial B(\varepsilon) \leq \text{Vol } S^{n-1}_\varepsilon$$

and also the n -dimensional volume $\text{Vol } B(\varepsilon)$ does not extend the volume of the Euclidean ε -ball in $\mathbb{R}^n$.

In fact the tube inequality $(+*)''$ tells us more. Namely, if we integrate it over ε we obtain the following bound on the growth of the $(n-1)$ -dimensional volume of the boundary spheres $\partial B(\varepsilon)$ in V ,

$$\text{Vol } \partial B(\lambda\varepsilon) \leq \lambda^{n-1} \text{Vol } \partial B(\varepsilon),$$

where λ is an arbitrary number ≥ 1 . Then the second integration over ε yields the following very useful

BISHOP INEQUALITY. *If a complete n -dimensional Riemannian manifold V without boundary has $\text{Ricci } V \geq 0$, then every two concentric balls in V of radii $\varepsilon \geq 0$ and $\lambda\varepsilon \geq \varepsilon$ satisfy*

$$(+++) \quad \text{Vol } B(\lambda\varepsilon) \leq \lambda^n \text{Vol } B(\varepsilon).$$

This can be thought of as a relation on the pushforward of the Riemannian measure by the function

$$\text{dist}(v_0, \cdot) : V \rightarrow \mathbb{R}.$$

For example, this inequality provides an upper bound on the number of disjoint ε -balls inside a larger ball of radius ϱ which leads to a non-trivial restriction on the distances between finite configurations of points in V (see the discussion in § 3½ around Alexandrov-Toponogov).

The Bishop inequality becomes sharper for $\text{Ricci} > 0$, as for $\text{Ricci} \geq (n-1)\varrho^{-2}$ the rate of growth of spheres and balls in V is dominated by that in the round sphere $S^n(\varrho) \subset \mathbb{R}^{n+1}$. It follows, that the volume $\text{Vol } B(\varepsilon)$ does not increase at all for $\varepsilon \geq \pi\varrho$ and so the diameter of V is bounded by $\pi\varrho$, which generalizes Bonnet theorem (see § 3). In particular, the universal covering of every closed manifold V with strictly positive Ricci curvature is compact and $\pi_1(V)$ is finite exactly as in the case $K > 0$ we studied earlier in § 3. Moreover, even in the non-strict case the structure of the fundamental group

$\pi_1(V)$ for $\text{Ricci} \geq 0$ is similar to that for $K \geq 0$. Namely, π_1 contains a free Abelian subgroup of rank $\leq \dim V$ having finite index in π_1 . This follows from the Cheeger-Gromoll splitting theorem applied to the universal covering of V . (This theorem, in its general form, claims that every complete Riemannian manifold X without boundary and with $\text{Ricci } X \geq 0$ admits an isometric splitting, $X = Y \times \mathbb{R}$, provided X contains a line ℓ that is a geodesic which minimizes the distance between each pair of points in ℓ).

On the other hand, there seems to be no restriction on the simply connected part of the topology forced by $\text{Ricci} \geq 0$. (An exception is discussed in § 6 in the framework of the positive scalar curvature). For example, a recent construction by Sha and Yang (refined by Anderson) provides manifolds V of a given dimension $n \geq 4$ with $\text{Ricci} > 0$ and with arbitrary large Betti numbers. (These manifolds can not have metrics with $K \geq 0$ by the discussion in § 3). To obtain some perspective one may compare the manifolds with $\text{Ricci} \geq 0$ to subharmonic functions while $K \geq 0$ goes parallel to convexity. This analogy suggests Ricci flat manifolds (i.e. with $\text{Ricci} = 0$) as the counterparts to harmonic functions (which are by far more numerous than linear functions corresponding to $K = 0$) and one may expect that a great deal of simply connected manifolds admit Ricci flat metrics. But even a single example of this kind is not at all easy to produce. Yet, Yau has proven the existence of these for all even $n = \dim V$, as he has produced a Riemannian (even Kähler) metric with $\text{Ricci} = 0$ on every smooth complex projective hypersurface V of degree $m + 1$ in $\mathbb{C}P^m$. (Notice that all such V are mutually diffeomorphic for a fixed m and they are simply connected for $m \geq 3$). Yau has also shown that the hypersurfaces of degree $\leq m$ have metrics with $\text{Ricci} > 0$ and those of degree $\geq m + 2$ with $\text{Ricci} < 0$. Moreover, in the latter case Yau has proven the existence of Einstein metrics g on V satisfying the equation $\text{Ricci}_g = -g$.

Among more elementary examples of manifolds with $\text{Ricci} \geq 0$ we mention again homogeneous spaces G/H for compact G , whose homogeneous metrics often have $\text{Ricci} > 0$ while K is, typically, only non-strictly positive. The simplest among them are Cartesian products, like those of spheres; $S^k \times S^\ell$ for $k, \ell \geq 2$, and compact semi-simple Lie groups G with biinvariant metrics (only $SU(2)$ and $SO(3)$ have $K > 0$). Notice that the tube related inequalities provide non-

trivial information on the geometry of G expressed by *Paul Levy's isoperimetric inequality* which generalizes the classical isoperimetric inequality in S^n . This inequality is especially interesting when $\text{Ricci } G \rightarrow \infty$ along with $\dim G \rightarrow \infty$ as it implies the following remarkable concentration of values of functions $f_i : G_i \rightarrow \mathbb{R}$ with $\|\text{grad } f_i\|_{g_i} \leq \text{const.}$

The pushforwards of the Haar measures on G_i weakly converge to the Dirac δ -measure on $\mathbb{R}$ for $\dim G_i \rightarrow \infty$, provided the biinvariant metrics are normalized to have $\text{Diam}(G_i, g_i) = 1$ and f_i are normalized by $\int_{G_i} f_i = 0$, (See [Mi-S]).

SINGULAR SPACES WITH $\text{Ricci} \geq 0$. The theory of these spaces does not exist yet. It seems hard (if at all possible) to express adequately the inequality $\text{Ricci} \geq 0$ by some universal distance inequality similar to Alexandrov-Toponogov for $K \geq 0$. Although the condition $\text{Ricci} \geq 0$ does imply some relations on the distance, e.g. those related to balls inside a larger ball (see above), these are not strong enough to characterize $\text{Ricci} \geq 0$. Let us state a finer inequality of this kind, that appears very much similar to $A-T$ but still does not furnish a characterization of $\text{Ricci} \geq 0$. We consider, as in the case $K \geq 0$, four points $v_i \in V$, $i = 0, 1, 2, 3$ where v_3 lies on a minimizing segment $[v_1, v_2]$ (see Fig. 15 in § 3½) and we want to give the *lower bound* on the distance $|v_0 - v_3|$ between v_0 and v_3 . Denote by E the *excess* in the triangle inequality for (v_0, v_1, v_2) , that is

$$E = |v_0 - v_1| + |v_0 - v_2| - |v_1 - v_2|$$

and let

$$s = \min(|v_0 - v_1|, |v_0 - v_2|).$$

Then, if $\text{Ricci} \geq 0$, the distance $|v_0 - v_3|$ satisfies the following

ABRESCH-GROMOLL INEQUALITY.

$$|v_0 - v_3| \geq (s(E/4)^{n-1})^{\frac{1}{n}}.$$

(See [Che] for a proof).

Recall that the distance inequalities, in general, characterize the image of the distance function on V^N , denoted $M_N(V^N) \subset \mathbb{R}^{N'}$, $N' = \frac{N(N-1)}{2}$ (see the discussion preceding Alexandrov-Toponogov

in § 3½). Now, the Ricci curvature manifests itself via the tube inequalities for the volume behavior of the distance function. Thus one may expect the desired abstract characterization of $\text{Ricci} \geq 0$ in terms of the push-forward of the Riemannian measure to $\mathbb{R}^N$. (For example, the Bishop inequality $(++)$ is of this nature). Notice that for general metric spaces there is no distinguished measure, and so the (hypothetical) theory of $\text{Ricci} \geq 0$ must *include* a measure as a *given* element of the structure along with the metric.

An important feature of the theory of spaces with $K \geq 0$ defined by A - T is the good behavior under the *Haussdorf limits* of sequences of spaces, where the Hausdorff convergence $V_i \rightarrow V$ roughly corresponds to the convergence, for every $N = 2, 3, \dots$, of the subsets $M_N(V_i^N) \subset \mathbb{R}^N$ to $M_N(V^N)$ for the Hausdorff metric (see [G-L-P] more about it). Now, in the Ricci curvature case one probably should allow *weak Haussdorf limits* corresponding to weak limit of the M_N -pushforward measures on $\mathbb{R}^N$.

There is another option for the abstract theory of $\text{Ricci} \geq 0$ where instead of the metric one emphasizes the *heat flow (diffusion)* on V , but at this stage it is unclear whether the two approaches are equivalent and if not which one is better for applications.

Let us indicate a specific problem giving more substance to the above discussion. We recall that *smooth* manifolds with $\text{Ricci} \geq \varrho > 0$ satisfy Paul Levy's inequality which implies, in turn, certain bounds on the spectrum of the Laplace operator Δ and on the heat kernel on V . (See [Mi-S], [Gal]). Now we ask if similar bounds remain valid on singular spaces with $K \geq 0$ where the extra condition $\text{Ricci} \geq \varrho$ is enforced in an appropriate way. For example one can strengthen the A - T -inequality in § 3½ to make it equivalent to $K \geq \varrho/n - 1$ which would imply the above bound on Ricci. (For the meaning of the spectrum of Δ etc. on an abstract metric space we refer to [Gr]₁).

ON $\text{RICCI} \leq 0$. The traced tube formula does not provide much information if $\text{Ricci} \leq 0$ (unlike the case $K \leq 0$) and in fact nothing is known on the geometric meaning of this condition. It seems that the only known result is the old theorem by Bochner saying that a closed manifold V with $\text{Ricci} < 0$ must have *finite isometry group*.

Also no topological restriction on V seems to issue from the negative Ricci. One believes nowadays that every manifold of dimension $n \geq 3$ admits a complete metric with $\text{Ricci} < 0$. (For $n = 3$ this is a *theorem* of Gao and Yau) (*).

EXAMPLE. If V is immersed into $\mathbb{R}^m$ for some m as a *minimal* subvariety then the induced metric has $\text{Ricci} \leq 0$ (by an easy computation) and one may expect that every *open* manifold V admits a complete minimal immersion into some Euclidean space.

REMARKS ON PSEUDOCONVEXITY AND POSITIVE BISECTIONAL CURVATURE. The notion of *pseudoconvexity* of domains and hypersurfaces is $\mathbb{C}^m$ (see § 1½) interpolates between convexity and mean convexity. Then one may ask what are the manifolds endowed with complex structures and Riemannian metrics where the inward equidistant deformations preserve pseudoconvexity. Notice, that a priori there may be no such manifolds at all, but, in fact, they do exist. They are *Kähler* manifolds with a certain inequality on the curvature tensor called the *positivity* of the *bisectional curvature*. (For example every Kähler manifold with $K \geq 0$ has bisectional curvature ≥ 0). A theorem of Siu-Yau claims that *every closed manifold with strictly positive bisectional curvature is diffeomorphic (even biholomorphic) to $\mathbb{C}P^m$* . Yet there is no direct proof of this result using pseudoconvex deformations. (Siu and Yau use *harmonic maps* $S^2 \rightarrow V$, compare § 7½. There is another approach due to S. Mori who appeals to algebraic geometry over finite (!) fields). On the other hand the pseudoconvexity considerations are very useful in the study of complex subvarieties in V with positive bisectional curvature. For example, for every complex hypersurface $H \subset \mathbb{C}P^m$ the regions

$$V_\varepsilon^- = \{v \in \mathbb{C}P^m \mid \text{dist}(v, H) \geq \varepsilon\}$$

are pseudoconvex and, hence, by easy Morse theory $\mathbb{C}P^m - H$ has the *homotopy type of an m -dimensional polyhedron*. This is (the special case of) the famous theorem of Lefschitz.

Finally we notice for experts that similar positivity conditions can be introduced for other manifolds with *restricted holonomy*, but the usefulness of these is limited by the list of known examples.

(*) A strengthened version of this conjecture, namely the *h-principle* for $\text{Ricci} < 0$, is proven by Lohkamp for all $n \geq 3$.

§ 6. POSITIVE SCALAR CURVATURE.

The formal definition of the scalar curvature $Sc(V)$ is easy,

$$Sc(V) = \text{Trace}_g \text{Ricci } V.$$

Then, if we recall the definition of Ricci in terms of the sectional curvature, we can compute Sc at a given point $v \in V$ by using an orthonormal frame $\nu_1, \dots, \nu_n$ in $T_v(V)$ and adding together the sectional curvatures of the planes σ_{ij} , spanned by ν_i and ν_j for all $1 \leq i, j \leq n$ and $n = \dim V$, $Sc_v(V) = \sum_{i,j} K(\sigma_{ij})$. Thus the unit sphere

$S^n \subset \mathbb{R}^{n+1}$ has $Sc = n(n-1)$ (which gives 2 for the 2-sphere whose sectional curvature $K=1$). One can make the above look more geometric by using the integral of $\text{Ricci}(s, s)$ over the unit sphere

$S^{n-1} \subset T_v(V)$ instead of the trace (or the sum $\sum_{i,j} K(\sigma_{ij})$). Then, ac-

cording to the tube formula, $Sc_v(V)$ measures the *excess* of the total mean curvature of the ε -sphere $S(V, \varepsilon)$ in V around v for $\varepsilon \rightarrow 0$ over the total mean curvature of the Euclidean ε -sphere. To see this we observe that the *rescaled* spheres $\varepsilon^{-1}S(V, \varepsilon)$ form a *smooth* family in $\varepsilon \geq 0$ at $\varepsilon=0$, where $\varepsilon^{-1}S$ means $(S, \varepsilon^{-2}g_\varepsilon)$ for the induced Riemannian metric g_ε on the ε -sphere $S(V, \varepsilon) \subset V$ and where $\varepsilon^{-1}S(V, \varepsilon)$ for $\varepsilon=0$ refers to the unit Euclidean sphere $S^{n-1} \subset T_v(V)$. This makes sense as we identify S^{n-1} with $S(V, \varepsilon)$ by relating the tangent vectors $\tau \in S^{n-1}$ to the ε -ends of the geodesic ε -segments issuing from v and tangent to τ . (The smoothness of the family $\varepsilon^{-2}g_\varepsilon$ at 0 follows from the smoothness of g at $v \in V$). Now we formally expand g_ε into a power series in ε ,

$$g_\varepsilon = \varepsilon^2 g_0 + \varepsilon^3 g_1 + \varepsilon^4 g_2 + \dots,$$

where $g_i, i=0, 1, \dots$, are some quadratic differential forms on S^{n-1} with g_0 being the spherical metric. Then the shape operators A_ε of $S(V, \varepsilon)$ (defined by $\langle A_\varepsilon \tau, \tau' \rangle_{g_\varepsilon} = \Pi_\varepsilon(\tau, \tau')$, see §§ 0, 1, 2) are also expanded as

$$A_\varepsilon = \varepsilon^{-1} Id + A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots$$

where A_i and $A_i, i=0, 1, \dots$, are operators on the tangent bundle

$T(S^{n-1})$ for S^{n-1} identified with $S(V, \varepsilon)$. Then we invoke the basic tube formula

$$\frac{dA_\varepsilon}{d\varepsilon} = -A_\varepsilon^2 + B$$

(see (**) in § 2) and substitute the above series for A_ε . Thus we obtain

$$\begin{aligned} & -\varepsilon^{-2} Id + A_1 + 2\varepsilon A_2 + \dots = \\ & -(\varepsilon^{-2} Id + \varepsilon^{-1} 2A_0 + A_\varepsilon + 2A_1 + \varepsilon(A_0 A_1 + \dots) + \dots) + B, \end{aligned}$$

which implies

$$(*) \quad A = \varepsilon^{-1} Id + \frac{\varepsilon}{3} B + \dots,$$

where the omitted term is $O(\varepsilon^2)$. Then we take the traces of the operators in (*) (these are $T_s(S^{n-1}) \rightarrow T_s(S^{n-1}) s \in S^{n-1}$), and arrive at the following relation (for *functions*) on S^{n-1} ,

$$M_\varepsilon = (n-1)\varepsilon^{-1} - \frac{\varepsilon}{3} \text{Ricci} + \dots$$

where M_ε is the mean curvature of $S(V, \varepsilon)$ and Ricci stands for Ricci (s, s) , $s \in S^{n-1} \subset T_v(V)$. Now we want to evaluate the integral of M_ε over $S(V, \varepsilon)$ and we have to exercise some control over the *volume density* on $S(V, \varepsilon)$. We write this as $J_\varepsilon ds$ for the spherical measure ds on S^{n-1} and a (density) function J_ε on S^{n-1} , which is related to M_ε by the equation

$$\frac{dJ_\varepsilon}{d\varepsilon} = J_\varepsilon M_\varepsilon$$

(see § 5). We look for a solution in the form

$$J_\varepsilon = \varepsilon^{n-1} + b_n \varepsilon^n + b_{n+1} \varepsilon^{n+1} + \dots$$

(this is justified by the smoothness of g as earlier) and see that

$$\begin{aligned} & (n-1)\varepsilon^{n-2} + nb_n \varepsilon^{n-1} + (n+1)b_{n+1} \varepsilon^n + \dots = \\ & (\varepsilon^{n-1} + b_n \varepsilon^n + b_{n+1} \varepsilon^{n+1} + \dots) \left((n-1)\varepsilon^{-1} - \frac{\varepsilon}{3} \text{Ricci} + \dots \right). \end{aligned}$$

It follows, that $b_n = 0$ and

$$(n+1)b_{n+1} = -\frac{1}{3} \text{Ricci} + (n-1)b_{n+1}.$$

Therefore,

$$(**) \quad J_\varepsilon = \varepsilon^{n-1} - \frac{\varepsilon^{n+1}}{6} \text{Ricci} + \dots,$$

and so the $(n-1)$ -dimensional volume of the sphere $S(V, \varepsilon) \subset V$ is

$$\text{Vol } S(V, \varepsilon) = \varepsilon^{n-1} (1 - \varepsilon^2 \alpha_n \text{Sc}_v + \dots) \text{Vol } S^{n-1},$$

where

$$\alpha_n = (6n)^{-1},$$

(as the average of $\text{Ricci}(s, s)$ over S^{n-1} equals $\text{Trace Ricci}/n$). We also can write down the integral mean curvature of $S(V, \varepsilon)$ by

$$\overline{M}_\varepsilon = \int_{S^{n-1}} M_\varepsilon J_\varepsilon ds,$$

which gives us

$$\overline{M}_\varepsilon = \varepsilon^{-(n-2)} (n-1 - \varepsilon^2 \beta_n \text{Sc} + \dots) \text{Vol } S^{n-1},$$

where $\beta_n = n^{-1} \left(\frac{1}{3} + \frac{n-1}{6} \right)$. So, as we claimed, the scalar curva-

ture measures the excess of the integral mean curvature $\overline{M}_\varepsilon$ of $S(V, \varepsilon)$ over that for the Euclidean spheres $S(\mathbb{R}^n, \varepsilon)$ (where $\overline{M} = \varepsilon^{n-2}(n-1)$). In fact, the above formula for $\text{Vol } S(V, \varepsilon)$ gives a similar interpretation of Sc by the excess of the volume of the spheres $S(V, \varepsilon)$ and then by integrating over ε one gets yet another such relation, this time for the balls $B(V, \varepsilon) \subset V$ around v ,

$$\text{Vol } B(V, \varepsilon) = \varepsilon^n (1 - \varepsilon^2 \alpha_n \text{Sc}_v + \dots) \text{Vol } B^n,$$

where B^n denotes the unit ball in $\mathbb{R}^n$. For example, if $\text{Sc}_v(V) > 0$ then every sufficiently small ball $B(V, \varepsilon)$ has

$$\text{Vol } B(V, \varepsilon) < \varepsilon^n \text{Vol } B^n = \text{Vol } B(\mathbb{R}^n, \varepsilon).$$

Conversely, if

$$\text{Vol } B(V, \varepsilon) \leq \varepsilon^n \text{Vol } B^n$$

for all sufficiently small ε , then $Sc_v(V) \geq 0$. (Notice, that our discussion here and earlier only makes sense for $n \geq 2$. If $n = 1$ all Riemannian manifolds are locally isometric to $\mathbb{R}$ and there is no curvature to speak of).

It may seem at this stage that we have achieved a certain understanding of the scalar curvature of V . Yet the above infinitesimal relations for Sc do not integrate the way it was happening for the sectional curvature and Ricci. In fact, we are still nowhere as far as the geometry and topology of manifolds with $Sc \geq 0$ (or $Sc \leq 0$) is concerned. To see the problem from another angle let us look at certain examples of manifolds with $Sc \geq 0$. First we observe that the scalar curvature is additive for the Cartesian product of manifolds. Thus, if a manifold V has $\inf Sc > -\infty$, (e.g. V is compact) then the product of V with a small round sphere $S^2(\delta) \subset \mathbb{R}^3$ (which has $Sc(S^2(\delta)) = 2\delta^{-2}$) has positive scalar curvature. This product manifold $V \times S^2(\delta)$ is, on the other hand, as geometrically and topologically complicated as the underlying manifold V and it may appear hopeless to find any global pattern for $Sc > 0$.

The first global result for $Sc > 0$ was obtained by Lichnerowicz in 1963 who proved the following

LICHTNEROWICZ THEOREM. *If a closed $4k$ -dimensional spin manifold V admits a metric with $Sc > 0$, then a certain characteristic number of V , namely the $\hat{A}$ -genus, vanishes.*

The meaning of « spin » and « $\hat{A}$ -genus » will be discussed later on along with the idea of the proof (which uses in an essential way the Atiyah-Singer index theorem applied to the Dirac operator). Here we only indicate a particular example of V , where the theorem applies.

EXAMPLE. Let V be a smooth complex hypersurface of degree d in $\mathbb{C}P^{m+1}$. If m is even, then the (real) dimension of V is divisible by 4. Furthermore, if d is even, then V is spin. Finally if $d \geq m + 2$, then $\hat{A}(V) \neq 0$, and so such a V cannot have a Riemannian metric with $Sc > 0$. The simplest such manifold is quartic (i.e. $d = 4$) in $\mathbb{C}P^3$, which is a 4-dimensional simply connected manifold V^4 which by Lichnerowicz's theorem admits no metric with $Sc > 0$. (By Yau's

theorem mentioned in § 5 this V^4 admits a metric with $\text{Ricci} = 0$, and hence with $Sc = 0$. On the other hand, even if we are content to show «no metric with $\text{Ricci} > 0$ » or even less, «no metric with $K > 0$ », we are still unable to do it geometrically without an appeal to the deep analysis underlying the proof of the Lichnerowicz theorem).

The method of Lichnerowicz was extensively developed by N. Hitchin, who has shown, among other things, that *there exists an exotic 9-dimensional sphere V (i.e. a manifold which is homeomorphic but not diffeomorphic to S^9) which admits no metric with $Sc > 0$* . (In fact, half of the exotic spheres in dimension 1 and 2 (mod. 8) carry no such metrics by Hitchin's theorem). Here again, there is no alternative geometric approach even with $Sc > 0$ replaced by $K > 0$.

SCALAR CURVATURE AND MINIMAL HYPERSURFACES. The first geometric insight into $Sc \geq 0$ was achieved by Schoen and Yau in 1979 with the following innocuously looking modification of the traced tube formula (see (**)) in § 5) for surfaces W in a 3-dimensional manifold V . At every point $v \in W$ we consider the tangent plane $\sigma_v = T_v(W)$ and the unit normal vector ν_v to W . (We assume W is cooriented and stick to inward looking ν). First we observe that our formulas expressing Ricci and Sc in terms of K imply, that

$$Sc_v = 2K(\sigma_v) + 2 \text{Ricci}(\nu_v, \nu_v).$$

(In general, for hypersurfaces W in V^n for $n \geq 3$ the term $2K(\sigma_v)$ must be replaced by the sum of $K(\sigma_{ij})$ for some orthonormal basis $\nu_1, \dots, \nu_{n-1}$ in $T_v(W)$). Then we bring in the principal curvatures λ_1 and λ_2 of W at v and recall that the sectional curvature of W with the induced metric is expressed according to Gauss' formula (Teorema egregium in § 2) as

$$K(W, \sigma_v) = K(V, \sigma_v) + \lambda_1 \lambda_2,$$

which is equivalent to

$$K(W, \sigma_v) = K(V, \sigma_v) + \frac{1}{2} (M^2 - \text{Trace } A^2)$$

where A is the shape operator of W (whose eigenvalues are exactly λ_1 and λ_2) and $M = \text{Trace } A = \lambda_1 + \lambda_2$ is the mean curvature of W .

The tube formula (see (**) in § 5) expresses the derivative of M under the normal equidistant deformation W_ε of $W = W_0$ at $\varepsilon = 0$ as follows

$$\frac{dM}{d\varepsilon} = -\text{Trace } A^2 - \text{Ricci } (\nu, \nu).$$

Then we recall that M equals the (logarithmic) derivative of the volume density on W_ε at $\varepsilon = 0$ (see § 5). It follows by integration over W , that the derivative $-\text{Area } W_\varepsilon$ at $\varepsilon = 0$ equals the total mean curvature of W

$$\overline{M} = \int_W M \, dw.$$

(We say « Area » rather than « Vol » since $\dim W = 2$).

Then we observe that

$$\frac{d\overline{M}}{d\varepsilon} = \int_W \frac{dM}{d\varepsilon} \, dw + \int_W M^2 \, dw,$$

where the second summand is due to the variation of the volume (area) element $dw = \text{Vol}_{\varepsilon=0}$ expressed by the mean curvature. Then we substitute Ricci in the above tube formula for $\frac{dM}{d\varepsilon}$ by

$$-\frac{1}{2} \text{Sc}(V) + K(V|T(W))$$

and then we use the Gauss formula

$$K(V|T(W)) = K(W) + \frac{1}{2} (\text{Trace } A^2 - M^2).$$

Thus we obtain the following second variation formula for the area of $W = W_\varepsilon$ at $\varepsilon = 0$.

$$\frac{d^2 \text{Area } W}{d\varepsilon^2} = \frac{d\overline{M}}{d\varepsilon} = \frac{2}{1} \int_W (-\text{Sc } V + 2K(W) - \text{Trace } A^2 + M^2) \, dw =$$

$$(+) \quad \int_W \left(-\frac{1}{2} \text{Sc}(V) + K(W) + \lambda_1 \lambda_2 \right) dw$$

Now, we recall the *Gauss-Bonnet theorem*

$$\int_W K(W) = 2\pi\chi(W),$$

where χ denotes the Euler characteristic and W is assumed compact without boundary. Then, if $Sc(V) \geq 0$, we have the following inequality,

$$\frac{d^2 \text{Area } W}{d\varepsilon^2} \leq 2\pi\chi(W) + \int_W \lambda_1 \lambda_2 dw.$$

In particular, if $\chi(W) < 0$ and W is a *saddle surface*, i.e. if $\lambda_1 \lambda_2 \leq 0$, then

$$(\text{++}) \quad \frac{d^2 \text{Area } W}{d\varepsilon^2} < 0.$$

Notice, that as at the previous occasions, this conclusion only applies to small equidistant deformations which do not destroy the smoothness of W . Now, instead of extending the above computation to non-smooth W_ε as we did earlier for $K \geq 0$ and $\text{Ricci} \geq 0$, we follow the idea of Schoen and Yau and apply (++) to *smooth minimal surfaces* W in V . (Non-smooth extension of the above is questionable because of the saddle condition). The existence of such surfaces is insured by the following theorem known since long in the geometric measure theory (see, e.g. [Law]).

Every 2-dimensional homology class in a closed Riemannian 3 dimensional V can be represented by a smooth absolutely minimizing embedded oriented surface $W \subset V$.

Recall that « absolutely minimizing » means that every surface $W' \subset V$ homologous to W has

$$\text{Area } W' \geq \text{Area } W.$$

REMARK. A similar result remains valid for minimal hypersurfaces W in V^n for $n \geq 3$ but now these W may have singularities. One knows, that the singularity is absent for $n \leq 7$ and, in general, it has codimension ≥ 7 in W .

Now, since a minimizing surface W provides the minimum for the function $W \mapsto \text{Area } W$ on the space of surfaces in V , the first

variation of Area is clearly zero and the second is non-negative. In particular

$$\frac{d^2 \text{Area } W}{d\varepsilon^2} \geq 0.$$

Moreover, every connected component of W , say W_c , also has

$$\frac{d^2 \text{Area } W_c}{d\varepsilon^2} \geq 0.$$

Furthermore, minimal surfaces have $M = \lambda_1 + \lambda_2 = 0$ and so are saddle. Hence, the above inequality is incompatible with $Sc(V) \geq 0$ and the issuing inequality $(++)$ unless $\chi(V) \geq 0$. Thus we conclude to the following.

SCHOEN-YAU THEOREM. *Let V be a closed 3-dimensional Riemannian manifold with $Sc(V) \geq 0$. Then every homology class in $H_2(V)$ can be realized by an embedded oriented surface W whose every connected componenti has $\chi \geq 0$.*

EXAMPLE. Let $V = V_0 \times S^1$ where V_0 is an orientable surface of genus ≥ 2 (i.e. $\chi(V_0) < 0$). Then elementary topology tells us that $V_0 = V_0 \times s_0 \subset V$ is *not* homologous to a surface whose all components have genus ≤ 1 . Therefore this V admits no metric with $Sc \geq 0$. (By an obvious readjustment of the above discussion, one rules out $Sc > 0$ starting from $\text{genus}(V_0) = 1$).

Schoen and Yau have generalized their method to manifolds V^n with $n \leq 7$ and they proved that *if $Sc(V^n) \geq 0$, then every class in $H^{n-1}(V^n)$ can be realized by a hypersurface W which admits some metric with $Sc \geq 0$* . In fact, they take the valume minimizing hypersurface for $W \subset V^n$ and then modify the induced metric in W by a conformal factor to make $Sc \geq 0$. This does not work for $n > 7$ due to the (possible) presence of singularities on minimal W but later on Schoen and Yau indicated a way out of this problem. (See [Sch] for a brief account of these results).

The above theorem of Schoen-Yau shows (by a simple induction on n) that there are non-trivial topological restrictions on V^n with $Sc(V^n) \geq 0$. For example, the Cartesian product of surfaces of genus

≥ 2 admits no such metric. Furthermore, their method can be refined in order to provide non-trivial *geometric* restrictions on V as well. For example, let V^n be a complete non-compact oriented Riemannian manifold without boundary with uniformly positive scalar curvature, i.e. $Sc(V^n) \geq c > 0$. Then V^n admits no proper distance decreasing map to $\mathbb{R}^n$ of non-zero degree. In other words, V^n is no larger than $\mathbb{R}^n$.

EXAMPLE. Let $V^n = S_2 \times \mathbb{R}^{n-2}$ with the product metric. Clearly this V^n (which has $Sc \geq c > 0$) admits no above map to $\mathbb{R}^n$. But if we modify the product metric $g_s + g_E$ on $V^n = S^2 \times \mathbb{R}^{n-2}$ by introducing a so-called *warping* factor, that is a positive function $\varphi : \mathbb{R}^{n-2} \rightarrow \mathbb{R}_+$, and by making $g = \varphi g_s + g_E$, then for (V, g) we can easily produce a contracting proper map into $\mathbb{R}^n$ of degree one, provided the function $\varphi(x)$ satisfies the asymptotic relation

$$\varphi(x) \rightarrow \infty \text{ for } x \rightarrow \infty.$$

It follows, that such a warping φ necessarily makes

$$\inf Sc g \leq 0,$$

though it is not hard to achieve $Sc g > 0$ non-uniformly on V .

§ 6 $\frac{1}{2}$. SPINORS AND THE DIRAC OPERATOR.

Now we return to Lichnerowicz' approach. First we recall that the fundamental group of the special orthogonal group is

$$\pi_1(SO(n)) = \begin{cases} \mathbb{Z} & \text{for } n = 2 \\ \mathbb{Z}_2 & \text{for } n \geq 3. \end{cases}$$

Thus $SO(n)$, for all $n \geq 2$, admits a unique double cover denoted

$$\text{Spin}(n) \rightarrow SO(n),$$

where $\text{Spin}(n)$ carries a natural structure of a Lie group such that the above covering map is a homomorphism. (This is quite obvious for $SO(2) = S^1$ and easy but not all obvious for $n \geq 3$).

Next, for a Riemannian manifold V , we look at the orthonormal frame bundle $SO(V)$, that is the principal bundle with the fiber $SO(n)$, $n = \dim V$, associated to $T(V)$, and we ask ourselves if there exists a double cover

$$\text{Spin}(V) \rightarrow SO(V)$$

which reduces over each point $v \in V$ to the above $\text{Spin}(n) \rightarrow SO(n)$. This is clearly possible if the tangent bundle (and, hence, $SO(V)$ as well) is *trivial*, $T(V) = V \times \mathbb{R}^n$, as one can take $V \times \text{Spin}(n)$ for $\text{Spin } V$. In general, there is a topological obstruction for the existence of $\text{Spin}(V)$ which can be easily identified with the second *Stiefel-Whitney class* $w_2(V)$. This is a certain cohomology class in $H^2(V, \mathbb{Z}_2)$ which measures non-triviality of $T(V)$ and which is also known to be a homotopy invariant of V . In any case, $w_2 = 0$ if $H^2(V, \mathbb{Z}_2) = 0$ and then $\text{Spin}(V) \rightarrow SO(V)$ does exist.

The space $\text{Spin}(V)$, whenever it exists, has a natural structure of a principal $\text{Spin}(n)$ bundle over V and then one may look for associated vector bundles. These come along with linear representations of the group $\text{Spin}(n)$. There are, for even $n = 2r$, two distinguished faithful (spin) representations of $\text{Spin}(n)$ of (lowest possible) dimension 2^{r-1} , for which the corresponding vector bundles, denoted $S_+ \rightarrow V$ and $S_- \rightarrow V$, are called the *positive* and *negative spin bundles*, whose sections are called (positive and negative) *spinors* on V . Atiyah and Singer have discovered a remarkable elliptic differential operator between the spinors, i.e.

$$D_+ : C^\infty(S_+) \rightarrow C^\infty(S_-),$$

which they call the *Dirac operator*. This operator is constructed with the connection ∇_+ in S_+ induced by the Levi-Civita connection in V , where the connection in S_+ is thought of as an operator from spinors to spinor valued 1-forms on V , i.e.

$$\nabla_+ : C^\infty(S_+) \rightarrow \Omega^1(S_+).$$

Then D_+ is obtained by composing ∇ with a certain canonical vector bundle homomorphism $\Omega^1(S_+) \rightarrow S_-$ coming from some algebraic manipulations with spin representations. (We somewhat abuse the

notations by using $\Omega^1(S_+)$ for the bundle of spinor-forms as well as for the sections of this bundle). Notice that D_+ is defined locally and needs no « spin condition » $w_2 = 0$, but if $w_2 \neq 0$, then spinors are globally defined only up to $\pm$ sign. The interested reader may look to the book [L-M] for an actual construction of the spinors and Dirac. Here we just *assume* the existence of certain bundles S_+ and S_- and an operator D_+ with the properties stated below.

Besides the operator D_+ we need its twin, called

$$D_- : C^\infty(S_-) \rightarrow C^\infty(S_+)$$

which is constructed in the same way as D_+ and which can be defined as the *adjoint* operator to D_+ for the natural Euclidean structure on the spin-bundles. Then one looks at the *index* of D_+ , i.e.

$$\text{Ind } D_+ = \dim \ker D_+ - \dim \ker D_- ,$$

where the dimensions of the kernels of D_+ and D_- are finite if V is a closed manifold, since the operators D_+ and D_- are elliptic. The remarkable (and easy to prove) property of the index is the invariance under the deformations of D_+ in the class of *elliptic* operators between spinors. In particular, this index does not depend on the Riemannian metric used for the definition of D_+ and so it represents a *topological invariant* of V . The famous theorem of Atiyah and Singer identifies $\text{Ind } D_+$ with a certain characteristic number called *$\hat{A}$ -genus* of V , but for our present purpose we may *define* $\hat{A}$ -genus as $\text{Ind } D_+$. The only serious property of D_+ we need at the moment is *non-vanishing* of $\hat{A}(V)$ for certain manifolds V . (Otherwise, what follows will be vacuous).

Now we need another operator associated with ∇_+ , called *Bochner Laplacian*,

$$\Delta_+ = \nabla_+^* \nabla_+ : C^\infty(S_+) \rightarrow C^\infty(S_+),$$

where

$$\nabla_+^* : \Omega^1(S_+) \rightarrow C^\infty(S_+)$$

is the adjoint to ∇_+ . This Laplacian makes sense for an arbitrary

vector bundle with a Euclidean connection over V and an important property of Δ_+ is *positivity*, i.e.

$$\int_W \langle \Delta_+ s, s \rangle \geq 0$$

for all spinors $s : V \rightarrow S_+$. (The Bochner Laplacian for the trivial 1-dimensional bundle reduces to the classical Laplace Beltrami operator $\Delta = d^*d$ on functions whose positivity follows by integration by parts, as $\int f \Delta f = \int \langle df, df \rangle$, and a similar consideration proves positivity of Δ_+ on spinors).

Now the scalar curvature enters the game via the following LICHNEROWICZ FORMULA. *The operator $D_- D_+ + D_+ D_-$ on $S_+ \oplus S_-$ is related to $\Delta_+ + \Delta_-$ by*

$$D_- D_+ + D_+ D_- = \Delta_+ + \Delta_- + \frac{1}{4} Sc Id.$$

Recall that $Sc = Sc(V)$ denotes the scalar curvature which is a function on V and Id is the identity operator on $C^\infty(S_+ \oplus S_-)$.

The proof of the Lichnerowicz formula consists of a straightforward (infinitesimal) algebraic computation which is quite easy with the definitions (we have not given) of D_+ and D_- . Yet the geometric meaning of the formula remains obscure.

COROLLARY. *If a closed Riemannian spin (i.e. with $w_2 = 0$) manifold V has $Sc(V) > 0$ then every harmonic spinor on V vanishes and so*

$$\hat{A}(V) = \text{Ind } D_+ = 0.$$

Here, «harmonic spinor» means a spinor $s = (s_+, s_-) \in C^\infty(S_+ \oplus S_-)$, such that

$$D_{\text{def}} s = D_+ s_+ + D_- s_- = 0.$$

The proof of the corollary is obvious.

$$\int_V \langle Ds, s \rangle = \int_V \langle \Delta s, s \rangle + Sc \langle s, s \rangle,$$

which implies by positivity of Δ that

$$\int_V \langle Ds, s \rangle \geq \int_V Sc \langle s, s \rangle$$

and then for $Ds = 0$ we get

$$\int_V Sc \langle s, s \rangle \leq 0$$

which is possible only for $s = 0$ since $Sc > 0$.

Notice again, that this Corollary is non-vacuous since there exist spin manifolds with $\hat{A} \neq 0$, e.g. complex hypersurfaces in $\mathbb{C} P^{m+1}$ mentioned earlier in § 6. This property of non-vanishing of $\text{Ind } D$ for some V and the Lichnerowicz formula is all which is needed from spinors and Dirac in order to show that some manifolds V admit no metric with $Sc > 0$.

Although we do not quite understand the geometry behind the Lichnerowicz formula, we can use this formula to reveal some geometry of V with $Sc|_V \geq c > 0$. Namely, we want to show that such a V cannot be « too large ». For example, it cannot be much larger than the unit sphere S^n . Indeed imagine that V is much larger than S^n in the sense that there exists a smooth map $f : V \rightarrow S^n$ of degree $d \neq 0$, such that the differential of f is everywhere small,

$$\|Df\|_v \leq \varepsilon, \quad v \in V.$$

Then we pull-back to V some fixed vector bundle E_0 with a Euclidean connection over S^n . The pull-backed bundle, say E over V , is « ε -flat », that is locally ε -close to a trivial bundle. In particular, the *twisted Dirac operator*, denoted $D_+ \otimes E : C^\infty(S_+ \otimes E) \rightarrow C^\infty(S_- \otimes E)$ is locally ε -close to the direct sum of k copies of D_+ for $k = \text{rank } E$. (If $E = \mathbb{R}^k \times V \rightarrow V$ with the trivial connection then

$$S_+ \otimes E = \underbrace{S_+ + S_+ + \dots + S_+}_k$$

and the twisted Dirac is $D_+ + D_+ + \dots + D_+$. The definition of the twisting with a non-trivial connection is such that the ε -flatness of E

makes the twisted Dirac ε -close to $D_+ + D_+ + \dots + D_+$). It follows, that is ε is small compared to $c = \inf_V Sc(V) > 0$, then the twisted Dirac operator $D_+ \otimes E$ has

$$\text{Ind } D_+ \otimes E = 0$$

by an ε -perturbed version of the Lichnerowicz formula.

Now, one can find in certain cases a bundle E_0 such that $\text{Ind } D \otimes E_0 \neq 0$. In fact, one can always produce such a (complex vector bundle) E_0 over an *even* dimensional sphere, as follows from the Atiyah-Singer index theorem applied to $D_+ \otimes E$. Therefore, no spin manifold V with $Sc(V) \geq c > 0$ can be ε^{-1} times greater than S^n for $\varepsilon \ll c$, (where the odd-dimensional case reduces to the even dimensional one by multiplying V by a long circle S^1).

The reader may be justly dissatisfied at this point as the discussion was incomplete and quite formal. A detailed exposition can be found in the book [L-M] but filling in the details does not seem to reveal extra geometry.

CONCLUDING REMARKS. The existence of two so different approaches to $Sc > 0$ has no rational explanation at the present state of art. In general terms, the Schoen-Yau method appeals to the (non-linear) analysis in the space of submanifolds in V while the Dirac operator approach uses the linear analysis (of spinors) over V . One may hope for the existence of a unified general theory which would treat simultaneously non-linear objects inside V as well as linear ones over V in a way similar to what happens in algebraic geometry. Probably, such a unification may be possible only in an infinite dimensional framework.

SCALAR CURVATURE ≤ 0 . This condition has no topological effect on V by a theorem of Kazdan and Warner which claims the *existence of a metric* $Sc < 0$ on every manifold of dimension $n \geq 3$. Probably, the global geometry of V is also unsensitive to $Sc < 0$ (though the condition $Sc \geq c$ for $c < 0$ does have non-trivial corollaries) (*).

(*) The flexibility and the h -principle (in the sense of [Gro]) for $Sc < 0$ is proven by Lohkamp.

§ 7. THE CURVATURE OPERATOR AND RELATED INVARIANTS.

We have mentioned in the end of § 2 that the sectional curvature function on the space of 2-planes of V , i.e.

$$K : Gr_2 V \rightarrow \mathbb{R}$$

uniquely extends to a quadratic form (function) Q on the bundle $\Lambda^2 T(V)$, and the symmetric operator $R : \Lambda^2 T(V) \rightarrow \Lambda^2 T(V)$ corresponding to Q is called the *curvature operator*. The condition $K \geq 0$ can be expressed in terms of Q by

$$Q(\tau \wedge \nu, \tau \wedge \nu) \geq 0$$

for all tangent vectors τ, ν in $T_v(V)$, $v \in V$, while strict positivity $K > 0$ corresponds to $Q(\tau \wedge \nu, \tau \wedge \nu) > 0$ for all pairs of *linearly independent* pairs (τ, ν) .

From the point of view of Q a more natural condition is $Q \geq 0$ which means $Q(\alpha, \alpha) \geq 0$ for all $\alpha \in \Lambda^2 T(V)$ (that may be sums $\alpha = \sum_{i=1}^k \tau_i \wedge \nu_i$ for $k > 1$) which is called *positivity of the curvature operator* R . Then *strict positivity* of R refers to positive definiteness of Q . Similarly one introduces the (strict and non-strict) *negativity* of Q and R .

The above positivity of Q and R is a significantly more restrictive condition than $K \geq 0$. Yet, the basic examples of manifolds with $K \geq 0$ also have $Q \geq 0$. Namely, convex hypersurfaces in $\mathbb{R}^{n-1}$ and compact symmetric spaces have $Q \geq 0$. Also Cartesian products of manifolds with $Q \geq 0$ have $Q \geq 0$.

To see the point of departure between $K \geq 0$ and $Q \geq 0$ we look at the complex projective space $\mathbb{C}P^n$ with a $U(n+1)$ -invariant Riemannian metric g for a natural action of the unitary group on $\mathbb{C}P^n$. It is not hard to see that such a g (which exists because $U(n+1)$ is compact) is unique up to a scalar multiple and $(\mathbb{C}P^n, g)$ is a *symmetric space of rank one* which is equivalent for compact symmetric spaces to $K > 0$. In fact, we already know (see § 3 $\frac{3}{4}$) that the sectional curvatures of g are pinched between $\frac{1}{4} a$ and a

for some constant $a > 0$ depending on (normalization of) g . (An inquisitive reader would be happy to learn that $a = \pi^{-1} (\text{Diam}(\mathbb{C}P^n, g))^{-2}$).

On the other hand, the curvature operator R is only non-strictly positive, i.e. the form Q is only *semi*-positive definite. Thus a small perturbation of g may easily break the condition $R \geq 0$ without destroying $K \geq 0$.

The above example of $\mathbb{C}P^n$ is especially interesting in view of the following well known

CONJECTURE. If a closed n -dimensional Riemannian manifold has $R > 0$ then its universal covering is diffeomorphic to the sphere S^n .

The positive solution is classical for $n=2$ where R is the same as K and $\int_V K > 0$ implies $\chi(V) > 0$ by the Gauss-Bonnet theorem.

The cases $n=3, 4$ are due to R. Hamilton whose proof uses a deep analysis of a *heat flow on the space of metrics*. Namely, Hamilton considers the following differential equation for a one-parameter family of metrics g_t on V ,

$$\frac{dg_t}{dt} = \alpha_n g_t - 2 \text{Ricci}(g_t)$$

for $\alpha_n = 2n^{-1} \int_V \text{Sc}(g_t) / \text{Vol}(V, g_t)$, and he proves the solvability of this for a given initial metric $g = g_0$. Then he shows that the resulting heat flow preserves the subspace of metrics g with $R(g) > 0$. (This is called « heat flow » since the correspondence $g \mapsto \text{Ricci}(g)$ is a differential operator on quadratic differential forms on V which is in many respects similar to the Laplace operator on functions. Notice that Ricci is a non-linear operator but it has a remarkable (albeit obvious) property of commuting with the action of the group of diffeomorphisms of V on the space of metrics).

Finally, for $n=3$ and 4 Hamilton proves that the solution g_t of his equation with $R(g_0) > 0$ converges as $t \rightarrow \infty$ to a metric g^∞ of constant positive curvature which makes the universal covering

$(\tilde{V}, \tilde{g}_\infty)$ obviously isometric to S^n . Notice that Hamilton's proof yields the solution of the *strengthened conjecture* which claims the existence of a $(\text{Diff } V)$ -invariant contraction of the space of metrics with $R \geq 0$ to the subspace with $K = 1$.

Also notice that for $n = 3$ Hamilton only needs $\text{Ricci } V > 0$ in order to make his method work.

The basic point in Hamilton's approach is the study of the evolution of the curvature tensor under the heat flow, where the condition $R > 0$ becomes crucial because it is invariant under the flow. (Notice that the term $\alpha_n g_t$ in Hamilton's equation is brought in for the purpose of a normalization, while the curvature discussion applies to the equation $\frac{dg_t}{dt} = -2 \text{Ricci}(g_t)$).

There are other more stringent curvature conditions which are also invariant under the heat flow and for some cases one is able to prove the eventual contractibility to constant curvature. For example one has as a corollary the following result for metrics g with point-wise pinched sectional curvature.

(RUH-HUISKEN-MARGARIN-NISHIKAWA) *Let the sectional curvature $K : \text{Gr}_2(V) \rightarrow \mathbb{R}$ of a closed Riemannian n -dimensional manifold V be pinched (i.e. restricted) at each point $v \in V$ by*

$$c_n a(v) \leq K(o) \leq a(v),$$

where a is a positive function on V and $c_n = 1 - 3(2n)^{-\frac{3}{2}}$, while σ stands for an arbitrary 2-plane in $T_v(V)$. Then V is diffeomorphic to S^n .

Notice that the above theorem is quite non-trivial for any $c_n < 1$. For example, such a condition is satisfied for $n = 2$ (where there is only one σ at each v) by every metric with $K > 0$ and so the corresponding heat flow does not amount to a small perturbation of the original metric. (See [Bou] for an exposition of the heat flow method).

BOCHNER FORMULAS. Many natural (but usually complicated) curvature expressions go along with natural differential operators on V .

For example, we could *define* the scalar curvature of V with the Dirac operator $D = D_+ \oplus D_-$ by

$$ScId = 4(D^2 - \nabla^* \nabla)$$

where Id is an identity operator on the spin bundle $S = S_+ \oplus S_-$ (compare Lichnerowicz' formula in § 61½).

Now we want to do a similar comparison between the *Hodge-de Rham Laplacian* Δ on k -forms on V and the *rough* (Bochner) Laplacian $\nabla^* \nabla$, where ∇ denotes the Levi-Civita connection of V extended to the bundle $\Lambda^k T^*(V)$ of k -forms on V .

It is not hard to see that the two operators coincide if V is flat (i.e. locally Euclidean). Then we recall that every metric g can be infinitesimally first order approximated at each point by a (osculating, see § 2) flat metric. Then the following result comes as no surprise.

The differential operator $\Delta - \nabla^ \nabla$ has zero order and is given by a symmetric endomorphism R_k of the bundle $\Lambda^k T^*(V)$, where R_k is algebraically (even linearly) expressible by the curvature tensor of V . (Here « symmetric » means that R_k is a symmetric operator on every fiber of the bundle).*

Another way to put it is by writing

$$\Delta = \nabla^* \nabla + R_k,$$

which is called the Bochner (or Bochner-Weitzenböck) formula for Δ . The expression of R_k in terms of the curvature operator R is rather complicated for $k \geq 2$ (see, e.g. [Bes]) but for $k=1$ it is quite transparent. Namely

$$R_1 = \text{Ricci}^*,$$

that is the symmetric operator on the cotangent bundle $T^*(V)$ associated with the quadratic form Ricci in $T(V)$ in the natural manner via the underlying metric g . It is worth observing that this Bochner formula

$$\Delta \omega = \nabla^* \nabla \omega + \text{Ricci}^*(\omega)$$

applied to *exact* forms $\omega = df$, where the function f has *unit gra-*

dient, i.e.

$$||\omega|| = ||df|| = ||\text{grad } f|| = 1,$$

is essentially the same thing as the traced tube formula from § 5 applied to the levels $W_\varepsilon = \{f(x) = \varepsilon\}$.

The Bochner formula with Ricci* immediately implies that if Ricci > 0 , then *every harmonic 1-form on V vanishes* (compare the proof of Lichnerowicz' theorem in § 6½) and thus

$$H^1(H, \mathbb{R}) = 0.$$

(We have indicated another proof of this in § 5 using the more powerful Cheeger-Gromoll splitting theorem, but the above analytic proof by Bochner is older by a quarter of a century).

The operators R_k for $k \geq 2$ are significantly more complicated than Ricci*. Yet one has the following result of Bochner-Yano-Berger-Meyer, (see [L-M]).

If $R > 0$ then $R_k > 0$ (i.e. *positive definite*) for all $k \neq 0$, $n = \dim V$. Thus *every closed Riemannian manifold with positive curvature operators has $H^k(V, \mathbb{R}) = 0$ for $1 \leq k \leq n - 1$.*

This result shows that $R > 0$ implies that V is a *rational homology sphere* which is significantly weaker than being *diffeomorphic* to the sphere required by the conjecture stated above. Now, a recent theorem by Micallef and Moore claims that the universal covering of V is, in fact a *homotopy sphere*, and hence *homeomorphic* to the sphere by the Poincaré conjecture (solved for $n \geq 5$ by S. Smale and for $n = 4$ by M. Freedman. The remaining case $n = 3$ for Ricci > 0 is taken care of by Hamilton's theorem cited earlier). The method of Micallef-Moore is similar to that employed by Siu and Yau in their study of Kähler manifolds with positive bisectional curvature (see the end of § 5). Both methods make an essential use of *harmonic maps* of the sphere S^2 into V and the curvature appears in the *second variation formula* for the energy of a harmonic map as we are going to explain next.

§ 71½. HARMONIC MAPS AND THE COMPLEXIFIED CURVATURE $K_{\mathbb{C}}$.

The energy of a smooth map between Riemannian manifolds, say

$$f : W \rightarrow V$$

is defined by

$$E(f) = \frac{1}{2} \int_W ||Df(w)||^2 dw$$

where the squared norm of the differential

$$D = Df(w) : T_w(W) \rightarrow T_v(V), v = f(w),$$

at each point $w \in W$ is

$$||Df||^2 = \text{Trace } D^* D$$

where $D^* : T_v(V) \rightarrow T_w(W)$ is the adjoint operator.

A map f is called *harmonic* if it is stationary (or critical) for the energy thought of as a smooth function on the space of maps $W \rightarrow V$. The stationary condition for E at f , i.e. $dE(f) = 0$, says in plain words that for every smooth one-parameter deformation f_t of $f = f_0$ the derivative $\frac{dE(f_t)}{dt}$ vanishes at $t = 0$. Notice that this derivative of E at $t = 0$ depends only on the « direction » of the deformation f_t at $t = 0$, that is the vector fields $\delta = \frac{\partial f}{\partial t}$ in V along $f(W)$. More precisely, δ is a section of the induced bundle $T^* = f^*(T(V)) \rightarrow W$.

Harmonic maps can also be defined as solutions of a certain system of non-linear partial differential equations, namely the Euler-Lagrange equations corresponding to E . This system can be written as $\Delta f = 0$ where the operator Δ generalizes the classical Laplace operator. In fact, if one takes geodesic coordinates $x_1, \dots, x_m$ at $w \in W$ and $y_1, \dots, y_n$ at $v = f(w)$ and represents f by n functions $y_i = y_i(f(x_1, \dots, x_m))$, then the above $\Delta f(w)$ becomes equal to the ordinary Laplacian of the vector-function $y_1, \dots, y_n$ at zero, that is

$$\left(\sum_j \frac{\partial^2 y_1(0)}{\partial x_j^2}, \dots, \sum_j \frac{\partial^2 y_n(0)}{\partial x_j^2} \right)$$

EXAMPLES. (a) If V is the circle S^1 then every map $f : W \rightarrow S^1$ is locally represented by a function $\varphi : W \rightarrow \mathbb{R}$ defined up to an additive constant and

$$E(f) = \frac{1}{2} \int_W \|\text{grad } \varphi\|^2.$$

Then the equation $\Delta f = 0$ is the same as $\Delta \varphi = 0$ for the ordinary Laplace-Beltrami operator on W .

(b) Now, let $W = S^1$ and V be arbitrary. Then

$$\Delta f = \nabla_\tau \tau,$$

where

$$\tau \stackrel{\text{def}}{=} \frac{df}{ds} \stackrel{\text{def}}{=} D(f) \left(\frac{\partial}{\partial s} \right)$$

for the standard (cyclic) parameter s on S^1 , where $\frac{\partial}{\partial s}$ denotes the correspondence (coordinate) vector field on S^1 , and where ∇ is the covariant derivative in V . Harmonic maps $f : S^1 \rightarrow V$ are those where $\nabla_\tau \tau = 0$. These are exactly *geodesic maps*: the image of f is a geodesic in V and the parameter is a multiple of the length.

The curvature of the ambient manifold V enters the picture once we look at the *second variation*

$$\delta^2 E(f) \stackrel{\text{def}}{=} \frac{d^2 E(f_t)}{dt^2} \text{ at } t = 0.$$

In general, the second derivative in t along $f_0(W) \subset V$ at $t = 0$ (i.e. $\frac{d^2 E(f_t)}{dt^2}$ at $t = 0$) depends not only on the field $\delta = \frac{\partial f}{\partial t}$ at $t = 0$ but also on the derivative $\Delta_\delta \frac{\partial f}{\partial t}$. However, if $f = f_0$ is harmonic, then this derivative depends only on δ which justifies the notation $\delta^2 E$ in this case, in fact, if a function E has zero (first) differential at some point f , then there is a well defined *second differential* (or *Hessian*) H of e at f which is a quadratic form on the vector fields δ at f , such that

$$\delta(\delta E(f)) = H(\delta, \delta).$$

THE SECOND VARIATION FORMULA FOR HARMONIC MAPS. If f is a smooth harmonic map, then

$$(*) \quad \delta^2 E(f) = \int_W (||\nabla \delta||^2 + \tilde{K}(\delta^2)) dw,$$

where ∇ is the covariant derivative of δ in V along W and $\tilde{K}(\delta^2)$ is an algebraic quadratic expression in δ involving the curvature of V . (Notice that in our earlier formulas for the second variations for areas and volumes the field δ was unit and normal to a hypersurface W and the $\nabla \delta$ -term was zero). Let us make the above precise. First we recall that δ is, in fact, a section of the induced bundle $T^* \rightarrow W$ and denote by ∇ the connection in T^* induced from the Levi-Civita connection in $T(V)$. Then $||\nabla \delta||^2$ makes sense as ∇ is a differential operator with values in the bundle $\Omega^1 T^* \stackrel{\text{def}}{=} \text{Hom}(T(W), T^*)$ which has a natural Euclidean structure coming from those in T^* and $T(W)$.

Now we take care of the curvature term. First we extend the sectional curvature K by bilinearity to all pairs (τ, ν) of vectors in V . In terms of the form Q on $\Lambda^2 T(V)$ this reads

$$K(\tau \wedge \nu) = Q(\tau \wedge \nu, \tau \wedge \nu).$$

Then the curvature $\tilde{K}(\delta^2)$ is expressed at every point $w \in W$ with an orthonormal frame $\tau_1, \dots, \tau_m$ in $T_w(W)$, $m = \dim W$, by

$$\tilde{K}(\delta^2) = - \sum_{i=1}^m K((D\tau_i) \wedge \delta), \text{ for } D = Df,$$

where the result does not depend on the choice of the frame. (See [E-L]₁ and [E-L]₂ for an extensive discussion of these matters). Now, we see that $K(V) \leq 0$ makes $\delta^2 E(f) \geq 0$ and so one may expect that every harmonic map provides a local minimum for the energy. In fact, every harmonic map of a compact manifold W into a complete V with $K \leq 0$ gives the absolute minimum for the energy function (see [E-L]₁).

If $K(V) \geq 0$ one may expect $\delta^2 E$ to be negative for those fields δ whose (covariant) derivatives along W are small. For example, if δ is ∇ -parallel along W , i.e. $\nabla \delta = 0$, then $\delta^2(E) \leq 0$.

EXAMPLE. Let $W = S^1$, then $f(S^1)$ is a geodesic in V and every vector $x \in T_v(V)$, $v = f(w)$ admits an extension to a ∇ -parallel field along this geodesic. When we go around the circle, the vector x does not go, in general, into itself, but into another vector, say $x' \in T_v(V)$. The resulting map say $L : T_v(V) \rightarrow T_v(V)$ for $L(x) = x'$, is called the *holonomy transformation* (or *parallel transport*) along the loop $f(S^1)$ and is known (by the basic property of the Levi-Civita connection) to be an orthogonal linear map. Since the curve $f(S^1)$ is geodesic (and has $\nabla_\tau \tau = 0$ as we have seen above) the tangent vector $\tau_v = \frac{df}{ds} = (Df) \frac{\partial}{\partial s} \in T_v(V)$ is invariant under L . Then we look at the orthogonal complement $N_v \subset T_v(V)$ of τ_v and observe that the operator $L|_{N_v} : N_v \rightarrow N_v$ fixes a unit vector ν_v , i.e. $L(\nu_v) = \nu_v$, in the following two cases

(i) $n = \dim V = \dim N_v + 1$ is even and L is an *orientation preserving* map, i.e. $\text{Det } L = +1$;

(ii) n is odd and L is *orientation reversing*, i.e. $\text{Det } L = -1$.

Notice, that L is orientation preserving for all loops in V if the manifold V is orientable. But if V is non-orientable then there exists a homotopy class of loops in V , such that L is orientation reversing for all loops in this class.

If $L(\nu_v) = \nu_v$ then the vector ν_v extends to a global (periodic) parallel field ν in V along W for which the second variation of E is

$$\nu^2 E(f) = - \int_{S^1} K(\tau \wedge \nu) ds.$$

If $K(V) > 0$ this variation is strictly negative and so f is not a local minimum of the energy. On the other hand it is not hard to show that if V is a closed manifold, then every homotopy class of maps $S^1 \rightarrow V$ contains a smooth harmonic (i.e. geodesic) map f giving the absolute minimum to the energy on this class of maps. Thus we obtain the classical

SYNGE THEOREM (See [Mil]). *Let V be a closed Riemannian manifold with $K(V) > 0$. If $n = \dim V$ is odd, then V is orientable and if n*

is even, then the canonical oriented double cover of V is simply connected (e.g. if V is orientable then it is simply connected).

Notice that the proof of this theorem uses the positivity of K along some (non-specified) closed geodesic in V and no geometric information is needed (nor revealed) away from this geodesic. This sharply contrasts with our study of $K \geq 0$ by means of the tube formula (though the second variation formula for ν follows from the tube formula applied to (germs of) hypersurfaces normal to the geodesic in question).

THE SECOND VARIATION OF THE ENERGY FOR $\dim W = 2$. If $\dim W \geq 2$ then, generically, there is no ∇ -parallel field δ along W as the system $\nabla \delta = 0$ on W is overdetermined. In fact, the operator ∇ applies to sections δ of the induced bundle $T^* = f^*(T(V)) \rightarrow W$ (where $f: W \rightarrow V$ is our harmonic map) which are locally given by $n = \text{rank } T^*$ functions on W while the target bundle for $\Omega^1 T^* = \text{Hom}(T(W), T^*)$, has rank $= n \dim W$ which is $> n$ for $\dim W \geq 1$. Now let W be an oriented surface and let $S \rightarrow W$ be a complex vector bundle with a complex linear connection ∇ . The Riemannian metric in W together with the orientation defines a complex structure in the bundle $T(W)$. Namely, the multiplication by $i = \sqrt{-1}$ is given by rotating tangent vectors by 90° counter clockwise. Each fiber $\Omega^1 S_w$ of the bundle $\Omega^1 S$, which consists of $\mathbb{R}$ -linear maps $T_w(W) \rightarrow S_w$, splits into the sum of two subspaces, Ω' and Ω'' in $\Omega^1 S_w$, where Ω' consists of $\mathbb{C}$ -linear maps $\ell': T_w(W) \rightarrow S_w$ i.e. commuting with multiplication by $\sqrt{-1}$, which means $\ell'(\sqrt{-1}x) = \sqrt{-1}\ell'(x)$, while the maps $\ell'' \in \Omega''$ anti-commute with $\sqrt{-1}$, i.e. $\ell''(\sqrt{-1}x) = -\sqrt{-1}\ell''(x)$. This gives us a splitting of ∇ into the sum of two operators, $\nabla = \nabla' + \nabla''$ for $\nabla': C^\infty(S) \rightarrow C^\infty(\Omega')$ and $\nabla'': C^\infty(S) \rightarrow C^\infty(\Omega'')$. Notice that the bundles Ω' and Ω'' have the same rank over $\mathbb{R}$ as S and so the systems $\nabla' \varphi = 0$ and $\nabla'' \varphi = 0$ are determined. Now there is a good chance they are solvable. In fact, there is an important case where solutions are known to exist; Namely, let S be the complexification of a real vector bundle T with a Euclidean connection over W . Then one has the following

PROPOSITION (See [M-M]). If W is homeomorphic to the sphere S^2 then the equation $\nabla'' \varphi = 0$ has (at least) n linearly independent

over $\mathbb{C}$ solutions $\varphi_1, \dots, \varphi_n : W \rightarrow S = T \oplus \sqrt{-1} T$ for $n = \text{rank } T$, which, moreover span in each fiber S_w a (complex) subspace of dimension $\geq n/2$.

IDEA OF THE PROOF. There is a natural complex analytic structure on the total space of S for which holomorphic sections are exactly those φ which satisfy $\nabla'' \varphi = 0$. Furthermore, the vector bundle $S \rightarrow W$ is *self-dual* as a complex analytic bundle, because the Euclidean structure on T (which is a quadratic form on T) extends, by $\mathbb{C}$ -linearity to a *non-singular quadratic form* on S which is parallel and, hence, *holomorphic* on S . Then the existence of the required n holomorphic sections follows from Riemann-Roch combined with the Birkhoff-Grothendieck theorem on splitting holomorphic vector bundles over S^2 into line bundles.

Next one shows that the Laplace operator associated to ∇'' , that is

$$\Delta'' = (\nabla'')^* \nabla'',$$

is related to $\Delta = \nabla^* \nabla$ by the following Bochner-Weitzenböck formula

$$\Delta = 4\Delta'' = -\sqrt{-1} K''$$

where K'' is the (skew-Hermitian) endomorphism of S associated with the curvature of ∇ and the unit bivector field (codensity) on W associated with the metric in W . (We recall that the curvature of ∇ is a 2-form on W with values in $\text{End } S$ and K'' equals the value of this form on $\tau_1 \wedge \tau_2$ for orthonormal tangent vectors at every point in W). In particular, one relates the integrals

$$\int_W \|\nabla \varphi\|^2 dw = - \int_W \langle \Delta \varphi, \varphi \rangle dw$$

and

$$\int_W \|\nabla'' \varphi\|^2 dw = - \int_W \langle \Delta'' \varphi, \varphi \rangle dw$$

by

$$\int_W \|\nabla \varphi\|^2 dw = 4 \int_W \|\Delta'' \varphi\|^2 dw + \sqrt{-1} \int_W \langle K'' \varphi, \varphi \rangle dw.$$

(The above integral formulae for $-\langle \Delta \varphi, \varphi \rangle$ and $-\langle \Delta'' \varphi, \varphi \rangle$ are obtained by integration by parts. In fact one could *define* $\Delta = \nabla^* \nabla$ and $\Delta'' = (\nabla'')^* \nabla''$ as well as the adjoint operators ∇^* and $(\nabla'')^*$ by postulating these integral formulae for all smooth sections φ . Also notice that $\langle, \rangle$ denotes the Hermitian inner product in $S = T \oplus \sqrt{-1} T$ associated to the Euclidean structure in T).

We want to rewrite the second variation formula (*) for $H(\delta, \delta) = \delta^2 E(f)$ (for harmonic maps f) with $\|\nabla'' \varphi\|^2$ instead of $\|\nabla \varphi\|^2$. First of all we extend the formula to the complexified bundle $S^* = T^* \oplus \sqrt{-1} T^*$, where it expresses the Hermitian extension of the Hessian of E at f . On every complex field φ which is a formal combination of two real fields, $\varphi = \delta_1 + \sqrt{-1} \delta_2$, this Hessian is

$$H(\varphi, \tilde{\varphi}) \underset{def}{=} H(\delta_1, \delta_1) + H(\delta_2, \delta_2) = \delta_1^2 E(f) + \delta_1^2 E(f).$$

Then by the second variation formula,

$$H(\varphi, \tilde{\varphi}) = \int_W \|\nabla \varphi\|^2 + \tilde{K}(\varphi^2) dw,$$

where

$$\|\nabla \varphi\|^2 = \|\nabla \delta_1\|^2 + \|\nabla \delta_2\|^2$$

and $\tilde{K}(\varphi^2) = \tilde{K}(\delta_1^2) + \tilde{K}(\delta_2^2)$ which is expressible in term of the sectional curvature K of V and orthonormal vectors τ_1, τ_2 at each point $w \in W$ by $\tilde{K}(\varphi) = - \sum_{1 \leq i, j \leq 2} K(D\tau_i \wedge \delta_j)$ for the differential

$D = Df(w)$, as we have seen earlier. (Here as always we identify δ_1 and δ_2 in T^* with their images in $T(V)$ under the tautological map $T^* = f^*(T(V)) \rightarrow T(V)$). Now we substitute $\int \|\nabla \varphi\|^2$ in $H(\varphi, \tilde{\varphi})$ by

$$4 \int_W \|\nabla'' \varphi\|^2 + \sqrt{-1} \int_W \langle K'' \varphi, \varphi \rangle$$

according to the previous formula and obtain the following expression for H ,

$$H(\varphi, \tilde{\varphi}) = 4 \int_W \|\nabla'' \varphi\|^2 dw + \int_W K''(\varphi^2) dw$$

for

$$\tilde{K}''(\varphi^2) = \tilde{K}(\varphi^2) + \sqrt{-1} \langle K'' \varphi, \varphi \rangle.$$

(Notice that $\tilde{K}''(\varphi^2)$ is real since $\langle K'' \varphi, \varphi \rangle$ is purely imaginary). We are going to apply this formula to fields φ satisfying $\nabla'' \varphi = 0$, which makes

$$H(\varphi, \bar{\varphi}) = \int_W \tilde{K}''(\varphi^2) dw,$$

and we want to know when $\tilde{K}''(\varphi^2)$ is negative (compare the Synge theorem). The answer is obtained with the following notion of the *complexified sectional curvature* $K_{\mathbb{C}}$ of V .

Extend the form Q by complex multilinearity to the complexified tangent bundle $\mathbb{C} T(V) = T(V) \oplus \sqrt{-1} T(V)$ and let

$$K_{\mathbb{C}}(\alpha \wedge \beta) = Q(\alpha \wedge \beta, \overline{\alpha \wedge \beta})$$

for α and β in $\mathbb{C} T(V)$ (and the obvious conjugation $z \mapsto \bar{z}$ in the complexified bundle). If we write $\alpha = t_1 + \sqrt{-1} t_2$ and $\beta = t_3 + \sqrt{-1} t_4$ for vectors $t_i \in T(V)$, then a trivial computation expresses $K_{\mathbb{C}}$ in real terms as follows,

$$\begin{aligned} K_{\mathbb{C}}(\alpha \wedge \beta) = & Q(t_1 \wedge t_3 - t_2 \wedge t_4, t_1 \wedge t_3 - t_2 \wedge t_4) + \\ & Q(t_1 \wedge t_4 - t_2 \wedge t_3, t_1 \wedge t_4 - t_2 \wedge t_3). \end{aligned}$$

With this one sees that the condition $K_{\mathbb{C}} \geq 0$ interpolates between $K \geq 0$ and $Q \geq 0$ (i.e. the positivity of the curvature operator). Also notice that the inequality $K_{\mathbb{C}} > 0$, by definition, says that $K_{\mathbb{C}}(\alpha \wedge \beta) > 0$ for all pairs of $\mathbb{C}$ -independent vectors α and β in $\mathbb{C} T(V)$.

A useful sufficient condition for positivity of $K_{\mathbb{C}}$ is the *1/4-pinching* of K . That is $K_{\mathbb{C}} > 0$ at every point v where the sectional curvatures $K(\sigma)$ satisfy

$$1/4 a < K(\sigma) < a$$

for some $a = a(v) > 0$ and all 2-planes $\sigma \in T_v(V)$. Similarly, the *negative 1/4-pinching* of $K(\sigma)$ (between $-a$ and $-1/4 a$) insures $K_{\mathbb{C}} < 0$. (This is due to Hernandez. Earlier Micallef and Moore proved

a slightly weaker result needed for the present application. Also notice that the pinching criterion is sharp: The complex projective space has $1/4 \leq K \leq 1$ and $K_{\mathbb{C}}$ is *non-strictly* positive).

Now we return to our map $f : W \mapsto V$, we take two orthonormal vectors τ_1 and τ_2 at some point $w \in W$ and let

$$D' \tau = \frac{1}{2} (D\tau_1 - \sqrt{-1} D\tau_2) \in \mathbb{C} T(W)$$

for $D = Df : T(W) \rightarrow T(V)$.

LEMMA. *The above curvature term $\tilde{K}''$ satisfies at each point $f(w) \in V$,*

$$K''(\varphi^2) = -4\tilde{K}_{\mathbb{C}}(D'\tau \wedge \varphi).$$

This is proven in [M-M] by a straightforward computation based on the definitions of the curvatures in question.

COROLLARY. *If $K_{\mathbb{C}} \geq 0$ then the (complexified) second variation of the energy is non-positive on the solutions φ of the equation $\nabla'' \varphi = 0$. Furthermore, if $K_{\mathbb{C}} > 0$ and φ is non-tangent to $D'\tau \in \mathbb{C} T(V)$ at some point $f(w_0) \in V$, then $H(\varphi, \bar{\varphi}) > 0$. It follows that f is not a local minimum of the energy function $f \mapsto E(f)$.*

PROOF. The only point which may need explanation is the relation of (the sign of) the complex Hessian with the real variation of the energy. But for $\varphi = \delta_1 + \sqrt{-1} \delta_2$ the complex Hessian is the sum of the two real ones

$$H(\varphi, \bar{\varphi}) = H(\delta_1, \delta_1) + H(\delta_2, \delta_2) = \delta_1^2 E(f) + \delta_2^2 E(f)$$

and so the negativity of $H(\varphi, \bar{\varphi})$ implies that for some of the two real variations $\delta_1^2 E(f)$ or $\delta_2^2 E(f)$. Q.E.D.

Now we assume that W is homeomorphic to S^2 and the map $f : S^2 \rightarrow V$ is non-constant. Then we have with our earlier proposition $n = \dim V$ linearly independent solutions $\varphi_1, \dots, \varphi_n$ of the equations $\nabla'' \varphi = 0$ which span a subspace of dimension $\geq n/2$ at some point $f(w_0) \in V$ where $D'\tau \neq 0$. Thus for $n \geq 4$ we obtain at least one field φ for which $H(\varphi, \bar{\varphi}) < 0$ and in general, for $n \geq 4$

we have at least $\frac{n}{2} - 1$ fields, such that H is negative definite on their span.

Recall, that all this needs $K_{\mathbb{C}} > 0$. In fact, Micallef and Moore prove the above under the (weaker) condition of positivity of $K_{\mathbb{C}}$ (only) on those complex 2-planes in $\mathbb{C}T(V)$ on which the complexified Riemannian metric (which is a $\mathbb{C}$ -quadratic form on $\mathbb{C}T(V)$) vanishes.

The above Corollary shows, in particular, that no smooth non-constant harmonic map $f : S^2 \rightarrow V$ is energy minimizing in its homotopy class for $n \geq 4$. On the other hand, a fundamental theorem of Sacks and Uhlenbeck claims the existence of such f whenever the second homotopy group $\pi_2(V)$ does not vanish (where V is otherwise an arbitrarily closed Riemannian manifold). Thus $K_{\mathbb{C}}(V) > 0$ implies $\pi_2(V) = 0$ for $\dim V \geq 4$ and this suffices for $n = 4$ (by elementary topology) to insure that the universal covering of V (which is compact as $K(V) > 0$) is a homotopy sphere.

REMARK. The subtlety of the Sacks-Uhlenbeck theorem is due to the fact that the space of maps $f : S^2 \rightarrow V$ with $E(f) \leq \text{const}$ is *non-compact*. Moreover, a simple computation (using the fact that $\dim S^2 = 2$) reveals that the energy is invariant under the (non-compact!) group of *conformal transformations* of S^2 and so even the space of *harmonic* maps with bounded energy is non-compact. Thus one can hardly expect convergence of any kind of a minimization process for obtaining a harmonic map with minimal energy in a given homotopy class of maps. In fact, one does have divergences were a map $S^2 \rightarrow V$ « bubbles » into several pieces, see fig. 20 below

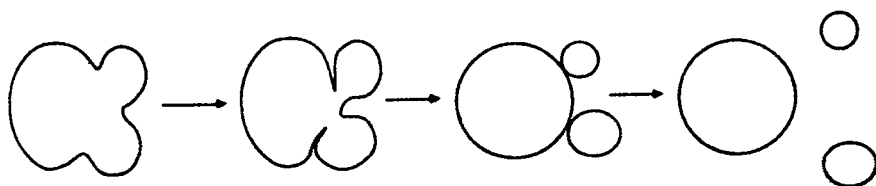


Fig. 20.

Such bubbling transforms a single map $f : S^2 \rightarrow V$ to a finite collection of maps $f_i : S^2 \rightarrow V$, $i = 1, \dots, k$, such that $\sum_{i=1}^k E(f_i) \leq E(f)$ and the homotopy classes of the maps f_i add up to that of f . This explains why one cannot have an energy minimizing map in every homotopy class. Yet this perfectly agrees with the Sacks-Uhlenbeck theorem claiming the existence of an energy minimizing map in some non-trivial homotopy class though this class is not known in advance.

Micallef and Moore generalize the Sacks-Uhlenbeck theorem by developing a limited Morse theory for the energy function E on the space of maps $S^2 \rightarrow V$ and showing for $n \geq 4$ that *either a closed n -dimensional simply connected Riemannian manifold V is a homotopy sphere or there exists a non-constant harmonic map $f : S^2 \rightarrow V$, which admits at most $k \leq \frac{n}{2} - 2$ fields φ_i , such that the Hessian $H(\varphi, \bar{\varphi})$ is negative on their span. Therefore, the condition $K_{\mathbf{c}}(V) > 0$ implies (by the above existence discussion for φ_i with negative H) V is a homotopy sphere.*

We have mentioned earlier that the strict $1/4$ -pinching condition on the sectional curvature K implies strict positivity of $K_{\mathbf{c}}$. Thus the above theorem of Micallef-Moore implies the Sphere Theorem (see § 3 $\frac{3}{4}$).

Notice, that Micallef and Moore need only *local* pinching, i.e. $\frac{1}{4} a(v) \leq K_v(V) \leq a(v)$ for some positive function $a(v)$ while in the sphere theorem one requires a is a positive *constant*. But Moore and Micallef do not directly produce (for locally pinched manifolds V) any explicit geometric homeomorphism between V and S^n as is done in the proof of the sphere theorem (see § 3 $\frac{3}{4}$) but appeal to the topological solution of the Poincaré conjecture for $n \geq 4$. In fact, one has no geometric picture at all of locally pinched manifolds (even with a constant c_n close to 1 instead of $\frac{1}{4}$), despite the remarkable success of Micallef-Moore's method on the topological side. Here again, the situation is parallel to the Synge theorem discussed earlier in this §.

§ 7 $\frac{2}{3}$. HARMONIC MAPS INTO MANIFOLDS WITH $K_{\mathbb{C}} \leq 0$.

First we only assume that $K \leq 0$ and recall the following basic existence theorem for harmonic maps (see [E-L]₁).

(Eells-Sampson) *Let V and W be closed Riemannian manifolds, where $K(V) \leq 0$. Then every continuous map $W \rightarrow V$ is homotopic to a smooth harmonic map (which is energy minimizing in its homotopy class).*

Notice, that for $K(V) \leq 0$ the bubbling phenomenon displayed in Fig. 20 is impossible and harmonic maps can be obtained (as is proven by Eells and Sampson) by a straightforward minimization process. The condition $K(V) \leq 0$ enters via a Bochner type formula for maps $f: V \rightarrow W$ which generalises the formula $\Delta = \nabla^* \nabla - \text{Ricci}^*$ on 1-forms (see § 7) and which is stated below in the special case where f is harmonic.

EELLS-SAMPSON FORMULA. (See [E-L]₁) *Every harmonic map $f: W \rightarrow V$ satisfies*

$$\Delta \|Df\|^2 = \|\text{Hess}_f\|^2 + \text{Curv},$$

where Hess is the totality of the second covariant derivatives of f and Curv is a curvature term described below.

First we describe Hess by interpreting the differential Df of f as the section of the bundle $\Omega^1 = \text{Hom}(T(W), T^*)$ for $T^* = f^*(T(V))$ and then by setting $\text{Hess}_f = \nabla Df$ where the connection ∇ in Ω^1 comes from those in $T(V)$ and T^* . Then we observe that the Ricci form on W defines together with the metric in T^* a quadratic form on Ω^1 also called Ricci^W . Furthermore, the differential $D: T(W) \rightarrow T(V)$ brings the quadratic form Q on $\Lambda^2 T(V)$ to a quadratic form on $\Lambda^2 T(W)$. The trace of this with respect to the metric of $T(W)$ is denoted $K^V((Df)^4)$. Notice that if $K(V) \leq 0$ then so is $K^V((Df)^4)$. Furthermore, if $K < 0$ and $\text{rank } Df \geq 2$ then $K^V((Df)^4) < 0$. With the above notations we can write down the explicit form of the curvature term in the Eells-Sampson formula

$$\text{Curv} = \text{Ricci}^W(Df, Df) - K^V((Df)^4).$$

In particular if $K(V) \leq 0$ and $\text{Ricci}(W) = 0$ (e.g. W is flat) then $\text{Curv} = K^V((Df)^4) \geq 0$.

By integrating the Eells-Sampson formula we obtain the following relation for *closed* manifolds W

$$\int_W \Delta \|Df\|^2 dw = 0 = \int_W (|\text{Hess}_f|^2 + \text{Curv}) dw,$$

which for $\text{Curv} \geq 0$ implies that $\text{Hess}_f = 0$ and so the map f is *geodesic* in an obvious sense. In particular, if $K(V) < 0$ and $\text{Ricci}W = 0$, then $\text{rank } Df \leq 1$ and so the image of f is either a point (we assume V is connected) or a closed geodesic in V . (Notice that all this remains valid for $\text{Ricci}(W) \geq 0$ instead of $\text{Ricci} = 0$).

The story becomes by far more interesting for manifolds W which are *Kähler* rather than flat. The corresponding Bochner type formula, due to Siu and refined by Sampson, generalises the *Hodge formula* for the Laplace operator on functions on Kähler manifolds, that is

$$\Delta \stackrel{\text{def}}{=} d^* d = 2\Delta'' \stackrel{\text{def}}{=} 2\bar{\partial}^* \bar{\partial}.$$

The Siu-Sampson formula is an (infinitesimal) identity which involves $K_{\mathbb{C}}(V)$ and the *complex Hessian* $H_f^{\mathbb{C}}$ defined as follows. First we introduce the operator d^{∇} from T^* -valued 1-forms on V (i.e. sections $W \rightarrow \Omega^1 = \text{Hom}(T(W), T^*)$) to T^* -valued 2-forms on W , which is obtained by the usual « twisting » of the exterior d on 1-forms on W with the connection in T^* . Then we put

$$\text{Hess}_f^{\mathbb{C}} = d^{\nabla} J Df,$$

where $J : \Omega^1 \rightarrow \Omega^1$ is the operator induced by the multiplication by $\sqrt{-1}$ in $T(V)$. Observe that the definition of $\text{Hess}^{\mathbb{C}}$ uses the complex structure in W and the Levi-Civita connection in V but not the metric (or connection) in W . Also notice that $H_f^{\mathbb{C}} = 0$ if and only if the restriction of f to every *holomorphic curve* in W is harmonic. Such maps are called *pluriharmonic* (and they are similar to geodesic maps of flat manifolds W to V). Also notice that this discussion for $\dim_{\mathbb{R}} W = 2$ shows the *conformal invariance* of the equation $\Delta f = 0$ for maps $f : W \rightarrow V$).

Next we complexify the differential of f and thus obtain a $\mathbb{C}$ -linear homomorphism $D^{\mathbb{C}} : T(W) \rightarrow \mathbb{C} T(V)$. This $D^{\mathbb{C}}$ pulls back

the form Q on $\mathbb{C} \Lambda^2 T(V) = \Lambda^2 \mathbb{C} T(V)$ to a form on $\Lambda^2 T(W)$ (here exterior power Λ^2 is meant over $\mathbb{C}$). In fact, what we need is the Hermitian form associated with the quadratic form Q , say $Q \cdot (\alpha, \beta) = Q(\alpha, \bar{\beta})$, for $\alpha, \beta \in \mathbb{C} \Lambda^2 T(V)$. (Recall that Q was originally defined on $\Lambda^2 T(V)$ and then extended to $\mathbb{C} \Lambda^2 T(V)$ by complex multilinearity; compare the earlier discussion in § 7½ around the theorem of Micallef-Moort). Then we pull-back the form $Q \cdot$ to $\Lambda^2 T(W)$ and we denote by $K_{\mathbb{C}}^V((Df)^4)$ the trace of this pull-back with respect to the Hermitian form in $\Lambda^2 T(W)$ induced by the Kähler metric in W . Notice that $K_{\mathbb{C}}(V) \leq 0$ implies $K_{\mathbb{C}}^V((Df)^4) \leq 0$. Furthermore, if $K_{\mathbb{C}}(V) < 0$ and $\text{rank } Df \geq 3$ then $K_{\mathbb{C}}^V((Df)^4) < 0$.

Now we write down (without proof) the following

INTEGRATED SIU-SAMPSON FORMULA. *Let W be a closed Kähler manifold and V a Riemannian manifold. Then every smooth harmonic map $f : W \rightarrow V$ satisfies*

$$(+)\quad \int_W \|\text{Hess}_f^{\mathbb{C}}\|^2 dw - \int_W K_{\mathbb{C}}^V((Df)^4) dw = 0.$$

COROLLARY. *If $K_{\mathbb{C}}(V) \leq 0$ then every harmonic map $f : W \rightarrow V$ has $\text{Hess}_f^{\mathbb{C}} = 0$ and hence, is pluriharmonic. Furthermore if $K_{\mathbb{C}}(V) < 0$ then $\text{rank } Df \leq 2$ at every point $w \in W$.*

Finally, we combine this corollary with the Eells-Sampson existence theorem for harmonic maps and arrive at the following

THEOREM (Siu, Sampson, Jost-Yau, Carlson-Toledo). *Let V be a closed manifold with $K_{\mathbb{C}}(V) < 0$. Then every continuous map of an arbitrary Kähler manifold W into V can be homotoped to a map of W into the 2-skeleton of some triangulation of V .*

This imposes a very strong (albeit weird) restriction on the topology of V , as there are many Kähler manifolds W to which the theorem may be applied. Important examples of W are compact quotients of bounded symmetric domains (such as the ball $B^{2n} \subset \mathbb{C}^n$) by discrete (holomorphic) automorphism groups.

Among manifolds V to which the above theorem applies the most important are the spaces with constant negative curvature.

Also there are examples of manifolds V with strictly $1/4$ -pinched curvature which are not homotopy equivalent to constant curvature manifolds.

The non-strict case $K_{\mathbb{C}}(V) \leq 0$ is especially important because all locally symmetric spaces of non-compact type satisfy this condition and the pluriharmonic conclusion of the above corollary plays a crucial role in the representation theory of the fundamental group $\pi_1(W)$ in the isometry groups (e.g. SL_n) of symmetric spaces (see [Cor], [G-Pa]).

Finally, we notice that the theory of harmonic maps extends to the case where the target is singular with $K \leq 0$ in the sense of Alexandrov-Toponogov (compare § 3 $\frac{2}{3}$). Then one tries to understand the (stronger) condition $K_{\mathbb{C}} \leq 0$ for singular spaces (such as the Bruhat-Tits buildings on which p -adic Lie groups act) and harmonic maps appear quite useful for this purpose (*).

§ 7 $\frac{3}{4}$. METRIC CLASSES DEFINED BY INFINITESIMAL CONVEX CONES.

Every subset C in the space of quadratic forms Q on $\Lambda^2 \mathbb{R}^n$ which is invariant under the orthogonal transformations of $\mathbb{R}^n$ defines a class $\mathcal{C}$ of metrics on every n -dimensional manifold V by requiring that the quadratic form Q on $T_v(V) = \mathbb{R}^n$ built with the curvature of this metric is contained in C for every $v \in V$ (Notice that the identification $T_v(V) = \mathbb{R}^n$ is unique only up to orthogonal transformations of $\mathbb{R}^n$ by the required $O(n)$ -invariance of C). All classes of metrics defined by $K \geq 0$, $K \leq 0$, Ricci ≥ 0 , etc., we have met so far could be obtained with such a C which is uniquely determined by the class of metrics in question. Moreover, the subset C in all our cases was a *convex cone* in the linear space of quadratic forms on $\Lambda^2 \mathbb{R}^n$. It is not clear at all why geometrically significant classes $\mathcal{C}$ must be generated by *convex* cones, but analytically this corresponds to quasi-linearity of the differential condition defining $\mathcal{C}$ (compare p. 24 in [Gro]).

The greatest cone we met was given by $Sc \geq 0$. In fact this condition defines a half-space in the space of Q 's. The smallest of

(*) See Gromov and Schoen in Publications Mathématiques IHES (1993).

our cones was $\{Q > 0\}$ corresponding to the strict positivity of the curvature operator. The closure of this cone (given by $Q \geq 0$) can be defined as the *minimal* closed convex $O(n)$ -invariant cone which contains the curvature Q of the product metric on $S^2 \times \mathbb{R}^{n-2}$. This suggests other definitions of interesting (?) classes of metrics defined with natural cones C . (Compare curvature positivity conditions in [Gro]₂ aimed at bounding the size of V). In the search of interesting cones C one may be guided by how C interacts with natural differential operators on V . (Compare the invariance of $\{Q > 0\}$ under R. Hamilton's heat flow on the space of metrics and various Bochner formulas we have seen in § 7). More geometrically one may look at $\mathcal{C}$ as a subset in the space $\mathcal{G}_+$ of Riemannian metrics g on a given manifold V which are considered as sections of the symmetric square $S^2 T^*(V)$. Then the above mentioned global analytic features of the underlying C can often be interpreted in terms of infinitesimal geometry of $\mathcal{C}$. It is worth noticing at this point that $\mathcal{C} \subset \mathcal{G}_+$ is a *cone* for every C and that $\mathcal{C}$ is invariant under the natural action of $\text{Diff } V$ on $\mathcal{G}_+$. But $\mathcal{C}$ is not a *convex* cone unless C is *empty* or equals the space of *all* forms Q . In fact, $\mathcal{G}_+$ (which itself is a convex Diff -invariant cone in the linear space of sections $V \rightarrow S^2 T^*(V)$) contains no non-trivial Diff -invariant convex subcones at all, if the underlying manifold V is compact connected without boundary (see p. 231 in [Gro]) and also pp. 24 and 111 in this book).

SUMMARY. — This is an expanded version of my « Lezione Leonardesca » given in Milano in June 1990. I try to reveal to non-initiates the inner working of the Riemannian geometry by following the tracks of relatively few ideas from the very bottom to the top of the edifice.

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