NY 2025 Lecture on Scalar Curvature

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Unlike manifolds with controlled sectional and Ricci curvatures, those with their *scalar curvatures bounded from below* are not configured in specific rigid forms but display an uncertain variety of flexible shapes similar to what one sees in geometric topology.

Yet, there are definite limits to this flexibility, where determination of such limits crucially depends, at least in the known cases, on two seemingly unrelated analytic means: *index theory of Dirac operators* and the *geometric measure theory*,[1]

The emergent picture of spaces with $Sc.curv \geq 0$, where topology and geometry are intimately intertwined, is reminiscent of *the symplectic geometry*,[2] but the former has not reached yet the maturity of the latter.

The mystery of the scalar curvature remains unsolved.

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[1] Spaces of metrics with $Sc \geq \sigma$ on 3-manifolds are amenable to the global study with the *Hamilton's Ricci flow*, which also applies, at the present moment only C^0-locally, in higher dimensions. Also, much topological and geometrical information on 4-manifolds with $Sc \geq \sigma$, for positive as well as negative σ, is obtained, exclusively, with the *Seiberg-Witten equations*.

[2] Geometric invariants associated with the scalar curvature, such as the *K-area*, are linked with the symplectic invariants (see [G(positive) 1996], [Polterovich(rigidity) 1996], [Entov(Hofer metric) 2001], [Savelyev(jumping) 2012[), but this link is still poorly understood.

1 Curvature Formulas for Manifolds and Submanifolds.

We enlist in this section several classical formulas of Riemannian geometry and indicate their (more or less) immediate applications.

1.1 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) Riemannian Variation Formula. Let h_t, $t \in [0, \varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0, \varepsilon]$.

Notice that an arbitrary Riemannian metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0, \varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = 2A^*,$$

where

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1, ..., n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y, y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

Sign Convention. The first derivative of h changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ of a hypersurface $Y \subset X$ depends on the *coorientation* of Y in X, where our convention is such that the boundaries of *convex* domains have *positive (semi)definite* second fundamental forms A^*, also denoted II_Y, hence, *positive* mean curvatures, with respect to *the outward* normal vector fields.[3]

(2.1.B) First Variation Formula. This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t{}^4$$

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for C^1-smooth functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t{}^5$$

Now comes the first formula with the Riemannian curvature in it.

1.2 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannian manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y, \tau) = \kappa(X, \tau) + \wedge^2 A^*(\tau),$$

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^*(y) \alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

• $\alpha_i^*(y)$, $i = 1, ..., n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;

• α^*-sum is taken over all ordered pairs (i, j) with $j \neq i$;

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν, τ_i) for ν being a unit (defined up to $\pm$-sign) normal vector to Y;

• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu, \nu).$$

[3] At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.

[4] This come with the *minus* sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal(2019].

[5] This remains true for Lipschitz functions but if ψ is (badly) non-differentiable, e.g. it is equal to the characteristic function of a domain $U \subset Y$, then the derivative $\partial_{\psi\nu} vol_{n-1}(Y_t)$ may become (much) larger than this integral.

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i \neq j} \alpha_i \alpha_j = \left(\sum_i \alpha_i \right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu, \nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

$$Sc(Y) \geq -2Ricci(\nu, \nu). \qquad (\mathrm{Sc} \geq -2\mathrm{Ric})$$

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n \max_i (c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape* that is the symmetric on $T(Y)$ associated with A^* via the Riemannian scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau, \tau) = \langle A(\tau), \tau \rangle_g \text{ for all } \tau \in T(Y).$$

1.3 Variation of the Curvature of Equidistant Hypersurfaces and Weyl's Tube Formula

(2.3.A) **Second Main Formula of Riemannian Geometry.**[6] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape operators $A_t = A(Y_t)$ satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where B_t is the symmetric associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $\left(\tau, \nu = \frac{d}{dt}\right)$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently

[6] The first main formula is *Gauss' Theorema Egregium.*

than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry. [7]

Tracing this formula yields

(2.3.B) Hermann Weyl's Tube Formula.

$$trace\left(\frac{dA_t}{dt}\right) = -||A^*||^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -||A^*||^2 - Ricci(\nu, \nu),$$

where

$$||A^*||^2 = ||A||^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y,y) = Sc(X,y) + \sum_{i\neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu,\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i\neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + ||A^*||^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $||A^*||^2 = \sum_i (\alpha_i^*)^2$.

1.4 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called *umbilic* if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $Y = (Y,h)$ be a smooth Riemannian (n-1)-manifold and $\varphi = \varphi(t) > 0$, $t \in [0,\varepsilon]$ be a smooth positive function. Let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0,\varepsilon]$.

[7] Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darmos(Gravitation einsteinienne) 1927] with the reference to Darboux' textbook of 1897.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'(t)}{\varphi(t)}$, $i = 1, ..., n-1$ for

$$A_t^* = \frac{\varphi'(t)}{\varphi(t)} h_t \text{ for } \varphi' = \frac{d\varphi(t)}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}.$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2}Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2}Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

(c) What is slightly less obvious, is that if

$$\varphi(t) = \exp\int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[8]

Cylindrical Extension Exercise. Let Y be a smooth manifold, $X = Y \times \mathbb{R}_+$, let g_0 be a Riemannian metric in a neighbourhood of the boundary $Y = Y \times \{0\} = \partial X$, let h denote the Riemannian metric in Y induced from g_0 and let Y has *constant mean curvature* in X with respect to g_0.

Let X' be a (convex if you wish) ball in the standard (i.e complete simply connected) space with constant sectional curvature and of the same dimension

[8]See §12 in [GL(complete) 1983].

n as X, let $Y' = \partial X'$ be its boundary sphere, let, let $Sc(h) > 0$ and let the mean and the scalar curvatures of Y and Y' are related by the following (comparison) inequality.

$$[<] \qquad \frac{|mean.curv_{g_0}(Y)|^2}{Sc(h,y)} < \frac{|mean.curv(Y')|^2}{Sc(Y')} \text{ for all } y \in Y.$$

Show that

if Y is compact, there exists a smooth positive function $\varphi(t)$, $0 \le t < \infty$, which is constant at infinity and such that the the warped product metric $g = \varphi^2 h + dt^2$ has

the same Bartnik data *as g_0, i.e.*

$$g|Y = h_0 \text{ and } mean.curv_g(Y) = mean.curv_{g_0}(Y),$$

Then show that

one *can't make* $Sc(g) \ge Sc(X')$ in general, if [<] is relaxed to the corresponding *non-strict* inequality, where an example is provided by the Bartnik data of $Y' \in X'$ itself.[9]

Vague Question. What are "simple natural" Riemannian metrics g on $X = Y \times \mathbb{R}_+$ with given Bartnik data $(Sc(Y), mean, curv(Y))$, where $Y \subset X$ is allowed *variable* mean curvature, and what are possibilities for lower bound on the scalar curvatures of such g granted $|mean.curv(Y,y)|^2 / Sc(Y,y) < C$, e..g. for $C = |mean.curv(Y')|^2 / Sc(Y')$ for Y' being a sphere in a space of constant curvature.

1.4.1 Higher Warped Products

Let Y and S be Riemannian manifolds with the metrics denoted dy^2 (which now play the role of the above dt^2) and ds^2 (instead of h), let $\varphi > 0$ be a smooth function on Y, and let

$$g = \varphi^2(y) ds^2 + dy^2$$

be the corresponding warped metric on $Y \times S$,

Then

$(\star\star)$

$$Sc(g)(y,s) = Sc(Y)(y) + \frac{1}{\varphi(y)^2} Sc(S)(s) - \frac{m(m-1)}{\varphi^2(y)} ||\nabla\varphi(y)||^2 - \frac{2m}{\varphi(y)} \Delta\varphi(y),$$

where $m = dim(S)$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace on Y.

To prove this, apply the above c $(\star)$ to $l \times S$ for naturally parametrised geodesics $l \subset Y$ passing trough y and then average over the space of these l, that is the unit tangent sphere of Y at y.

The most relevant example here is where S is the real line $\mathbb{R}$ or the circle S^1 also denoted $\mathbb{T}^1$ and where $(\star)$ reduces to

$$(\star\star)_1 \qquad Sc(g)(y,s) = Sc(Y)(y) - \frac{2}{\varphi} \Delta\varphi(y).^{10}$$

[9] It follows from [Brendle-Marques(balls in S^n)N 2011] that the the cylinder $S^{n-1} \times \mathbb{R}_+$ admits a complete Riemannian metric g cylindrical at infinity which has $Sc(g) > n(n-1)$, and which has the same Bartnik data as the boundary sphere X'_0 in the hemisphere X' in the unit n-sphere. But the non-deformation result from [Brendle-Marques(balls in S^n) 2011], suggests that this might be impossible for the Bartnik data of *small* balls in the round sphere.

[10] The roles of Y and $S = \mathbb{R}$ and notationally reversed here with respect to those in $(\star)$

For instance, if the $L = -\Delta + \frac{1}{2}Sc$ on Y is strictly positive, that is the lowest eigenvalue λ is strictly positive and if φ equals to the corresponding eigenfunction of L, then

$$-\Delta\varphi = \lambda\cdot\varphi - \frac{1}{2}Sc\cdot\varphi$$

and

$$Sc(g) = 2\lambda > 0,$$

The basic feature of the metrics $\varphi^2(y)ds^2 + dy^2$ on $Y\times\mathbb{R}$ is that they are $\mathbb{R}$-*invariant*, where the quotients $(Y\times\mathbb{R})/\mathbb{Z} = Y\times\mathbb{T}^1$ carry the corresponding $\mathbb{T}^1$-*invariant* metrics, while *the* $\mathbb{R}$-*quotients are isometric to* Y.

Besides $\mathbb{R}$-invariance, a characteristic feature of warped product metrics is *integrability* of the tangent hyperplane field normal to the $\mathbb{R}$-orbits, where $Y\times\{0\}\subset Y\times\mathbb{R}$, being normal to these orbits, serves as an integral variety for this field.

Also notice that $Y = Y\times\{0\}\subset Y\times\mathbb{R}$ is totally geodesic with respect to the metric $\varphi^2(y)ds^2+dy^2$, while the ($\mathbb{R}$-invariant) *curvature* (vector field) *of the* $\mathbb{R}$-*orbits* is equal to the *gradient field* $\nabla\varphi$ extended from Y to $Y\times\mathbb{R}$. coordinates

In what follows, we emphasize $\mathbb{R}$-invariance and interchangeably speak of $\mathbb{R}$-invariant metrics on $Y\times\mathbb{R}$ and metrics warped with factors φ^2 over Y.

Gauss-Bonnet $g^{\rtimes}$-*Exercise.* Let the above S be the Euclidean space $\mathbb{R}^N$ (make it $\mathbb{T}^n$ if you wish to keep compactness) with coordinates $t_1,...,t_N$, let

$$\Phi(y) = (\varphi_1(y),...,\varphi_i(y),...,\varphi_N(y))$$

be an N-tuple of smooth positive function on a Riemannian mnanifold $Y = (Y,g)$ and define the (iterated t warped product) metric $g^{\rtimes} = g^{\rtimes}_{\Phi}$ on $Y\times S$ as follows:

$$g^{\rtimes} = g(y) + \varphi_1^2(y)dt_1^2 + \varphi_2^2(y)dt_2^2 + ... + \varphi_N^2(y)dt_N^2$$

Show that the scalar curvature of this metric, which, being $\mathbb{R}^N$-invariant, is regarded as a function on Y, satisfies:

$$Sc(g^{\rtimes},y) = Sc(g) - 2\sum_{i=1}^N\Delta_g\log\varphi_i - \sum_{i=1}^N(\nabla_g\log\varphi_i)^2 - \left(\sum_{i=1}^N\nabla_g\log\varphi_i\right)^2,$$

thus

$$\int_Y Sc(g^{\rtimes},y)dy \le \int_Y Sc(g,y)dy,$$

and, following [Zhu(rigidity) 2019], obtain the following

"Warped" Gauss-Bonnet Inequality for Closed Surfaces Y:

$$\int_Y Sc(g^{\rtimes},y)dy \le 4\pi\chi(Y)$$

for the (iterated) warped product metrics $g^{\rtimes} = g^{\rtimes}_{\phi}$ for all positive N-tuples of Φ of positive functions on Y. [11]

[11]See [Zhu() 2019] and sections ??, ?? for applications and generalizations.

1.5 Second Variation Formula

The Weyl formula also yields the following formula for the *second derivative* of the $(n-1)$-volume of a cooriented hypersurface $Y \subset X$ under a normal deformation of Y in X, where the scalar curvature of X plays an essential role.

The deformations we have in mind are by vector fields directed by geodesic normal to Y, where in the simplest case the norm of his field equals one.

In this case we have an equidistant motion $Y \mapsto Y_t$ as earlier and the second derivative of $vol_{n-1}(Y_t)$, denoted here $Vol = Vol_t$, is expressed in terms of of the shape $A_t = A(Y_t)$ of Y_t and the Ricci curvature of X, where, recall $trace(A_t) = mean.curv(Y_t)$ and

$$\partial_\nu Vol = \int_Y mean.curv(Y)dy$$

by the first variation formula.

Then, by Leibniz' rule,

$$\partial^2_\nu Vol = \partial_\nu \int_Y trace(A(y))dy = \int_Y trace^2(A(y))dy + \int_Y trace(\partial_\nu A(y))dy,$$

and where, by Weyl's formula,

$$trace(\partial_\nu A) = -trace(A^2) - Ricci(\nu,\nu)$$

for the normal unit field ν.

Thus,

$$\partial^2_\nu Vol = \int_Y (mean.curv)^2 - trace(A^2) - Ricci(\nu,\nu),$$

which, combining this with the above expression

$$Ricci(\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + ||A^*||^2\right),$$

shows that

$$\partial^2_\nu Vol = \int \frac{1}{2}\left(Sc(Y) - Sc(X) + mean.curv^2 - ||A^*||^2\right).$$

In particular, if $Sc(X) \geq 0$ and Y is minimal, then,

$$(\textstyle\int \mathrm{Sc} \geq 2\partial^2 \mathrm{Vol}) \qquad \int_Y Sc(Y,y)dy \geq 2\partial^2_\nu Vol$$

(compare with the $(Sc \geq -2Ric)$ in 2.2).

Warning. Unless Y is minimal and despite the notation ∂^2_ν, this derivative depends on how the normal filed on $Y \subset X$ is extended to a vector filed on (a neighbourhood of Y in) X.

Illuminative Exercise. Check up this formula for concentric spheres of radii t in the spaces with constant sectional curvatures that are S^n, $\mathbb{R}^n$ and $\mathbf{H}^n$.

Now, let us allow a non-constant geodesic field normal to Y, call it $\psi\nu$, where $\psi(y)$ is a smooth function on Y and write down the full second variation formula as follows:

$$\partial^2_{\psi\nu} vol_{n-1}(Y) = \int_Y ||d\psi(y)||^2 dy + R(y)\psi^2(y)dy$$

for

$$[\circ\circ] \qquad R(y) = \frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - ||A^*(Y)||^2\right),$$

where $M(y)$ stands for the mean curvature of Y at $y \in Y$ and $||A^*(Y)||^2 = \sum_i (\alpha^*)^2$, $i = 1, ..., n-1$.

Notice, that the "new" term $\int_Y ||d\psi(y)||^2 dy$ depends only on the normal field itself, while the R-term depends on the extension of $\psi\nu$ to X, unless

Y is minimal, where $[\circ\circ]$ reduces to

$$[**] \qquad \partial^2_{\psi\nu} vol_{n-1}(Y) = \int_Y ||d\psi||^2 + \frac{1}{2}\left(Sc(Y) - Sc(X) - ||A^*||^2\right)\psi^2.$$

Furthermore, if Y is volume minimizing in its neighbourhood, then $\partial^2_{\psi\nu} vol_{n-1}(Y) \geq 0$; therefore,

$$[\star\star] \qquad \int_Y (||d\psi||^2 + \frac{1}{2}(Sc(Y))\psi^2 \geq \frac{1}{2}\int_Y (Sc(X,y) + ||A^*(Y)||^2)\psi^2 dy$$

for all non-zero functions $\psi = \psi(y)$.

Then, if we recall that

$$\int_Y ||d\psi||^2 dy = \int_Y \langle -\Delta\psi, \psi\rangle dy,$$

we will see that $[\star\star]$ says that

the $\psi \mapsto -\Delta\psi + \frac{1}{2}Sc(Y)\psi$ *is greater than*[12] $\psi \mapsto \frac{1}{2}(Sc(X,y) + ||A^*(Y)||^2)\psi$.

Consequently,

if $Sc(X) > 0$, then the $-\Delta + \frac{1}{2}Sc(Y)$ on Y is positive.

Justification of the $||d\psi||^2$ Term. Let $X = Y \times \mathbb{R}$ with the product metric and let $Y = Y_0 = Y \times \{0\}$ and $Y_{\varepsilon\psi} \subset X$ be the graph of the function $\varepsilon\psi$ on Y. Then

$$vol_{n-1}(Y_{\varepsilon\psi}) = \int_Y \sqrt{1 + \varepsilon^2 ||d\psi||^2} dy = vol_{n-1}(Y) + \frac{1}{2}\int_Y \varepsilon^2 ||d\psi||^2 + o(\varepsilon^2)$$

by the Pythagorean theorem

and

$$\frac{d^2 vol_{n-1}(Y_{\varepsilon\psi})}{d^2\varepsilon} = ||d\psi||^2 + o(1).$$

by the binomial formula.

This proves $[\circ\circ]$ for product manifolds and the general case follows by *linearity/naturality/functoriality* of the formula $[\circ\circ]$.

Naturality Problem. All "true formulas" in the Riemannian geometry should be derived with minimal, if any, amount of calculation – only on the basis of their "naturality" and/or of their validity in simple examples, where these formulas are obvious.

Unfortunately, this "naturality principle" is absent from the textbooks on differential geometry, but, I guess, it may be found in some algebraic articles (books?).

Exercise. Derive the second main formula 2.3.A by pure thought from its manifestations in the examples in the above *illuminative exercise.*[13]

[12] $A \geq B$ for selfadjoint operators signifies that $A - B$ is positive semidefinite.

[13] I haven't myself solved this exercise.

1.6 Conformal Laplacian and the Scalar Curvature of Conformally and non-Conformally Scaled Riemannian Metrics

Let (X_0, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and let $\varphi = \varphi(x)$ be a smooth positive function on X.

Then, by a straightforward calculation,[14]

$$◐ \qquad Sc(\varphi^2 g_0) = \gamma_n^{-1} \varphi^{-\frac{n+2}{2}} L(\varphi^{\frac{n-2}{2}}),$$

where L is the *conformal Laplace* on (X_0, g_0)

$$L(f(x)) = -\Delta f(x) + \gamma_n Sc(g_0, x) f(x)$$

for the ordinary Laplace (Beltrami) $\Delta f = \Delta_{g_0} f = \sum_i \partial_{ii} f$ and $\gamma_n = \frac{n-2}{4(n-1)}$.

Thus, we conclude to the following.

Kazdan-Warner Conformal Change Theorem. [15] Let $X = (X, g_0)$ be a closed Riemannian manifold, such the the conformal Laplace L is positive.

Then X admits a Riemannian metric g (conformal to g_0) for which $Sc(g) > 0$.

Proof. Since L is positive, its first eigenfunction, say $f(x)$ is positive[16] and since $L(f) = \lambda f$, $\lambda > 0$,

$$Sc\left(f^{\frac{4}{n-2}} g_0\right) = \gamma_n^{-1} L(f) f^{-\frac{n+2}{n-2}} = \gamma_n^{-1} f^{\frac{2n}{n-2}} > 0.$$

Example: Schwarzschild metric. If (X_0, g_0) is the Euclidean 3-space, and $f = f(x)$ is positive function, then

the sign of $Sc(f^4 g_0)$ is equal to that of $-\Delta f$.

In particular, since the function $\frac{1}{r} = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}$, is harmonic, *the Schwarzschild metric $g_{Sw} = \left(1 + \frac{m}{2r}\right)^4 g_0$ has zero scalar curvature.*

If $m > 0$, then this metric is defined for all $r > 0$ and it is invariant under the involution $r \mapsto \frac{m^2}{r}$.

If $m = 0$, this the flat Euclidian metric.

If $m < 0$, then this metric is defined only for $r > m$ with a singularity ar $r = m$.

Non-Conformal Scaling. Let $X = (X, g)$ be a smooth n-manifold, and let $\mathbb{R}_x^\times \subset GL_x(n)$, $x \in X$, be a smooth family of diagnosable (semisimple) 1-parameter subgroups in the linear groups $GL_x(n) = GL_n$ that act in the tangent spaces $T_x(X)$.

Then the the multiplicative group of functions $\phi : X \to \mathbb{R}^\times$ acts on the tangent bundle $T(X)$ by

$$\tau \mapsto= \phi(x)(\tau) \text{ for } \phi(x) \in \mathbb{R}^\times = \mathbb{R}_x^\times \subset GL_x = GL(T_x(X))$$

and, thus on the space of Riemannin metrics g on X.

[14] There must be a better argument.

[15] [Kazdan-Warner(conformal) 1975]: *Scalar curvature and conformal deformation of Riemannian structure.*

[16] We explain this in section 1.9.

The main instance of such an action is where the tangent bundle is orthogonally split, $T(X) = T_1 \oplus T_2$, and ϕ acts by scaling on the subbundle T_2.

It is an not hard to write down a formula for the scalar curvature of $g_1 + \phi^2 g_2$, but it is unclear what, in general, would be a workable criterion for solvability of the inequality $Sc(g_\varphi) > 0$ in φ, e.g. in the case where $X = X_1 \times X_2$ and the subbundles T_1 and T_2 are equal to the tangent bundles of submanifolds $X_1 \times x_2 \subset X$, $x_2 \in X_2$, and $x_1 \times X_2 \subset X$, $x_1 \in X_1$.

Yet, in the case of $rank(T_2) = 1$, this equation introduced, I believe, by Robert Bartnik in [Bartnik(prescribed scalar) 1993] was successfully applied to extension of metrics with $Sc > 0$ (see section **??**)[17]

1.7 Schoen-Yau's Non-Existence Results for $Sc > 0$ on SYS Manifolds via Minimal (Hyper)Surfaces and Quasisymplectic $[Sc \ngtr 0]$–Theorem

Let X be a three dimensional Riemannian manifold with $Sc(X) > 0$ and $Y \subset X$ be an orientable cooriented surface with minimal area in its integer homology class.

Then the inequality $(\int Sc \geq 2\partial^2 V)$ from section 1.5, which says in the present case that

$$\int_Y Sc(Y, y)dy > 2\partial_\nu^2 area(Y),$$

implies that

Y must be a topological sphere.

In fact, minimality of Y makes $\partial_\nu^2 area(Y) \geq 0$, hence $\int_Y Sc(Y, y)dy > 0$, and the sphericity of Y follows by the Gauss-Bonnet theorem.

And since all integer homology classes in closed orientable Riemannian 3-manifolds admit area minimizing representatives by the geometric measure theory developed by Federer, Fleming and Almgren, we arrive at the following conclusion.

★$_3$ **Schoen-Yau $3d$-Theorem**. *All integer 2D homology classes in closed Riemannian 3-manifolds with $Sc > 0$ are spherical.*

For instance, the 3-torus admits no metric with $Sc > 0$.

The above argument appears in Schoen-Yau's 15-page paper [SY(incompressible) 1979], most of which is occupied by an independent proof of the existence and regularity of minimal Y.

In fact, the existence of minimal surfaces and their regularity needed for the above argument has been known since late (early?) 60s[18] but, what was, probably, missing prior to the Schoen-Yau paper was the innocuously looking corollary of Gauss' formula in 2.2,

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu, \nu)$$

and the issuing inequality

$$Sc(Y) > -2Ricci(\nu, \nu)$$

[17] Other special cases of this are (implicitly) present in the geometry of Riemannin warped product, in the process of *smoothing corners with* $Sc \geq \sigma$ and in the *transversal blow up* of foliations with $Sc > 0$.

[18] Regularity of volume minimizing hypersurfaces in manifolds X of dimension $n \leq 7$, as we mentioned earlier, was proved by Herbert Federer in [Fed(singular) 1970], by reducing the general case of the problem to that of minimal cones resolved by Jim Simons in [Simons(minimal) 1968].

for minimal Y in manifolds X with $Sc(X) > 0$.

For example, Burago and Toponogov, come close to the above argument, where, they bound from below the injectivity radius of Riemannian 3-manifolds X with $sect.curv(X) \le 1$ and $Ricci(X) \ge \rho > 0$ by

$$inj.rad(X) \ge 6e^{-\frac{6}{\rho}},$$

where this is done by carefully analysing minimal surfaces $Y \subset X$ bounded by, a priori very short, closed geodesics in X, and where an essential step in the proof is the lower bound on the first eigenvalue of the Laplace on Y by $\sqrt{Ricci(X)}$.[19]

Area Exercises. Let X be homeomorphic to $Y \times S^1$, where Y is a closed orientable surface with the Euler number χ.

(a) Let $\chi > 0$, $Sc(X) \ge 2$ and show that there exists a surface $Y_o \subset X$ homologous to $Y \times \{s_0\}$, such that $area(Y_o) \le 4\pi$.[20]

(b) Let $\chi < 0$, $Sc(X) \ge -2$ and show that all surfaces $Y_* \in X$ homologous to $Y \times \{s_0\}$ have $area(Y_*) \ge -2\pi\chi$.

(c) Show that (a) remains valid for complete manifolds X homeomorphic to $Y \times \mathbb{R}$.[21]

$\bigstar^{codim1}$ **Schoen-Yau Codimension 1 Descent Theorem,** [SY(structure) 1979]. Let X be a compact orientable n-manifold with $Sc > 0$.

If $n \le 7$, then all integer homology classes $h \in H_{n-1}(X)$ are representable by compact oriented $(n-1)$-submanifolds Y in X, which admit metrics with $Sc > 0$.

Proof. Let Y be a volume minimizing hypersurface representing h, the existence and regularity of which is guaranteed by a Federer 1970-theorem[22] and recall that by $[\star\star]$ in 1.5 the $-\Delta + \frac{1}{2}Sc(Y)$ is positive. Hence, the conformal Laplace $-\Delta + \gamma_n Sc(Y)$ is also positive for $\gamma_n = \frac{n-2}{4n-1} \le \frac{1}{2}$ and the proof follows by Kazdan-Warner conformal change theorem.

$\bigstar_{\mathbb{T}^n}$ **Mapping to the Torus Corollary.** If a closed orientable n-manifold X admits a map to the torus $\mathbb{T}^n$ with *non-zero degree*, then X admits *no metric with* $Sc > 0$.

Indeed, if a closed submanifold Y^{n-1} is *non-homologous to zero* in this X then it (obviously) admits a map to $\mathbb{T}^{n-1}$ with non-zero degree. Thus, the above allows an inductive reduction of the problem to the case of $n = 2$, where the Gauss-Bonnet theorem applies.

SYS-Manifolds. Schoen and Yau say in [SY(structure) 1979] that their codimension 1 descent theorem delivers a topological obstruction to $Sc > 0$ on a class of manifolds, which is, even in the spin case, [23] is not covered by the twisted Dirac operators methods.

[19][BurTop(curvature bounded above)1973],*On 3-dimensional Riemannian spaces with curvature bounded above.*

[20]See [Zhu(rigidity) 2019] for a higher dimensional version of this inequality.

[21]I haven't solved this exercise.

[22][Federer(singular) 1970]: *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension.*

[23]A smooth connected n-manifolds X is *spin* if the frame bundle over X admits a double cover extending the natural double cover of a fiber, where such a fiber is equal to the linear group, (each of the two connected components of) which admits a a unique non-trivial double cover $\tilde{G}L(n) \to GL(n)$.

This claim was confirmed by Thomas Schick, who defined, in homotopy theoretic terms, integer homology classes in aspherical spaces, say $h \in H_n(\underline{X})$ and who proved using the codimension one descent theorem that these h for $n \le 7$ can't be dominated by compact orientable n-manifolds with $Sc > 0$.

In more geometric terms, the n-manifolds X, to which Schick's argument applies, we call them *Schoen-Yau-Schick*, can be described d as follows.

A closed orientable n-manifold is *Schoen-Yau-Schick* if it admits a smooth map $f : X \to \mathbb{T}^{n-2}$, such that the homology class of the pullback of a generic point,

$$h = [f^{-1}(t)] \in H_2(X)$$

is *non-spherical,* i.e. it is not in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.

Then Schick's corollary to Schoen-Yau's theorem reads.

$\bigstar_{SYS}$ **Non-existence Theorem for SYS Manifolds.** *Schoen-Yau-Schick manifolds of dimensions $n \le 7$ admit no metrics with $Sc > 0$.*

(b) *Exercises.* (b_1) Construct examples of SYS manifolds of dimension $n \ge 4$, where all maps $X \to \mathbb{T}^n$ have zero degrees.

Hint: apply surgery to $\mathbb{T}^n$.

(b_2) Show that if the first homology group $H_1(X)$ of a SYS-manifold has no torsion, then a finite covering of X admits a map with degree one to the torus $\mathbb{T}^n$.

(c) The limitation $n \le 7$ of the above argument is due a presence of singularities of minimal subvarieties in X for $dim(X) \ge 8$.

If $n = 8$, these singularities were proven to be unstable by Nathan Smale; this improves $n \le 7$ to $n \le 8$ in $\bigstar_{SYS}$

More recently, as we mentioned earlier, the dimension restriction was removed for all n by Lohkamp and by Schoen-Yau; the arguments in both papers are difficult and I have not mastered them.[24]

Although the Dirac operator arguments don't apply to SYS-manifolds, they do deliver topological obstructions to $Sc > 0$, which, according to the present state of knowledge, lie beyond the range of the minimal surface techniques. Here is an instance of this.

$\otimes_{\wedge^k \tilde{\omega}}$ **Quasisymplectic Non-Existence Theorem.** Let X be a compact $\otimes_{\wedge^k \tilde{\omega}}$*-manifold* of dimension $n = 2k$, i.e. X is orientable and it carries a *closed 2-form* ω (e.g. a symplectic one), such that $\int_X \omega^k \ne 0$, and such that the lift $\tilde{\omega}$ of ω to the universal covering $\tilde{X}$ is *exact*, e.g. $\tilde{X}$ is contractible.[25]

Then X *admits no metric with $Sc > 0$.*

This applies, for instance, to *even dimensional tori*, to *aspherical* 4*-manifolds with* $H^2(X, \mathbb{R}) \ne 0$ and to *products* of such manifolds[26] but *not* to general SYS-manifolds.

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[24] See [Smale(generic regularity) 2003], SY(singularities) 2017], [Lohkamp(smoothing) 2018] and section **??**.

[25] It's enough to have $\tilde{X}$ spin.

[26] Recently, Chodosh and Li proved that

compact aspherical manifolds of dimensions 4 and 5 admit no metrics with positive scalar curvatures. (See [Chodosh-Li(bubbles) 2020], [G(aspherical) 2020] and section**??**)

But this remains problematic for products of pairs of aspherical 4-manifolds.

Idea of the Proof. Assume without loss of generality that ω serves as the curvature form of a complex line bundle$L \to X$and let $\tilde{L} \to \tilde{X}$ be the lift of L to the universal covering $\tilde{X} \to X$.

Since the curvature $\tilde{\omega}$ of $\tilde{L}$, is exact the bundle $\tilde{L}$is topologically trivial, hence it can be represented by k-th tensorial power of another line bundle,

$$L = (L^{\frac{1}{k}})^{\otimes k},$$

where the curvature of $L^{\frac{1}{k}}$ is $\frac{1}{k}\tilde{\omega}$. By Atiyah's L_2-*index theorem*, there are *non-zero harmonic* L_2-*spinors* on $\tilde{X}$ *twisted with* $L^{\frac{1}{k}}$ for infinitely many k, but the twisted Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula applied to large k doesn't allow such spinors for $Sc(\tilde{X} \geq \sigma > 0$.[27]

Exercise. Show that if X is $\otimes_{\wedge^k\tilde{\omega}}$, then the classifying map $X \to \mathsf{B}(\Pi)$, where $\mathsf{B}(\Pi) = K(\Pi, 1)$ is the classifying space for the group $\Pi = \pi_1(X)$, sends the fundamental homology class $[X]$ to a *non-torsion* class in $H_n(\mathsf{B}(\Pi))$.

Problem. *Is there a unified approach that would apply to* SYS*-manifolds and to the above* $\otimes_{\wedge^k\tilde{\omega}}$*-manifolds* X*, e.g. symplectic ones with contractible universal coverings?*

For instance,

do products of SYS *and* $\otimes_{\wedge^k\tilde{\omega}}$*-manifolds ever carry metrics with positive scalar curvatures?*

1.8 Warped $\mathbb{T}^{\rtimes}$-Stabilization and Sc-Normalization

Many geometric properties of Riemannian manifolds $X = (X, g)$ implied by the inequality $Sc(g) \geq \sigma$ follow (possibly in a weaker form) from the same inequality for a larger manifold, say X^*, that, topologically, is the product of X with the a torus, $X^* = X \times \mathbb{T}^N$ for some $N = 1, 2, ...,$ where the Riemannian metric g^* on X^* is invariant under the action of $\mathbb{T}^N$ and where $X^*/\mathbb{T}^N$ is isometric to X.

Surface Examples. Let $X = (X, g)$ be a closed surface and g^* be a $\mathbb{T}^N$-invariant metric on $X \times \mathbb{T}^N$, such that

$$(X \times \mathbb{T}^N, g^*)/\mathbb{T}^N = (X, g).$$

(a) ***Sharp Equivariant Area Inequality.*** If $Sc(g^*) \geq \sigma > 0$, then a special case a theorem by Jintian Zhu,[28] says that

the area of X *is bounded the same way as it is for* $Sc(g) \geq \sigma$,

$$area(X) \leq \frac{8\pi}{\sigma}.$$

Moreover,

the equality holds only if X^* *is the* isometric *product* $X \times \mathbb{T}^N$.

[27]Atiyah's theorem from [Atiyah(L2) 1976] needs a slight adjustment here, since the action of the fundamental group $\Gamma = \pi_1(X)$ on $\tilde{X}$ doesn't lift to $L^{\frac{1}{k}}$; yet the fundamental group of the (total space) of the unit circle bundle of L does naturally act on $L^{\frac{1}{k}}$. Also, there is no difficulty in extending Lichnerowicz' vanishing argument to the L_2 case, see §$9\frac{1}{8}$ in [G(positive) 1996].

[28]See [Zhu(rigidity) 2019] and **??**, **??** for related inequalities.

(b) (Weakened) $\mathbb{T}^*$-***Stable 2d Bonnet-Myers Diameter Inequality***. If $Sc(g^*) \geq \sigma$, then

$$[BMD] \qquad diam(X) \leq 2\pi\sqrt{\frac{N+1}{(N+2)\sigma}} < \frac{2\pi}{\sqrt{\sigma}}.$$

Proof. Given two points $x_1, x_2 \in X$, take two small ε-circles Y_{-1} and Y_{+1} around them, let $X_\varepsilon \subset X$ be the band between them and and apply (the relatively elementary $\mathbb{T}^N$-invariant case of) the $\frac{2\pi}{n}$-*Inequality* from section **??**.[29]

Non Trivial Torus Bundles. The inequality [BMD] is valid for (all) Riemannian $(N+2)$-manifolds X^* with free isometric $\mathbb{T}^N$-actions:

if $Sc(X^*) \geq \sigma > 0$, *then* $diam(X^*/\mathbb{T}^N) \leq 2\pi\sqrt{(N+1)/(N+2)\sigma}$.

In fact, the above proof applies, since, *topologically*, the part of X^* that lies over the band $X_\varepsilon \subset X$ *is* the product, $X_\varepsilon \times \mathbb{T}^N$.

It is *unclear*, however, if the areas of $X^*/\mathbb{T}^N$ are bounded in terms of $Sc(X^*)$ for all such X^*.

And, as we shall see later, possible non-triviality of torus bundles create complications for other problems with scalar curvature.

General Question. The above examples suggests that quotients X of manifolds X^* with $Sc(X^*) \geq \sigma$ under free isometric actions of tori have *similar* geometric properties to those of manifolds which have $Sc \geq \sigma$ themselves. But it is unclear how far this similarity goes.

Example. let X be a closed surface and $X^\rtimes = X \rtimes \mathbb{T}^1$ be a warped product as described below.

Does the inequality $Sc(X^\rtimes) \geq 2$ yield an upper bound on *all of geometry* of X?

For instance,

is there a bound on the number of unit discs needed to cover X?

(If $Sc(X) \geq 2$, then X admits a *distance decreasing* homeomorphism from the unit sphere S^2, that can be constructed using the family of boundary curves of concentric discs with center at some point in X.)

Warped Products. As far as geometric applications are concerned, the relevant X^* are (iterated) *warped products*, we denote them $X^\rtimes$ and call *warped* $\mathbb{T}^N$-*extensions of* X, that are characterized by the existence of *isometric* sections $X \to X^\rtimes$ for $X^\rtimes \to X = X^\rtimes/\mathbb{T}^N$.

Clearly, metrics $g^\rtimes$ on these $X^\rtimes$ are

$$g^\rtimes = g + \varphi_1^2(x)dt_1^2 + \varphi_2^2(x)dt_2^2 + ... + \varphi_N^2(x)dt_N^2$$

for some positive functions φ_i on X.

Among these we distinguish $O(N)$-*invariant* warped extensions, where the $\mathbb{Z}^N$ covering manifolds $\tilde{X}^\rtimes = X \times \mathbb{R}^N$, where

$$\tilde{X}^\rtimes/\mathbb{Z}^N = X^\rtimes,$$

are invariant under the action of the orthogonal group $O(N)$. Thus, $\tilde{X}^\rtimes$ are acted upon by the full isometry group of $\mathbb{R}^N$, that is $\mathbb{R}^N \rtimes O(N)$.

[29]Also see §2 in [G(inequalities) 2018] and the proof of theorem 10.2 in [GL(complete) 1983].

Equivalently, the metric in such an $X^{\rtimes}$ is a "simple" warped product: $g^{\rtimes} = g + \varphi^2 d||\bar{t}||^2$ for $\bar{t} = (t_1, t_2, ..., t_N)$, the scalar curvature of which, as we know, 1.4 is

$$Sc(g^{\rtimes})(x, \bar{t}) = Sc(X)(x) - \frac{2N}{\varphi(x)}\Delta_g\varphi(x) - \frac{N(N-1)}{\varphi^2(x)}$$

and which is most simple (and useful) for $N = 1$, where

$$[\rtimes_\varphi] \qquad Sc(g^{\rtimes})(x, \bar{t}) = Sc(X)(x) - \frac{2}{\varphi(x)}\Delta_g\varphi(x).$$

for the Laplace (Beltrami) Δ_g on $X = (X, g)$.

$[\rtimes_\varphi]^N$**-Symmetrization Theorem.** Let $X = (X, g)$ be a closed oriented Riemannian manifold of dimension $n = m + N$ and let

$$X \supset X_{-1} \supset ... \supset X_{-i} \supset ... \supset X_{-N},$$

be a descending chain of closed oriented submanifolds, where each $X_{-i} \subset X$ is equal to a transversal intersection of $X_{-(i-1)}$ with a smooth closed oriented hypersurface $H_i \subset X$,

$$H_i \cap X_{-(i-1)} = X_{-i}.$$

If $n \leq 7$, then

there exists a closed oriented m-dimensional submanifold $Y \subset X$ homologous to X_{-N} and a warped product $\mathbb{T}^N$-extension $Y^{\rtimes}$ of $Y = (Y, h)$ for the Riemannian metric h on Y induced from g on X, such that the scalar curvature of $Y^{\rtimes}$, that is, being $\mathbb{T}^N$-invariant, is represented by a function on Y, is bounded from below by the Scalar curvature of X on $Y \subset X$,

$$Sc(Y^{\rtimes}, y) \geq Sc(X, y), \ \ y \in Y.$$

Proof. Proceed by induction on codimension $i = 1, 2, , ...N$ and construct submanifolds

$$X \supset Y_1 \supset ... \supset Y_i \supset ... \supset Y_N = Y \subset X$$

as follows.

At the first step, let $Y_1 \subset X$ be a volume minimizing, hence stable, hypersurface homologous to X_{-1} where, the positivity of the second variation implies the positivity of the

$$-\Delta + \frac{1}{2}(Sc(Y_1) - Sc(X)|_{Y_1},$$

for the Laplace $\Delta = \Delta_{h_1}$ on Y_1 with the metric h_1 induced from X and let $\psi_1 > 0$ be the first eigenfunction of this with the positive eigenvalue λ_1, thus

$$-\Delta\psi = \left(\lambda - \frac{1}{2}(Sc(Y, h_1) - Sc(X))\right) \cdot \psi_1.$$

Here, let $h_1^{\rtimes}(y) = h_1(y) + \psi^2 dt^2$ be the warped product metric on $Y_1 \times \mathbb{T}^1$ and observe

$$Sc(h_1^{\rtimes}, y) = Sc(h_1, y) - \frac{2}{\psi}\Delta\psi_1 = Sc(X, y) + 2\lambda_1.$$

Then, at the second step, let $Y_2 \subset Y_1$ be a hypersurface, such that $Y_2 \times \mathbb{T}^1 \subset Y_1 \times \mathbb{T}^1$ is volume minimizing for the metric $h_1^{\rtimes}$, which is equivalent for Y_2 to be volume minimizing in Y_1 with respect to the metric $\psi_1^{l_1} h_1$ for $l_1 = \frac{2}{n-1}$.

Thus we obtain Y_2', where the corresponding metric on $Y_2' \times \mathbb{T}^2$ is

$$h_2' + \psi_1^2 dt_1^2 + \psi_2^2 dt_2^2.$$

Repeating this $N-2$ more times, we arrive at Y_N' and an (iterated) warped product metric

$$h_N' + \sum_{i=1}^{N} \psi_i^2 dt_i^2 \text{ on } Y_N' \times \mathbb{T}^N,$$

which can be symmetrised further to the required $h^{\rtimes}$ by applying the above infinitely many times to hypersurfaces $Y_N' \times T^{N-1} \subset Y_N' \times T^N$ for all subtori $T^{N-1} \subset Y_N' \times T^N$.[30] (The luxury of the extra $O(N)$-symmetry is unneeded for most purposes.)

Exercise. Apply $[\rtimes_\varphi]^N$-symmetrization to n-manifolds with isometric $\mathbb{T}^{n-2}$-actions and prove the above equivariant area inequality by reducing it to the warped product case that was already settled in section 1.4.1.

Symmetrization by Reflections and Convergence Problem. Let Y be a closed minimal co-orientable (i.e. two sided) hypersurface in a Riemannian manifold. If Y is locally volume minimizing, then it admits arbitrarily small neighbourhoods $V_\varepsilon \supset Y$ in X with smooth *strictly mean convex* boundaries. Then by reflecting such a varepsilon in the two boundary components, one obtains manifolds $\hat{V}_\varepsilon$ with isometric actions of $\mathbb{Z} \rtimes \mathbb{Z}_2$.

If these Y are non-singular, e.g. if $dim(X) \leq 7$, then one can take solutions of the isoperimetric problem for these V_ε, where one minimize the volumes of both components of the boundaries of V_ε per given (small) volume contained between them and Y. In this case, $\hat{V}_\varepsilon$, $\varepsilon \to 0$, converge to smooth Riemannian manifolds $V^{\rtimes}$ with isometric actions of $\mathbb{R}$ and with their scalar curvatures bounded from below by $Sc(X)|_Y$.

If Y is singular, the boundaries of these V_ε, even if singular, [31] can be smoothed with positive mean curvatures, but it is unclear if they converge to a reasonable object for $\varepsilon \to 0$: what is *missing for convergence* is a *Harnack type inequality* for the boundary components of $\partial_1, \partial_2 \subset \partial V_\varepsilon$, that is a uniform bound for the ratios of the distances

$$\frac{dist(y, \partial_i)}{dist(y', \partial_i)}, y, y' \in Y,$$

$i = 1, 2$, and /or of distances $dist(x, x', Y)$, $x, x' \in \partial_i$.

Notice, that "symmetrization by reflections", albeit open to generalizations to singular Y, is not, apparently, applicable, to stable μ-bubbles Y, where the warped product construction does apply. [32]

[30]See in, §12[GL(complete)1983], [G(inequalities) 2018] and also the sections **??**, **??** for details of this argument and for generalizations.

[31]If $n = 8$, then, by adapting Nathan Smale's argument, one can show that these V_ε are non-singular for an open dense set of values of ε; but this is problematic for $n \geq 9$.

[32]See §8 in [G(billiards) 2014], §4.3 in [G(inequalities) 2019] and section **??** for more about all this.

Symmetrization versus Normalization. $\mathbb{T}^{\rtimes}$-Symmetrization of metrics g typically) makes their scalar curvatures *constant* by paying the price of modification of the topology of the underlying manifolds, $X \rightsquigarrow X \times \mathbb{T}^1$.

As far as sets of "interesting" maps between Riemannian manifolds are concerned a similar effect effect is achieved by keeping the same manifold X but modifying the metric by $g = g(x) \rightsquigarrow g^\circ = g^\circ(x) = Sc(X,x)g(x)$.

In fact, we shall see later in many examples, that

there is a close (but not fully understood) similarity between the sets of λ°-*Lipschitz* maps $(X, g^\circ) \to (Y, h^\circ)$ and of $\mathbb{T}^1$-*equivariant* $\lambda^{\rtimes}$-*Lipschitz* maps $(X \times \mathbb{T}^1, g^{\rtimes}) \to (Y \times \mathbb{T}^1, h^{\rtimes})$ for λ° and $\lambda^{\rtimes}$ related in a certain way.

1.9 Positive Eigenfunctions and the Maximum Principle

Let X be a *compact connected* Riemannian manifold and let

$$\Delta f = \sum_i \nabla_{ii} f = \mathsf{trace\,Hess} f = \mathsf{div\,grad} f$$

denote the Laplace (Beltrami) on X, which, recall, is a *negative* , since

$$\int_X \langle f, \Delta f\rangle dx = -\int_X ||\mathsf{grad} f||^2 dx \le 0$$

by Green's formula.

Non-Vanishing Theorem. Let $s(x)$ be a smooth function, such that the

$$L = L_s : f(x) \mapsto -\Delta f(x) + s(x) f(x)$$

is *non-negative*, that is $\int_X \langle f(x), Lf(x)\rangle dx \ge 0$ for all f or, equivalently, if L the lowest eigenvalue $\lambda = \lambda_{min}$ is ≥ 0.[33]

Then

the eigenfunction $f(x)$ associated with λ doesn't vanish anywhere on X.

Start with two lemmas.

1. *C^1-Lemma.* If the minimal eigenvalue of the $f(x) \mapsto Lf(x) = -\Delta f(x) + s(x) f(x)$ on a compact Riemannian manifold is *non-negative*, $\lambda = \lambda_{min} \ge 0$, then the *absolute value* $|f(x)|$ of the eigenfunction f associated with λ is *C^1-smooth.*

2. *Δ-Lemma.* Let $f(x)$ be a *non-negative* continuous function on a Riemannian manifold, such that

(i) $f(x)$ *vanishes* at some point in X,

$$f(x_0) = 0,\ x_0 \in X,$$

(ii) $f(x)$ is *not identically zero* in any neighbourhood of the point $x_0 \in X$,

(iii) $f(x)$ is everywhere C^1-*smooth* and it is C^2-*smooth* at the points x where it doesn't vanish.

Then there exists a sequence of points $x_1, x_2, \ldots \in X$ convergent to x_0, *where $f(x_i) > 0$ and such that*

$$\frac{\Delta f(x_i)}{f(x_i)} \to \infty,\ \textit{for } i \to \infty.$$

[33] This is equivalent since our L has *discrete spectrum.*

Derivation of Non-vanishing Theorem from the Lemmas. Since $|f|$ is C^1 by the first lemma, the Δ-lemma, applied to $|f(x)|$, shows that there exists a point x, where $f(x) \neq 0$ and

$$\frac{\Delta f(x)}{f(x)} = \frac{\Delta|f(x)|}{|f(x)|} > |s(x)|,$$

that is incompatible with $-\Delta f(x) + s(x)f(x) = \lambda f(x) \geq 0$ for $\lambda \geq 0$.

Proof of C^1-Lemma. Recall that the eigenvalues of the $L = L_s = -\Delta + s$ are equal to the critical values of *the energy functional*

$$E(f) = \int_X (||\mathsf{grad} f(x)||^2 + s(x))f^2(x)dx$$

on the sphere

$$||f||^2 = \int_X f^2(x)dx = 1$$

in the Hilbert space $L_2(X)$ and the critical points of E are represented by eigenfunctions

Indeed,

$$E(f) = \langle f, Lf \rangle = \int_X \langle f(x), Lf(x) \rangle dx$$

by Green's formula and the differential of the quadratic function $f \mapsto \langle f, Lf \rangle$ on the sphere $||f||^2 = 1$ is

$$(dE)_f(\tau) = \langle \tau, Lf \rangle \text{ for all for all } \tau \textit{ normal to } f.$$

Thus, vanishing of dE at f on the unit sphere says, in effect, that Lf is a multiple of f, i.e. $Lf = \lambda f$.

All this makes sense in the present case, albeit the space $L_2(X)$ is *infinite dimensional* and L an *unbounded*, because L is an *elliptic* operator, which implies, for compact X, that

the spectrum of L is discrete, bounded from below and all eigenfunctions are smooth.

In particular – this is all we need,

all minimizes of $E(f)$ on the unit sphere, that are, a priori, only Lipschitz continuous, are smooth.[34]

Now, observe that,

taking absolute values of smooth functions $f(x) \mapsto |f(x)|$ doesn't change their energies, as well as their L_2-norms,

$$|||f||| = \|f\| = \sqrt{\int_X |f|^2(x)dx},$$

$$E(|f|) = E(f) = \int_X (||\mathsf{grad}|f|(x)||^2 + s(x))|f|^2(x)dx,$$

[34] Recall that our "smooth" means C^∞ and all our Riemannian manifolds are assumed smooth.

Indeed, absolute values $|f|(x)$ are Lipschitz for Lipschitz f, hence, they are almost everywhere differentiable functions, such that $\mathsf{grad}|f|(x) = \pm\mathsf{grad}f(x)$ at all differentiability points x of $|f|$.

It follows that the absolute value of the eigenfunction f with the smallest energy $E(f) = \lambda_{min}$ is also a minimizer; hence, *this* $|f|$ *is smooth.* QED.

Poof of Δ*-Lemma.* The common strategy for locating points $x \in X$ with "sufficiently positive" second differential of a function $f(x)$ is by using simple auxiliary functions $e(x)$ with this property and looking for minima points for $f(x) - e(x)$.

The basic example of such a function $e(x)$ in one variable is e^{-Cx}, $x > 0$, for large C, where $\frac{e''}{e} = C^2$, and where observe that the ratio $\frac{e''}{e'} = C$ also becomes large for large C.

It follows that that the Laplacians of the corresponding radial functions in small R-ball $B_y(R)$ in Riemannian manifolds X,

$$e(x) = e_C(x) = e_{y,C}(x) = e^{-C\cdot r_y(x)} \text{ for } r_y(x) = dist(y,x) \le R$$

satisfy

$$\Delta e(x) \ge C^2 e(x) - C \cdot mean.curv(\partial B_y(r), x) \text{ for } r = r_y(x) = dist(y,x)$$

Now, in order to find a point x close to a given $x_0 \in X$ where $f(x) = 0$, take $y \in X$ very close to x_0, where $f(y) > 0$, let $B_y(R) \subset X$ be the *maximal* ball, such that $f(x) > 0$ in its interior, let

$$e(x) = e_C(x) = e^{-C\cdot r_y(x)} - e^{-C\cdot R}$$

and observe that $e(x)$ vanishes on the boundary of the ball $B_y(R)$ and is strictly positive in the interior. Moreover

$$e(x) \ge \varepsilon\rho,$$

for all x on the geodesic segment between y and x_0 within distance $\ge \rho$ from x_0 for all $\rho_0 \le R$.

Notice that this $\varepsilon = \varepsilon_C$ albeit *strictly positive*, tends to zero for $C \to \infty$.

Assume without loss of generality that x_0 is the only point in $B_x(R)$ where $f(x)$ vanishes (if not, move y closer to x_0 along the geodesic segment between the two points), let C be *very very large* and see what happens to $f(x)$ and $e(x)$ in the vicinity of $x_0 \in \partial B_y(R)$, say in the intersection

$$U_0 = B_y(R) \cap B_{x_0}(R/3).$$

Observe the following.

• Since $f(x) > 0$ for $x \in B_y(R)$, $x \neq x_0$, and since $e_C(x) \to 0$ for $C \to \infty$ for $r_y(x) = dist(y,x) \ge r_0 > 0$, *the function* $e(x) = e_C(x)$*, for large* C*, is bounded by* $f(x)$ *on the boundary of* U_0,

$$e(x) \le f(x), \ x \in \partial U_0,$$

where $e(x) < f(x)$ *unless* $x = x_0$.

• Since f is differentiable at x_0 and assumes minimum at this point, the differential df vanishes at x_0, which makes $f(x) = o(\rho)$ for $\rho = dist(x, x_0)$, *there is a part of (the interior of)* U_0*, where* $e(x) > f(x)$.

Hence, the difference $f(x) - e(x)$ assumes minimum at an interior point $x = x_{y,C} \in U_0$, such that $x = x_{y,C} \to x_0$ for $C \to \infty$ and

$$\frac{\Delta f(x)}{f(x)} \geq \frac{\Delta e(x)}{e(x)} \to \infty.$$

The proof of the Δ-lemma and of the non-vanishing theorem are thus concluded.

Discussion. The non-vanishing theorem, which, probably, goes back to Rayleigh, is often used without being even explicitly stated as, for instance, by Kazdan and Warner in their "conformal change" paper. But I couldn't find an explicit reference on the web, except for the paper by Doris Fischer-Colbrie and Rick Schoen, where they prove such a non-vanishing for non-compact manifolds needed for their

non-existence theorem for non-planar stable minimal surfaces in $\mathbb{R}^3$.

Their argument relies on the *"strong maximum principle"* for the L, for which they refer to pp. 33-34 of the canonical Gilbarg-Trudinger textbook, where the relevant case of this principle is stated (on p. 35 in the 1998 edition which is available on line) after the proof of theorem 3.5 as follows.

"Also, if $u = 0$ at an interior maximum (minimum), then it follows from the proof of the theorem that u = 0, irrespective of the sign of c."

(The assumptions of the theorem specifically rule out c with variable signs, where this $c = c(x)$ is the coefficient at the lowest term in the equation $Lu = a^{ij}(x)D_{ij}u + b^i D_i u + c(x)u = 0$ introduced on p. 30.)

What is actually proven in this book on about twenty lines on p. 34, is a version of "Δ-lemma" for L.

In our proof, we reproduce what is written on these lines, except for "*direct calculation gives*" that is replaced by an explicit evaluation of $\Delta e(x)$ [35]

The following (obvious) corollary to the non-vanishing theorem will be used for construction of stable symmetric μ-bubbles in sections **??**, **??**.

Uniqueness/Symmetry Corollary. *If X is compact connected, then the lowest eigenfunction f of the L is unique up to scaling. Consequently, if L is invariant under an action of an isometry group on X, then, even if X is disconnected, there exists a positive f invariant under this action.*

Exercises. (a) **Multi-Dimensional Morse Lemma.** Show that two *non-coinciding* volume minimizing hypersurfaces in the same indivisible homology integer homology class of an orientable manifold X have *empty intersection and*

[35] In truth, the only non-evident aspect of the argument resides with the identities $(e^{-Cx})' = -Ce^{-Cx}$ and $(e^{-Cx})'' = (-Ce^{-Cx})' = C^2 e^{-Cx}$ with the issuing inequalities $(e^{-Cx})'' >> e^{-Cx}$ and $(e^{-Cx})'' >> |(e^{-Cx})'|$, which *can't be done by just staring* at the exponential function. (The appearance of e^x, that is an isomorphism between the *additive* $\mathbb{R}$ and *multiplicative* $\mathbb{R}_+^\times$ with all its counterintuitive properties, is amazing here – there is nothing visibly multiplicative in Δ; besides, the geometric proof of the existence of e^x via the *conformal* infinite cyclic covering map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ and analytic continuation is non-trivial.)

The rest of the proof is geometrically effortless: you *just look* at the graph Γ_e of the function $e(x) = \exp -C \cdot dist(y, x)$ in a small R-ball $B \subset X$ outside zero set of f with the center of your choice, such that B touches this set at x_0, and let $C = C_i \to \infty$. Then you *see* a tiny region in this ball close to x_0, where Γ_e mounts above Γ_f, and you take the point in X just under the *top* of this mountain, i.e. where the distance measured vertically between the two graphs is maximal, for you $x = x_i$.

that, consequently, volume minimizing hypersurfaces must be invariant under symmetries of X.[36]

(b) Generalize this to μ-bubbles, that are boundaries of domains V in a Riemannian manifold X that *minimize* the functional

$$V \to vol_{n-1}(\partial V) - \int_V \mu(x)dx$$

for a smooth function $\mu(x)$. (Unit spheres $S^{n-1}\mathbb{R}^n$ are *not minimizing* μ-bubbles for $\mu = (n-1)dx$.)

(b) **Courant's Nodal Theorem.** Show that the that is the number of connected components of the complement to the "k-th nodal set", i.e. the zero set of the k-th eigenfunction of $L = L_s = \Delta + s$ on a compact connected manifold, can't have more than k connected components.

Question. Is there a counterpart to this for non-quadratic functionals in spaces of functions, or, even better, spaces of hypersurfaces?

[36]This was used by Marston Morse to show that

if the $(n-1)$-dimensional homology group of some covering of a compact Riemannian n-manifold, doesn't vanish then the universal covering $\tilde{X}$ of X contains an infinite minimal hypersurface the image of which under the covering map $\tilde{X} \to X$ is compact.

Morse was concerned in his paper "*Recurrent Geodesics on a Surface of Negative Curvature*" with the case of $n = 2$ but his argument, transplanted to the environment of the geometric measure theory, applies to manifolds of all dimensions n.