Lectures on Curvature unedited

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Historical Preamble: Heinz Hopf: Selected Topics in Geometry, New York University 1946, Notes by Peter Lax.[1]

Heinz Hopf (19 November 1894 – 3 June 1971)

In 1925, he proved that any simply connected complete Riemannian 3-manifold of constant sectional curvature is globally isometric to Euclidean, spherical, or hyperbolic space.

In 1931, Hopf discovered the Hopf invariant of maps $S^3 \to S^2$ ("element of the architecture of our world" in the words of Penrose) and proved that the Hopf fibration has invariant 1. This:

(1) disproved the then standing intuitive conjecture that the continuous maps between spheres $S^N \to S^n$, $N > n$, are contractible;

(2) Opened the door to the world of vector bundles and the topology of spinors, where the curvature of the Hopf bundle is 1/2 curvature of the 2-sphere.

(Hopf bundle and Dirac Monopole `https://personal.math.ubc.ca/~mihmar/HopfDirac.pdf`,`https://www.sciencedirect.com/science/article/abs/pii/S0393044002001213` `https://ncatlab.org/nlab/show/Hopf%20fibration`)

Peter David Lax (1 May 1926 – 16 May 2025)

After the war ended, Lax remained with the Army at Los Alamos for another year and eventually returned to NYU for the 1946–1947 academic year.

Contents

[1] `https://link.springer.com/book/10.1007/3-540-39482-6` Among many other things, there is a proof of Legendre-Cauchy-A. Schur "Arms-Bow-Lemma" on pp 31-32 in these lecture (attributed by Hopf to E. Schmidt), which has been reproduced in all further publication concerning this theorem. e.g. in `https://www.scribd.com/document/759520702/Chern-Curves-and-surfaces-in-Euclidean-spaces`

1 Definitions, Problems and First Examples

Notation: $\overrightarrow{\mathbf{curv}}_\tau$, $\mathbf{curv}^\perp_\mathbf{x}$ and $\mathbf{curv}^\perp(\mathbf{X})$. Let $f : X \to \mathbb{R}^N$ be a smooth immersion, let $\tau = \tau_x \in T_x(X)$ be a tangent vector and let $\gamma_\tau \subset X$ be a geodesic in X issuing from x with the speed τ. Then the *normal curvature vector*

$$\overrightarrow{curv}_\tau(X) \in \mathbb{T}_{f(x)}(X) \in \mathbb{R}^N) = \mathbb{R}^N$$

is equal the acceleration (the second derivative) at $f(x)$ of a point moving along the curve $f(\gamma)$ in $\mathbb{R}^N$.

Granted this, define

$$curv^{\perp}_x(X) = \sup_{\|\tau_x\|=1} curv_{\tau_x}(X) \text{ and } curv^{\perp}(X) = \sup_{x\in X} curv_x(X).$$

If $dim(X) = 1$, say $X = [0,1]$ and the curve $X \overset{f}{\hookrightarrow} \mathbb{R}^N$ is parametrized by arc length, that is

$$\left\| \frac{df(x)}{dx} \right\| = 1,$$

then this is the usual curvature of a curve,

$$\overrightarrow{curv}(X,x) = \frac{d^2f(x)}{dx^2} \text{and } curv^{\perp}(X) = \sup_{x\in X} \left\| \tfrac{d^2f(x)}{dx^2} \right\|.$$

Thus,

the normal curvature $curv^{\perp}(X \overset{f}{\hookrightarrow} \mathbb{R}^N)$ *is equal to the supremum of the normal curvatures of the* f*-images in* $\mathbb{R}^N$ *of the geodesics from* X*.*

"Immersion" signifies a C^1-map $f : X \to Y$ between smooth manifolds, such that the differentials $df : T(X) \to T(Y)$ nowhere vanishes, $df(\tau) = 0 \implies \tau = 0, \tau \in T(X)$.

Immersions are locally one-to-one maps, but globally they may have self intersections; immersions *without self intersections* are called *embeddings*, where, for non-compact X, one usually require the induced topology in X to be equal the original one.

"Geodesics" $\gamma \hookrightarrow X \hookrightarrow \mathbb{R}^N$ are locally shortest among curves in X between pairs of points in X.

Locality of the Curvature and Curvature of Submanifolds. Since curvature of an immersion at a point $x \in X$ is a local invariant and since immersions locally are embeddings, the definition and many properties of curvatures of immersions formally follow from those for submanifolds $X \subset \mathbb{X}^N$. In this in mind, we may often (but not always) speak of curvatures of "immersed submanifolds", and, accordingly to simplify our notation.

Curvatures of Spheres. Spheres $S^n(R)$ of radius R of all dimensions n in the N-space $\mathbb{R}^N$, $N > n$, satisfy

$$curv^{\perp}(S^n(R)) = \|\overrightarrow{curv}^{\perp}_{\tau}(S^n(R))\| = 1/R \text{ for all unit tangent vectors } \tau \in T(S^n(R)).$$

❂ The unit n-spheres $S^n(R = 1) \subset B^N(1)$, are *the only closed immersed* n-sub-manifolds $X \hookrightarrow B^N(1)$ for $n \geq 2$ with curvatures ≤ 1, which are contained in the unit Euclidean N-balls and multiple covering of the unit circle are also such manifolds for $n = 1$.

This follows by the *maximum principle* applied to the distance function from X to the boundary $\partial B^N(1))$ or equivalently to the squared distance to the center of the ball B^N denoted $r^2(x)$.

In fact, since $curv^{\perp}(X) \leq 1$, the second derivatives of r^2 along geodesics parametrized by the arc length satisfy:

$$\|r''r\| \leq 1 \text{ and } (r^2)'' = 2(r''r + \|r'\|^2) \geq 0, \text{ since } \|r'\|^2 = 1.$$

This says that r^2 is *a convex*, hence constant=1 function on X. Thus, X is contained in the unit sphere $S^{N-1}(1) = \partial B(1)$, where it has zero normal curvature (see ???), i.e. totally geodesic. (compare with ???focal.raD)

Compact, Closed, Complete. Curvature has a limited effect on topology and on global geometry of immersion of *open manifolds*, i.e. those which contain *no compact connected components without boundaries*, called *closed manifolds.*

For instance, according to the generalized *Smale-Hirsch h-principle*[2], an arbitrary immersion f of open manifold X to an open subset $U \subset \mathbb{R}^N$ admits a homotopy (even a regular homotopy[3] to an immersion f_ε, such that

$$curv^\perp(X \overset{f_\varepsilon}{\hookrightarrow} U) \leq \varepsilon \text{ for a given } \varepsilon > 0.$$

Yet many global features of *closed* immersed manifolds influenced by their curvatures often remain valid for *complete* immersed manifolds $X \hookrightarrow \mathbb{R}^N$, i.e. where the induced Riemannian metrics in X, sometimes called *inner metrics*, are *geodesically complete*: geodesics starting at all point $x \in X$ extend infinitely in all directions $\tau_x \in T_x(X)$.

Exercise. Generalize ❂ to complete immersed $X \hookrightarrow B^N(1)$.

$curv^\perp$***-Extremal Immersions*** between Riemannian manifolds, $f : X \hookrightarrow Y$, e.g. for $Y = \mathbb{R}^N$, are those which minimise some geometric size invariant of the image $f(X) \subset Y$, such as $diam_Y(f(X))$, among all immersions with $curv^\perp \leq c$[4] or among all such immersion *regularly homotopic* to a given one. Beside the diameter, it may be, some kind of *width*, the *radius of the minimal ball* which contained $f(X)$, etc.

If we don't ¯specify any invariant, we call an immersion $f_0 : X \hookrightarrow Y$ *simple extremal* if it admits *no regular homotopy* $f_t : X \hookrightarrow Y$, such that $curv^\perp(f_1) < curv^\perp(f_0)$, where the local version of this says that all regular homotopies f_t, satisfy $curv^\perp(f_t) \geq curv^\perp(f_0)$ for $t > 0$.

If $Y = \mathbb{R}^N$, then this may be applied to the convex hull $Y_0 = conv(f(X)) \supset f(X)$ and then an immersion $f_0 : X \hookrightarrow \mathbb{R}^N$ is called *conv-curv*$^\perp$*-extremal* if one can't decrease the normal curvature of f_0 by a regular homotopy of immersions $f_t : X \hookrightarrow conv(f_0(X))$.

Basic Spherical Example. By ❂, spheres $S^n(1/c) \subset \mathbb{R}^N$ are extremal with respect to all above criteria.

Piecewise C^2 ***Circular Example.*** Some naturally arising submanifolds with bounded normal curvatures, e.g. many extremal ones are C^1-smooth and only *piecewise* C^2.[5]

For instance, immersed closed curves, which go around several circles (possibly going around each circle many times) in the figure below, have $curv^\perp$ equal to the curvature of the smallest circle.

These curves are C^1-smooth but they are not C^2: their curvatures jump as they switch the tracks from one circle to another at the contact points between circles.

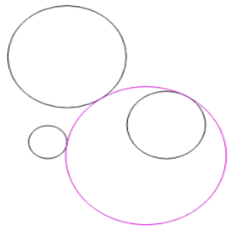
○○***-Subexample.*** Let $f : S^1 \hookrightarrow B^2(1) \subset \mathbb{R}^2$ be a C^1 immersion with

[2]See [El-Mi], section ??? and references therein

[3]A regular homotopy is a path in the space of C^1 immersions with the usual C^1-topology, that is $f_t : X \to Y$, $t \in [0,1]$, where the differential df_t of f_t in x-variables, $x \in X$, is continuous in t.

[4]Our definitions of $curv^\perp$ naturally generalize to all Riemannian manifold Y receiving immersions from X.

[5]This is a well know phenomenon in the *optimal control theory*, where one is predominantly concerned with $n = 1$, [Feld], compare ???.



curvature

$$curv^\perp(S^1 \overset{f}{\hookrightarrow} \mathbb{R}^2) \leq 2.$$

If the corresponding oriented Gauss map to the unit circle

$$\overrightarrow{G}_f = \frac{df}{\|f\|} : S^1 \to S^1 \subset \mathbb{R}^2$$

has degree zero (hence contractible), then the image of f *is equal to the union of two circles of radii 1/2, which meet at the center of the disc* $B^2(1)$*, where they are tangent one to another.* Thus the figure ∞ immersion is "radially extremal": it *minimises the radius of the 2-ball.* around it. (We shall explain why this is so in section ???).

Bi-invariants $\mathbf{curv}^\perp_{\mathbf{min}}(\mathbf{X},\mathbf{Y})$ ***and*** $\mathbf{Imm}_{\perp\leq\mathbf{c}}(\mathbf{X},\mathbf{Y})$***.*** Let X be a smooth closed manifold and and Y a Riemannian manifold and let $curv^\perp_{min}(X,Y)$ be the infimum of normal curvatures of smooth immersions $X \hookrightarrow Y$.

Now, if we choose and fix a particular Y, e.g. the unit ball in $\mathbb{R}^N$, the number $curv^\perp_{min}(X,Y)$ becomes a *topological invariant* of X, the value of which is unknown for most n-manifolds and $N > n$.

Dually, given a topological n-manifold, e.g. (homeomorphic to) the product of spheres, the minimal $curv_{\min}(X,Y)$ of immersions $X \hookrightarrow Y$ appears as a *metric invariant* of Y, which is unknown in most cases, for instance, for the N-balls and cubes $Y \subset \mathbb{R}^N$.

The number $\mathbf{curv}^\perp_{\mathbf{min}}(\mathbf{X},\mathbf{Y})$ carries only a small part of the information about immersions $f : X \hookrightarrow Y$ with curvatures $curv^\perp(f) \leq c$.

A more comprehensive information is contained in the *homotopy types* of the spaces of immersions with $curv^\perp(f) \leq c$, denoted $Imm_{\perp\leq c}(X,Y)$ and the *homotopy classes* of the inclusion maps

$$Imm_{\perp\leq c_1}(X,Y) \subset Imm_{\perp\leq c_2}(X,Y),\ c_1 \leq c_2,$$

where much of this information is encoded by the diagram of the natual (co)homology homomorphisms between these spaces.

1.1 Alternative Definitions of Normal Curvature

The full *second order infinitesimal information* of a smooth submanifold X in a Riemannian manifold Y, e.g. in the Euclidean N-space, at a point $x \in X$ is algebraically represented by the *second fundamental form* that is a *symmetric bilinear* form on X with values in the normal vector space $T^\perp(X) \subset T(Y)$, denoted

$$\mathrm{II}(X,x) = \mathrm{II}(X,x,\tau_1,\tau_2) = \mathrm{II}_x(\tau_1,\tau_2),$$

where $\tau_1,\tau_2 \in T_x(X)$ are tangent vectors to X and where the value $\mathrm{II}(\tau_1,\tau_2)$ is a vector in $T_x(Y)$ normal to the tangent (sub)space $T_x(X) \subset T_x(Y)$. This form in the case $Y = \mathbb{R}^N$ is defined as the *second differential* of a vector function, say $\Phi : T_x(X) \to T_x^\perp(X)$, such that the graph of Φ in a neighbourhood of $x \in \mathbb{R}^N \supset X$ is equal to $X \subset \mathbb{R}^N = T_x(\mathbb{R}^N) = T_x(X) \oplus T_x^\perp(X)$,

$$\mathrm{II}_x(\tau_1,\tau_2) = \partial_{\tau_1}\partial_{\tau_2}\Phi(x)$$

In the general case, this definition applies by equating $T_x(Y))$ with a small neighbourhood in Y via the exponential map $\exp_x : T_x(Y) \to Y$.

Exercises. **1.1.A.** Show that $\mathrm{II}(\tau,\tau)$ is equal to the second (covariant) derivative in Y of the geodesic in X issuing from x with the velocity τ, and that

$$[\tau_1\tau_2]_\leq \qquad \|\mathrm{II}_x(\tau_1,\tau_2)\| \leq curv_x^\perp(X)$$

for all $x \in X$ and all unit tangent vectors $\tau_1,\tau_2 \in T_X(X)$.

1.1.G. Let $X \hookrightarrow Y \hookrightarrow Z$ be isometric embeddings (or immersions) between Riemannin manifolds, i.e the Riemannian metrics in Y and in X are induced from a Riemannian metric in Z. Show that

$$curv_\tau^\perp(X \hookrightarrow Z) = \sqrt{(curv_\tau^\perp(X \hookrightarrow Y))^2 + (curvv_\tau^\perp(Y \hookrightarrow Z))^2}$$

for all tangent vectors $\tau \in T(X) \hookrightarrow T(Y) \hookrightarrow T(Z)$.

For instance, if $X \hookrightarrow Y = S^{N-1}(1) \hookrightarrow Z = \mathbb{R}^N$ then

$$curv_\tau^\perp(X \hookrightarrow \mathbb{R}^N) = \sqrt{(curv_\tau^\perp(X \hookrightarrow S^{N-1}))^2 + 1}.$$

1.1.B. Geodesic free definition of $curv^\perp$. Show that the normal curvature $curv_\tau^\perp(X \hookrightarrow \mathbb{R}^N) = \|\overrightarrow{curv}_\tau\|$, $\|\tau\| = 1$, is equal to the infimum of the Euclidean $curv^\perp$-curvatures of the curves in X tangent to τ.

1.1.C. Metric Definition of II. Let $Y = (Y, g)$ be a Riemannian manifold, e.g. $Y = (\mathbb{R}^N, g = \sum_{j=1}^N dy_j^2)$, let $X \subset Y$ be a smooth submanifold, let $\nu \in T_x^\perp(X)$ be a normal vector to X at x and $\tilde\nu$ be a smooth vector field on Y, which extend ν_x.

Let $g_{|X}$ be the restriction of the Riemannian quadratic form g to X and let $\tilde g'_{|X}$ be the restriction of (Lie) *derivative of g by the field $\tilde\nu$ to X.*

Show that the value $\tilde g'_{|X}(\tau_1,\tau_2)$ for $\tau_1,\tau_2 \in T_x(X)$ depends only on ν but not on the extension $\tilde\nu$ of ν.

Moreoever, show that

$$\tilde g'_{|X}(\tau_1,\tau_2) = \langle \nu, \mathrm{II}_x(\tau_1,\tau_2)\rangle_g,$$

and that the second fundamental form II is *uniquely determined* by this identity.

(The definition of the second fundamental form II as the derivative $\tilde g'_{|X}$ of the induced Riemannian form uses no covariant derivatives or geodesics either in X or in Y.)

1.1.D. Normal Curvature Defined via the Gauss Map. Let $\mathcal{H} = Gr_n(N)$ be the space of n-dimensional linear subspaces $H \subset \mathbb{R}^N$ and naturally identify the tangent space $T_H(\mathcal{H}$ with the space of linear maps from H to the normal space $H^\perp \subset \mathbb{R}^N$,

$$T_H(\mathcal{H}) = hom(H, H^\perp).$$

Let $f : X \hookrightarrow \mathbb{R}^N$ be a smooth immersed submanifold and $\overleftrightarrow{G} : X \to Gr_n(N)$, $n = dim(X)$, be the (non-oriented) Gauss map where $\overleftrightarrow{G}(x)$ is the linear subspace parallel to tangent subspace of X in R^N (regarded as an affine subspace) at x.

Let $D_x\overleftrightarrow{G} : T_x(X) \to T_x(X)^\perp$ be the differential of the map $\overleftrightarrow{G}$ at $x \in X$ regarded as a linear operator $T_x(X) \to T_x^\perp(X)^\perp$.

Show that

the normal curvature of X at x is equal to the norm of the operator $D_x\overleftrightarrow{G}$,

$[D\overleftrightarrow{G}]^\perp$
$$curv_x^\perp(X) = \sup_{\tau \in T_x(X), \|\tau\}=1} \|D_x\overleftrightarrow{G}(\tau)\|$$

and derive from this the following corollary.

1.1.E. Angular Arc Inequality. If the (inner) distance between two points $x_1, x_1 \in X$ satisfies

$$dist_X(x_1, \underline{x}) \leq \alpha(curv^\perp(X))^{-1}, \ \alpha \leq \pi/2,$$

then the angles between vectors $\tau \in T_{x_1}(X)$ and their images $\bar{\tau}$ under the normal projection $T_{x_1}(X) \to T_{x_1}(X)$ satisfy

$$\angle(\tau, \bar{\tau}) \leq \alpha,$$

where the equality holds if and only if there exists a

planar α-arc of radius $\frac{1}{curv^\perp(X)}$, which is contained in X, which join x_1 with $\underline{x}$ and such that τ is tangent to this arc at its x_1-end.

Conversely, the inequality $\angle(\tau, \bar{\tau}) \leq \epsilon/c + o(\epsilon)$, $c \geq 0$, for all pairs of ϵ-infinitesimally closed points implies that $curv^\perp(X) \leq c$.

no non-zero tangent vector $\tau_1 \in T_{x_1}(X)$ is normal to $T_{\underline{x}}(X)$.

Moreover the same non-normality conclusion holds if

$$dist_X(x_1, \underline{x}) \leq \frac{\pi}{2}(curv^\perp(X))^{-1},$$

unless there exists a

planar semicircle of radius $\frac{1}{curv^\perp(X)}$ contained in X and joining x_1 with $\underline{x}$.

Polygonal ApproximationADD????

◯ Let P be a *closed spacial polygonal* curve with k vertices p_i. Then a decomposition of P into triangles $\triangle_j$ [6] shows that *the sum of the angles* between the edges of P at these vertices satisfies:

$$\sum_{i=1}^{k}(\pi - \angle_{p_i}) \geq 2\pi,$$

where the difference $\sum_{i=1}^{k}(\pi - \angle_{p_i}) - 2\pi$, is the sum of (positive!) excesses of the angles of triangles $\triangle_{j_i}$ adjacent to p_i with respect to the angles of the angles $\angle_{p_i}(P)$,

$$\sum_{i=1}^{k}(\pi - \angle_{p_i}) - 2\pi = \sum_i exc_i$$

[6] One can decompose P into $k-2$ triangles with a common vertex e.g. p_1 but one can do it more efficiently with about $log_2 k$ triangles.

where

$$exc_i = \sum_{j_i} \angle_{p_i}(\triangle_{j_i})$$

and where $exc_i = \sum_{j_i} = 0$ for all i if and only if P is a *planar convex* curve.

Application-Exercise: Fenchel $\geq 2\pi$-Inequality Let X, written as $x(s)$, be a closed smooth spacial curve parametrised by the ark length parameter, $s \in [0, l]$ $l = length(X)$. Approximate X by polygonal curves, prove the following *Fenchel's Inequality*

$$\int_0^l curv^\perp(x(s))ds \geq 2\pi,$$

generalize this to piecwise smooth curve and show that equality implies that X is a *planar convex curve.* [7]

2 Products of Spheres, Clifford's sub-Tori with Small Curvatures and Petrunin Inequality

The product X of spheres $S^{n_i}(R_i) \subset \mathbb{R}^{N_i = n_i + 1}$, $i = 1, ...m$,

$$X = S^{n_1}(R_1) \times S^{n_2}(R_2) \times ... \times S^{n_m}(R_m) \subset \mathbb{R}^{N=(n_1+n_2+...+n_m)+m},$$

has the curvature equal to the maximum of $1/R_i$, $i = 1...., m$, and if

$$R_1^2 + R_2^2 + ... + R_m^2 \leq 1,$$

then X is contained in the unit ball in $\mathbb{R}^N$. (If $R_1^2 + R_2^2 + ... + R_m^2 = 1$, then X is contained in the unit sphere $S^{N-1}(1) = \partial B^N(1) \subset \mathbb{R}^N$.)

For example, the product of m-copies of S^n admits an embedding to the unit ball in $\mathbb{R}^{mn+m}$, where

$$curv^\perp\big((S^n)^m \subset B^{mn+m}(1)\big) = \sqrt{m}$$

The main instance of this is the *Clifford n-torus*, that the product of n circles imbedded to the unit $2n$ ball, such that

$$curv^\perp\big(\mathbb{T}^n \subset \partial B^{2n}(1)\big) = \sqrt{n}.$$

It is conceivable that the above (Clifford's) products of spheres $S^{n_1}(R_1) \times S^{n_2}(R_2) \times ... \times S^{n_m}(R_m) \subset \mathbb{R}^N$ are *conv-curv$^\perp$-extremal,* where this seems realistic for $m < min_i n_i$, but we have no idea, for instance, if there are immersions of n-tori to $B^{2n}(1)$ with $curv^\perp < \sqrt{n}$.

Yet, if $N >> n$, then the n-torus can be immersed to the unit ball $B^N(1)$ with *unexpectdly small* curvature.

1.C. $\sqrt{3}$-Clifford Sub-Torus Theorem. (Section ?) **[a]** If N is much greater than n, then the Clifford torus

$$\mathbb{T}^N \subset S^{2N-1} \subset B^{2N}(1),$$

[7] See [Chern] and also section ???for other proofs and applications of this inequality; also see ?? for a Riemannian version of it.

contains an n-subtorus $\mathbb{T}^n_o \subset \mathbb{T}^N$, such that the normal curvature of this n-torus the ambient Euclidean space $\mathbb{R}^{2N} \supset B^{2N} \supset \mathbb{T}^n_0$ satisfies

$$\left[\tfrac{3n}{n+2}\right]_{\mathbb{T}^n} \qquad curv^{\perp}\big(\mathbb{T}^n_o \subset B^{2N}(1)\big) \leq \sqrt{\frac{3n}{n+2}}.$$

One has a poor bound on the best (i.e. the smallest) N for this purpose, (something like 10^{10^n}, see section ???) but

[b] *if $N \geq 8n^2 + 8$, then, there exists a a locally isometric (with respect to the Euclidean metrics in $\mathbb{R}^n$ and $\mathbb{T}^N$) map, that is a group homomorphism*

$$g : \mathbb{R}^n \hookrightarrow \mathbb{T}^N \subset B^{2N}(1),$$

such that

$$\left[\tfrac{3n}{n+2}\right]_{\mathbb{R}^n}. \qquad curv^{\perp}\big(\mathbb{R}^n \hookrightarrow B^{2N}(1)\big) \leq \sqrt{\frac{3n}{n+2}}.$$

[c] *It follows that for all $\varepsilon > 0$, there exists a sub-torus* $^-$

$$\mathbb{T}^n_\varepsilon \subset \mathbb{T}^N \subset B^{2N}(1),$$

such that

$$\left[\tfrac{3n}{n+2} + \varepsilon\right]_{\mathbb{T}^n} \qquad curv^{\perp}\big(\mathbb{T}^n_\varepsilon \subset B^{2N}(1)\big) \leq \sqrt{\frac{3n}{n+2}} + \varepsilon.$$

1.D. $\sqrt{3}$-Immersion Corollary. *Let $f : X \hookrightarrow \mathbb{R}^m$ be an immersion then, for all $\epsilon > 0$, there exist an immersion (actually an embedding) f_ϵ to the unit ball B^{16m^2+16m} with curvature*

$$curv^{\perp}\big(X \overset{f_\varepsilon}{\subset} B^{16m^2+16m}(1)\big) \leq \sqrt{\frac{3m}{m+2}} + \varepsilon.$$

Proof. Let λ be a large constant, $\lambda >> 1/\epsilon$, *scale the manifold* $X \overset{f}{\hookrightarrow} \mathbb{R}^m$ by λ and compose the scaled map $\lambda \cdot f : X \hookrightarrow \mathbb{R}^m$ with the map $g : \mathbb{R}^m \hookrightarrow \mathbb{T}^N \subset B^{2N}(1)$ from the above **[b]**.

Then, if one one wishes, one slightly perturbs the resulting immersion $X :\to\hookrightarrow \mathbb{T}^N$. and makes it an embedding.

the embedding $\mathbb{T}^m_\varepsilon \subset \mathbb{T}^{8m^2+8m}$ as in the theorem. make it embedding????

On sharpness of $\left[\frac{3n}{n+2}\right]$. It is not hard to show that the Euclidean curvatures of all Clifford subtori $\mathbb{T}^n \subset \mathbb{T}^N \subset \mathbb{R}^{2N}$ (these $\mathbb{T}^n$ are very special submanifolds in $B^{2N}(1) \supset \mathbb{T}^N$)) satisfy $curv^{\perp}_{\mathbb{R}^{2n}}(\mathbb{T}^n) \geq \sqrt{\frac{3n}{n+2}}$, but the following is not so obvious.

1.E. Petrunin's $\sqrt{3}$-Inequality. (Section???) *All immersions $\mathbb{T}^n \hookrightarrow B^N(1)$ satisfy*

$$curv^{\perp}\big(\mathbb{T}^n \hookrightarrow B^N(1)\big) \geq \sqrt{\frac{3n}{n+2}} \text{ for all } n \geq 1 \text{ and all } N.$$

It is unclear what is, in general, the geometry of immersions $\mathbb{T}^n \hookrightarrow B^N(1)$ with $curv^\perp \approx \sqrt{3}n$[8] depending on the ambient dimension N. Conceivably the n-tori admit no immersions $\mathbb{T}^n \hookrightarrow B^N(1)$ with $curv^\perp \leq \sqrt{3}$ for $N << n^2$, but we have no means to rule out such immersions, say for for $N \leq 3n$ and $n \geq 4$.

3 $\mathbf{X}_{+\rho} = \mathbf{T}^\perp_\rho(\mathbf{X})$ *the Focal Radius and* $+\rho$*-Encircling*

.

Let Y be a complete Riemannian manifold, let $X \hookrightarrow Y$ be a smooth embedded or immersed submanifold, let $x_0 \in X$, let $\nu_0 \in T^\perp_{x_0}(X)$ be a unit normal vector at the point x_0 and $\gamma_\nu \hookrightarrow Y$ be a geodesic ray issuing from x_0 in the ν_0-direction.

Define ν_0*-focal radius* $rad^\perp_{\nu_0}(X)$ as the supremum of $r \geq 0$, such that the the segment $[x_0, y] \subset \gamma_0$ *locally minimises* the length of curves in Y between y and X, that is all curves, which are sufficiently close to the segment $[x_0, y]$ in C^0-topology and which join y and X, have *length* $> r$.

Then let

$$rad^\perp_{x_0}(X) = \inf_{\nu_0 \in T_{x_0}(X)} rad^\perp_{\nu_0}(X) \text{ and } rad^\perp(X) = \inf_{x_0 \in X} rad^\perp_{x_0}(X).$$

1.I. *Example: Curvatures and Focal Radii in Spheres.* The spherical curvatures of immersions

$$X \hookrightarrow S^{N-1}(R) \subset \mathbb{R}^N$$

are related to the Euclidean curvatures by the Pythagorean formula:

$$\left(curv^\perp_{\mathbb{R}^N}(X \hookrightarrow \mathbb{R}^N)\right)^2 = \left(curv_{S^{n-1}}\left(X \hookrightarrow S^N(R)\right)\right)^2 + \left(curv^\perp(S^N \subset \mathbb{R}^N)\right)^2 =$$

$$= \left(curv_{S^{n-1}}\left(X \hookrightarrow S^N(R)\right)\right)^2 + 1/R^2,$$

(see section»>???) while the Euclidean focal radii are related to the Euclidean one by the realition

$$focrad^\perp_{\mathbb{R}^N}(X) = 2R\sin\frac{1}{2}focrad^\perp_{S^{N-1}}(R)(X).$$

For instance,

• the spherical focal radii of the equatorial subspheres (with zero spheriacal curvatures) in the unit sphere $S^{N-1}(1)$ are equal to $\pi/2$, while their Euclidean focal radii are equal to one;

• the spherical focal radii of the subspheres with spherical radii $\pi/4$ are also $\pi/4$, while their spherical curvatures are equal to one and the Euclidean curvatures $\sqrt{2}$ with agreement with he identity $\frac{\sin\pi}{4} = 1/\sqrt{2}$.

Exercises.(OOO???) Let $rad^\perp_{x_0}(X) \geq r$ and let $B(R) \subset Y$ be an R-ball, which contains a (small) neighbourhood $V_0 \subset X$ of x_0 and such that the boundary sphere $S(R) = \partial B(R)$ contains x_0. Then

• $R \geq r$,

• if $R = r + \varepsilon$ for a small $\varepsilon \geq 0$, then the sphere $S(R)$ is *smooth* at the point x_0,

[8]Petrunin informed me that there exist extremal tori in $B^N(1)$. which are not contained in S^{N-1}.???

• if $S(R)$ is smooth at x_0, then the *radial component* of the second fundamental form of X at x_0 is greater than that of $S(R)$,

$$\langle \mathrm{II}_X(\tau,\tau),\nu\rangle \geq \langle \mathrm{II}_{\mathrm{S(R)}}(\tau,\tau)\nu\rangle,$$

where ν is the inward looking unit normal vector to $S(R)$ at x_0 and $\tau \in T_{x_0}(X) \subset T_{x_0}(S(R))$.

(If the sphere $S(R)$ is convex at x_0, then $0 \leq \langle \mathrm{II}_{\mathrm{S(R)}\,X}(\tau,\tau),\nu|\rangle = \sqrt{\|\mathrm{II}_{\mathrm{S(R)}\,X}(\tau,\tau)\|}$)

(i) Show that the focal radii of submanifolds in Euclidean spaces are equal to reciprocals of their normal curvatures.

$$rad^{\perp}_x(X \hookrightarrow \mathbb{R}^N) = \frac{1}{curv^{\perp}_x(X \hookrightarrow \mathbb{R}^N)}.$$

(ii) Show that the focal radius of $X \hookrightarrow Y$ is equal to the supremum of r, such that the normal exponential map $\exp T^{\perp}(X) \to Y$ is an *immersion* on the r-ball subbundle $B^{\perp}_X(r) \subset T^{\perp}(X)$.

Given an immersed $X \hookrightarrow \mathbb{R}^N$ let $T^{\perp}_{\rho}(X) \to \mathbb{R}^N$, $\rho > 0$, be the *normal exponential* (tautological) map from the ρ-spherical normal bundle of X to $\mathbb{R}^N$, where this "ρ-spherical normal bundle $T^{\perp}_{\rho}(X)$" is the set of vectors normal to X of length ρ.

For instance if $X \hookrightarrow \mathbb{R}^N$ is an embedding and $\rho > 0$ is small then the image of this map is equal the *the boundary of the ρ-neighbourhood of* X, denoted

$$X_{+\rho} = \partial U_{\rho}(X) = \{y \in \mathbb{R}^N\}_{dist(y,X)} = \rho.$$

In general, if $X \overset{f}{\hookrightarrow} \mathbb{R}^N$ is an immersion and if $\rho < \left(curv^{\perp} X \hookrightarrow \mathbb{R}^n\right)^{-1}$ then the exponential map is also an immersion and we abbreviate this by writing

$$X_{+\rho} \overset{f_{+\rho}}{\hookrightarrow} \mathbb{R}^N$$

and observe that

$$[\rho^{-1}] \qquad curv^{\perp}(X_{+\rho} \overset{f_{+\rho}}{\hookrightarrow} \mathbb{R}^N) = \max\left(\rho^{-1}, \left(\left(curv^{\perp}(X \hookrightarrow \mathbb{R}^n)\right)^{-1} - \rho\right)^{-1}\right)$$

and that if $X \hookrightarrow \mathbb{R}^N$ is contained in R-ball, then $X_{+\rho} \hookrightarrow \mathbb{R}^N$ is contained in the $(R+\rho)$- ball.

1.G. $[1+2c]$-*Example.* Let $curv^{\perp}\left(X \hookrightarrow B^N(1)\right) \leq c$ and move X to the smaller ball $B^N(r)$ by scaling $X \mapsto X' = rX$ for $r = 1-\rho$, for some $0 < \rho < 1/c$.

Then $X'_{+\rho}$ is contained in the unit ball,

$$\left(curv^{\perp}\left(X'_{+\rho} \hookrightarrow B^N(1)\right)\right)^{-1} \geq \min\left(\frac{1}{\rho}, \left(\frac{r}{c} - \rho\right)^{-1}\right)$$

and if ρ is such that $\rho = \frac{r}{c} - \rho$, then

$$curv^{\perp}\left(X'_{+\rho} \hookrightarrow B^N(1)\right) = 1/\rho \leq 1 + 2c = 1 = 1 + 2 \cdot curv^{\perp}(X).$$

1.H. *Focal Riemannian Remark.* Much of the above make sense for an arbitrary ambient Riemannian manifold Y instead of $\mathbb{R}^N$, e.g. for $Y = S^{N-1} \subset \mathbb{R}^N$, where $\frac{d^2 f(x)}{dx^2}$ in the definition of the curvature for curves in Y is understood as a *covariant derivative* and where curvatures immersions $X^n \hookrightarrow Y$ for $n > 1$ are defined accordingly.

If Y is complete, e.g. compact without a boundry, then the normal exponential map

$$\exp^{\perp}_{\rho} : T^{\perp}_{\rho}(X) \hookrightarrow Y$$

for am immersed $X \hookrightarrow Y$ is defined for all $\rho > 0$; if Y has a boundary, then normal exp-map is defined for $\rho \leq dist(X, \partial Y)$.

Then *focal radius* of $X \hookrightarrow Y$, sometimes denoted $rad^{\perp}(X \hookrightarrow Y)$, is the supremum of $r > 0$, such that the map $\exp_{\perp,\rho}$ is defined (i.e. $r \leq dist(X, \partial Y)$) and is an immersion for all $\rho < r$.

One knows that if the sectional curvature of a complete Y is ≤ 0, then

$$rad^{\perp} \geq (curv^{\perp}(X \hookrightarrow Y))^{-1}$$

and, this is obvious, the equality holds for Riemannian flat manifolds.

Thus the above [ρ^{-1}] for immersions $X \hookrightarrow \mathbb{R}^N$ can be rewritten in more transparent form in terms of focal radii:

$$rad^{\perp}(X_{+\rho}) = \min(\rho, rad^{\perp}(X) - \rho).$$

4 Focal Radius and the Maximum Principle

(iii) Let $x_0 \in X$ be a *local maximum* point in X for the distance function $x \mapsto dist_Y(x, y_0)$ for some $y_0 \in Y$. Show that

$$dist(x_0, y_0) \geq rad^{\perp}_{x_0}(X).$$

(iv) **maxrad$^{\perp}$ and the Maximum Principle.** Let Y be a metric space, let $X \subset Y$ be a subset and let $x_0 \in X$.

Define $maxrad^{\perp}_{x_0}(X)$ as the *infimum of the numbers* R, such that there exists a point $y_0 \in Y$ such that $dist(x_0, y_0) \leq R$ and the distance function $x \mapsto dist_Y(x, y_0)$ assumes *local maximum* at x_0.

Reformulate the above inequality $dist(x_0, y_0) \geq rad^{\perp}_{x_0}(X)$ as

$$maxrad^{\perp}_{x_0}(X) \geq rad^{\perp}_{x_0}(X),$$

for smooth submanifolds X in Riemannin manifolds Y.

(vi) Show that

$$maxrad^{\perp}_{x_0}(X) = rad^{\perp}_{x_0}(X) \text{ for } dim(X) = 1,$$

for smooth submanifolds X in Riemannian manifolds Y, provided the normal exponential map $\exp : T^{\perp}x_0(X) \to Y$ is immersion on the R-ball $B^{N-n}_{0=x_0}(T^{\perp}_{x_0}(X)$. Show that the condition $dim(X) = 1$ is necessary.

(vii) Show that if a compact subset $X \subset Y$ is *contained in an R-ball* $B_{y_0}(R) \subset Y$, then

$$\inf_{x \in X} maxrad^{\perp}_x(X) \leq R$$

Show that the inequality $\inf_{x\in X} maxrad^{\perp}_x(X) \leq R$ remains valid for smooth immersed *complete, possibly non-compact,* submanifolds $X \hookrightarrow Y$, provided $curv^{\perp}(X) < \infty$.

(The condition $curv^{\perp}(X) = \sup_x curv^{\perp}(X) < \infty$ is necessary: there are examples due to Rosendorn [???] of complete surfaces X in the unit 3-ball with negative Gauss curvatures, hence with $maxrad_x(X) = \infty$ for all $x \in X$.) [9]

(vi) Let $D(\rho) \subset B^N(1) \subset \mathbb{R}^N$, $N \geq 3$ be the *boundary of the convex hull of a truncated unit ball*, where $D(\rho)$ is equal to the union of a spherical cap $C^{N-1}(\rho) \subset S^{N-1}(1) = \partial B^N(1)$, $0 < \rho < \pi$ and a flat $(n-1)$-ball $B^{N-1}(r = \sin\rho) \subset B^N(1)$,

$$D(\rho) = C^{N-1}(\rho) cup B^{N-1}(r),$$

where ρ is the radius of $C^{N-1}(\rho)$ regarded as a ball in the spherical geometry in $S^{N-1}(R)$, and where the (edge-like) intersection E of the two parts of $D(\rho)$,

$$E(r) = C^{N-1}(\rho) \subset S^{N-1}(R) \cap B^{N-1}(r) = (\partial C^{N-1} = \partial B^{N-1}$$

is an $(N-1)$-sphere contained in $S^{N-1}(R)$ of (Euclidean) radius r.

Let $x_0 \in E_r$ and show that

$\bullet_{conv}$ if $\rho \leq \frac{\pi}{2}$ then $maxrad^{\perp}_{x_0}(D(\rho)) = r = \sin\rho$,

$\bullet_{concv}$ if $\rho \geq \frac{\pi}{2}$ then $maxrad^{\perp}_{x_0}(D(\rho)) = R$.

(vi) **Non-Smooth Maximum Principle.** Let $X \subset B^N(R) \subset \mathbb{R}^N$ be a closed connected subset in an R ball, such that

$$maxrad^{\perp}_x(X) \geq r \text{ for some } r \leq R \text{ qnd all } x \in X.$$

Observe that if $r = R$ then ??? implies that the intersection $X \cap \partial B^N(R) \subset S^N(R) = \partial B^N(R)$ is non-empty and show that *no connected component* of this intersection $X \cap \partial B^N(R) \subset S^N(R) = \partial B^N(R)$ is *contained in a spherical cap*

$$C^{N-1}\left(\rho < \frac{\pi}{2} r\right) \subset S^{N-1}(R).$$

Consequently, this intersection has no isolated ponts moreover,

the topological dimension of all connected components of $X \cap \partial B^N(R)$ satisfy

$$dim(compX \cap \partial B^N(R)) \leq 1.$$

(viiii????) Show that there exists a smooth convex (topologically spherical) rotationally symmetric surface in the unit 3-ball, $X \subset B^N(1) \subset \mathbb{R}^3$, which is *not equal to the boundary sphere* $S^2(1) = B^3(1)$ and such that $maxrad^{\perp}_x(X) \geq 1$ for all $x \in X$.

(....) Generalise the above (??) to subsets X (e.g. smooth submanifolds) in balls $B(R)$ in Riemannian manifolds Y, where the boundary of B, as well as the boundaries of concentric balls of radii $0 < r \leq R$ are smooth and where the inequality $\rho \geq \frac{\pi}{2}$ should be be replaced by $\rho \geq \delta = \delta(B) > 0$.

Thus show that if a compact connected subset $X \subset B(R)$ satisfies

$$maxrad^{\perp}_x(X) \geq R$$

for all $x \in X$, then

[9] Nadirashvilly etc???

• the intersection $X \cap \partial B^N(R)$ is non-empty,
• the connected components of this intersection satisfy

$$dim(compX \cap \partial B^N(R)) \leq 1,$$

• no connected component of $X \cap \partial B^N(R))$ can be diffeomorphic to segments $[0,1]$ and/or $(0,1]$.

Consequently,

if *all geodesics* γ in a smoothly immersed closed submanifold in a ball $B^N(R)) \subset Y$ satisfy $focrad^{\perp}(\gamma \hookrightarrow Y) \geq R$, then X *is contained in the boundary* $\partial B(R)$.

$(////????)$ Let X be a smoothly immersed *complete connected* submanifold in a ball $B(R)) \subset Y$, such that the intersection X with the boundary sphere $S(R) = \partial B(R)$ is*nonempty* and such that each point $x_0 \in X \cap S(R)$ admits a neighbourhood $X_0 \subset X$ such that radial component of the second fundamental form of X at all $x \in X_0$ is non greater than that of the concentric sphere $S(r)$, which contains x,

$$\langle \mathrm{II}_X(\tau,\tau),\nu\rangle \leq \langle \mathrm{II}_{S(r)}(\bar{\tau},\bar{\tau}),\nu\rangle,$$

where τ and ν(as in ???) ... and $\bar{\tau} \in T_x(S(r)$ is the normal projection of $\tau \in T_x(X) \subset T_x(Y) \supset T_x(S(r))$ to $T_x(S(r))$.

Then X is contained in the boundary of the ball, $X \subset \partial B(R)$.

Hint. Prove convexity of a $\phi(dist(x, S(R))$ for a suitable function $\phi(d)$. (Compare with ✪ in section 1.)

Question Is there a better verson of the 'maximum principle' which would incorporate ??? and ??.

where

• $y'(t)) \in T_{y(t)}(Y)$b is the unit tangent vector to the curve $y(t)$.
• $Hess_,(yh(\tau+\nu,\tau+\nu)$

where, observe, the gradient of h is a unit vector field normal to the concentric spheres $S_{y_0}(h)$ with the centre y_0 and the Hessian of $h^2(y)$ restricted to a sphere S_{y_0} equal the second fundanetal

$(h\circ f(t))" = (h'f')' = hess(h)(f') + h'f''$

4.1 Topological Definition of Focal Radius

Let Y be a complete Riemannian manifold, $X \subset Y$ a smooth immersed submanifold and let us *define* the focal curvature of X in Y as the reciprocal of the focal radius of X,

$$curv_x^{foc}(X \hookrightarrow Y) = \frac{1}{rad_x^{\perp}(X \hookrightarrow Y))}.$$

Let $X^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and let $x \in X$ Then
such that $curv_x^{\perp}(X) \leq c$, $c \geq 0$ and let
is equal the infimum of the curvatures c of the spheres $S^n_{\pm}(1/c)$, which are:

(i) tangent to X at x_0,

(ii) the balls bounded by these spheres do not intersect (small) neighbourhoods of $f(x_0)$ in $f(X)$ minus $f(x_0)$ itself,

(iii) do not mutually intersect away from x_0.

Generalise this to submanifolds $X^n \subset \mathbb{R}^N$ for all $N \geq n+1$ as follows.

Let $\mathcal{B}(c)$ be a family of balls $B_y^N(1/c)R^N$ with centers $y \in \mathbb{R}^N$ such that

(i)' all balls from $\mathcal{B}(c)$ contain $x \in X$,

(ii)' the balls do not intersect (small) neighbourhoods of x_0 in X minus x_0 itself,

(iii)' for all $\varepsilon > 0$, there exists a family of points in $\mathbb{R}^N$ continuously parametrized by $\mathcal{B}(c)$, say

$$\phi_\varepsilon : \mathcal{B}(c) \ni B \to \mathbb{R}^N,$$

such that

$\phi_\varepsilon(B) \in B$ for all $B \in \mathcal{B}(c)$,

$dist(\phi_\varepsilon(B), x_0) \leq \varepsilon$ for all $B \in \mathcal{B}(c)$,

the set $\mathcal{B}(c)$ contains an $(N-n-1)$-cycle the ϕ-imagef this cycle. is non-trivilally linked with X for all sufficiently small ε.[10]

Then show that $curv^{\perp}_{x_0}(X)$ is equal to

● *the infimum of $c > 0$, such that a family $\mathcal{B}(c)$ with all these properties exists.*

1.1.D.. Use ○ as a definition of curvature, observe that it doesn't need X to be smooth and show that if this ●-curvature of a submanifold $X \subset \mathbb{R}^N$ is finite at all $x \in X$, then X is C^1-smooth, moreover, it is $C^{1,1}$-smooth–the partial derivatives are *Lipschitz*. ADD: Rotation of segments of curves in $\mathbb{R}^3$ around tangent lines and folding polypeptide chains to proteins, https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/proteins-crystals-isoper.pdf.

1.1.E.. Express ●-curvature of a smooth submanifold in a complete Riemannian manifold, $X \subset Y$ in terms of the focal radius $rad^{\perp}(X)$ (see section ???).

1.1.F. Maximum Principle. Assume that all balls in Y are smooth and strictly convex, and show that that if a closed immersed submanifold in Y contained in a ball of radius R has

This description of $curv^{\perp}$, which doesn't refer to geodesics, has an advantage of being applicable to mechanical systems with *non-holonomic* constrains that are submanifolds in the tangent bundle of the ball, rather than the ball itself.

??? maximum principle for the ball curvature

ıExercises (a) If the signed (one sided) curvature of a closed planar curve X is$\leq 1/R$ then the (closed) domain X_+ bounded by X contains a disk of radius R. In fact there are at least two such discs unless X itself is a circle of radius R.

(b) if X is convex, then every circle of radius $\leq R$ tangent to X a point $x \in X$, either is contained in X_+ or intersect X_+ only at x. (c) Find a counter example to (a) for surfaces in $\mathbb{R}^3$ and prove a version of (b). (see "A Reverse Isoperimetric Inequality, Stability and Extremal Theorems For Plane-Curves With Bounded Curvature" by Howard and Treibergs https://scholarcommons.sc.edu/cgi/viewcontent.cgi?article=1024&context=math_facpub#:~:text=This%20gives%20a%20preliminary%20reverse,over%20to%20general%20Riemannian%20surfaces. and references therein.

[10] Think of X as a relative n-cycle in the pair $(B^N_{x_0}(2\varepsilon), \partial(B^N_{x_0})(2\varepsilon))$.

5 Products of Spheres in B^{n+1} with Small Curvatures

1.J. PRODUCTS OF SPHERES REPRESENTED BY HYPERSURFACES Let X be a product of m spheres and $k \geq m-1$. Then $X_m \times S^k$ admits a codimension one embedding to the unit ball with normal curvature $1+2\sqrt{m}$.

Proof. Imbed X to $B^{N+m}(1) \subset B^{N+k+1}(1)$ for $N = dim(X)$ with curvature $c = \sqrt{m}$ (see 1.A), let $\rho = 1+2\sqrt{m}$ and observe that $X'_{+\rho} \subset B^{N+k+1}(1)$, (this is the boundary of the ρ-neighbourhood of $X' \subset B^{N+k+1}(1)$ in the present case) is diffeomorphic to $X \times S^k$. Since $curv^\perp(X'_{+\rho}) \leq 1+2c$ (see ???) the proof follows.

TWO EXAMPLES AND ONE THEOREM.

($\bullet_1$) *Products of two spheres admit codimension one embeddings to the unit balls with normal curvatures 3:*

$$[2/3]\times[1/3]. \qquad curv^\perp\left(S^{n_1}\times S^{n_2} = S^{n_1}_{+1/3}(2/3) \subset B^{n_1+1+n_2}(1)\right) = 3,$$

($\bullet_2$) *Products of three spheres $S^{n_1}\times S^{n_2}\times S^{n_3}$, e.g. 3-tori $\mathbb{T}^3$, admit codimension one embeddings to the unit balls with curvatures $1+2\sqrt{2} < 4$.*

We don't know answers to the following questions:

are there immersions $S^{n_1}\times S^{n_2} \hookrightarrow B^{n_1+n_2}(1)$ with $curv^\perp < 3$?

are there immersions immersions $S^{n_1}\times S^{n_2}\times S^{n_3} \hookrightarrow B^{n_1+n_2+n_3}(1)$ with $curv^\perp < 1+2\sqrt{2}$.

But the situation changes starting from $m=4$ and $C = 1+3\sqrt{2} = 5.24264....$ with the following.

1.K. Codimension one Immersion Theorem. *Let X be a compact orientable n-manifold, which admits an immersion to $\mathbb{R}^{n+1}$, e.g. X is (diffeomorphic to) a product of spheres S^{n_i} of dimensions n_i, $\sum_i n_i = n$.*

Then, for all $\varepsilon > 0$, the product $S^{20n^2}\times X$ admits an immersion f_ε to the $(20n^2+n+1)$-ball, such that

$$(???) \qquad curv^\perp\left((S^N \times X) \overset{f_\varepsilon}{\hookrightarrow} B^{20n^2+n+1}(1)\right) \leq 1+2\sqrt{\frac{3(n+1)}{n+3}}+\varepsilon < 4.5.$$

Proof. The $\sqrt{3}$-immersion corollary 1.C with $m = n+1$ delivers an immersion $X \to B^{20n^2}(1)$ with $curv^\perp \leq \sqrt{\frac{3(n+1)}{n+3}}+\varepsilon$ and the manifold X'_ρ as in $[1+2c]$-example (1.G) does the job since it is diffeomorphic to $X\times S^{20n^2}$ in the present case.

$[X = \mathbb{T}^n]$**-Case.** If $N >> n$, then the $\sqrt{3}$-Clifford sub-torus theorem **1.C** implies that $S^N\times\mathbb{T}^n$ admits an immersion to the $(N+n+1)$-ball, such that

$$(???) \qquad curv^\perp\left((S^N\times X) \overset{f_\varepsilon}{\hookrightarrow} B^{N+n+1}(1)\right) \leq 1+2\sqrt{\frac{3n}{n+2}}.$$

Embedding Remark. Unlike how it is in ($\bullet_1$) and ($\bullet_2$), the construction of f_ε in 1.K creates self-intersection of $S^k\times X$ in the ball.

Sharpness Conjectures. The constant $1+2\sqrt{\frac{3n}{n+2}}$, probbaly, is optimal for tori $\mathbb{T}^n$ of dimension $n \geq 3$

We also conjecture that there are *no embeddings* $\mathbb{T}^n \times S^k \to B^{n+k+1}(1)$ with $curv^\perp \leq 1+2\sqrt{\frac{3n}{n+2}}+\varepsilon$ for all $n \geq 3$ and $\varepsilon < 1/n^2$.

But it is hard to say if the constant $\sqrt{\frac{3(n+1)}{n+3}}$ for general orientable $X^n\mathbb{R}^{n+1}$ can be improved, even to $\sqrt{\frac{3n}{n+2}}$.

Also it is unclear what to expect in this regard from *non-orientable immersed* hypersurface $X^n \hookrightarrow \mathbb{R}^{n+1}$

Products of Equidimensional Manifolds. The codimension one immersion theorem doesn't deliver immersions of products of equidimensional manifolds with "interesting" curvature bounds, while by arguing as in $(\bullet_1)$ and $(\bullet_2)$ we show the following.

$(\bullet_3)$ *The product of* $(m+2)$ *copies of* S^m *admits an embedding to the ball* $B^{m(m+2)+1}(1)$ *with* $curv^\perp \leq 1+2\sqrt{m+1}$.

For instance, (as in $\bullet_2$) the 3-torus embeds to the unit 4-ball, such that

$$curv^\perp\big(\mathbb{T}^3 \subset B^4)\big) \leq 1+2\sqrt{2} < 4.$$

Conjectrally, *the constant* $1+2\sqrt{m+1}$ *is optimal for all* $m = 1, 2, ...4$, *possibly, not only for embedding but also for immersions*

$$(S^m)^{m+2} \hookrightarrow B^{m(m+2)+1}(1).$$

6 Extremality, Rigidity, Stability: Spheres and Veronese Varieties

The natural candidates for *extremal immersions* $X \hookrightarrow B^N$, which implement maximal topological complexity with minimal curvatures are the most symmetric ones that are immersions, which are equivariant under large isometry groups G acting on X and B^N

For instance the standard ($O(n)$-equivariant) embedding $S^n \hookrightarrow B^N \subset B^{n+1} \times \mathbb{R}^{N-n-1}$ is extremal.

1.3. A. *All closed immersed n-submanifold* $X \overset{f}{\hookrightarrow} B^N$ *have* $curv^\perp \geq 1$, *where the n-dimensional spheres of radius one, are the only ones with* $curv^\perp \leq 1$. (If $n = 1$ these may be multiple coverings of the circle).

This follows by the maximum principle applied to the distance function from X to the boundary $\partial B^N(1))$ or equivalently to the squared distance to the center of the ball B^N denoted $r^2(x)$.

Since, $curv^\perp(X) \leq 1$, the second derivatives of r^2 along geodesics parametrized by the arc length satisfy: $\|r''r\| \leq 1$ and $(r^2)'' = r''r + \|r'\|^2 \geq 0$, since $\|r'\|^2 = 1$.

This says that r^2 is *a convex*, hence constant=1 function on X. Thus, X is contained in the unit sphere $S^{N-1}(1) = \partial B(1)$, where it has zero normal curvature (see ???), i.e. totally geodesic. QED

Rigidity and Stability. Most (all?) sharp geometric inequalities are accompanied by the rigidity/stability of the extremal objects[11].

To establish stability in this in the present case we start by observing that the above argument equally applies to all *complete* (for the induced Riemannian metrics) manifolds $C^{1,1}$-immersed to $B^n(1)$ and that the space of $C^{1,1}$-|immersions $curv^\perp \leq const$ of immersed complete manifolds to the ball is compact.

Thus we conclude that there exists $\varepsilon > 0$, such that if a closed immerses submanifold satisfies

$$curv^\perp\big(X^n \hookrightarrow B^N(1)\big) \leq 1+\varepsilon \text{ and } n \geq 2,$$

then X can be obtained by a δ-small C^1-perturbation of a unit n-sphere $S^n \subset B^N$, where $\delta \to 0$ for $\varepsilon \to 0$.

A priori, this ε could depend on n and N, but the above proof of the curvature $curv^\perp$-extremality of the unit spheres shows that this in not so; moreover, this, essentially 1-dimnsional proof suggests an effective, albeit rough, bound on ε, e.g. $\varepsilon = 0.01$ will do. (See below and section ??? for Petrunin's sharp results in this regard.)

Immersions to Tubes. The maximum principle applied to closed immersed n-submanifolds in "unit tubes" $B^N(1) \times \mathbb{R}^k \subset \mathbb{R}^{N+k}$ shows that

$$curv^\perp\big(X^n \hookrightarrow B^N(1) \times \mathbb{R}^k\big) \geq 1 \text{ for } k \leq n+1.$$

Extremal X, i.e. where $curv^\perp\big(X^n \hookrightarrow B^N(1) \times \mathbb{R}^k = 1$ for $k \geq 1$ are not unique, for $k \geq 1$; yet, the aspects of extremal geometry, which are dictated by the rigidity of half circle lemma (???) are stable, i.e. traceable in X with $curv^\perp(X) \leq 1+\epsilon$ (compare with section???) of where much of geometry of extremal X, where $curv^\perp\big(X^n \hookrightarrow B^N(1) \times \mathbb{R}^k = 1$ is dictated by the half circle lemma (see ??? ?? ????)

About Mean Curvature. The maximum principle argument also applies to immersed n-submanifolds X in B^N with $mean.curv \leq n-1$ (compare with ?? in section???) and shows that these X lie in S^{N-1}, where they are *minimal*, i.e. have zero mean curvatures.

There are lots of such submanifolds in S^{N-1} and the unit subspheres *are not* mean curvature stable and it is probably not hard to show that all n-manifolds admit δ-dense immersions $X \hookrightarrow B^N(1)$, $N \geq 2n$, with $mean.curv(X) \leq 1+\varepsilon$ for all $n \geq 2$ and $\varepsilon, \delta > 0$.

N. Nadirashvili, Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces. Invent. Math. 126 (1996), 457-465. MR 98d:53014

E. R. Rozendorn, The construction of a bounded, complete ‘ surface of nonpositive curvature, Uspekhi Mat. Nauk, 1961, Volume 16, Issue 2, 149–156

Veronese Manifolds.(*Elements of the architecture of our world?*) Besides n-spheres, there are other $O(n+1)$-equivariant immersion $S^n \hookrightarrow B^N(1)$, where the most interesting ones are the (quadratic) *Veronese maps.*

These are (minimal) isometric immersions of the n-spheres of radii $R_n = \sqrt{\frac{2(n+1)}{n}}$ to the unit balls, which factors through embeddings of the projective spaces $\mathbb{R}P^n = S^n(R_n)/\{\pm 1\}$ to the balls $B^{\frac{m(m+3)}{2}}$, where these embedding have *amazingly small* curvatures:

[11] See stability Gr. for a general discussion

$$curv(Ver_n) = curv^{\perp}\left(\mathbb{R}P^n_{Ver} \hookrightarrow B^{\frac{n(n+3)}{2}}\right) = \sqrt{\frac{2n}{n+1}}, \text{ e.g.}$$

$$curv(Ver_2) = curv\left(\mathbb{R}P^2_{Ver} \hookrightarrow B^5\right) = 2\sqrt{\frac{1}{3}} < 1.155,$$

Observe that the radii R_n of the Veronese n-spheres, which covers $\mathbb{R}P^n_{Ver}$, satisfy

[$2/curv^{\perp}$] $$R_n = \frac{2}{curv(Ver_n)}.$$

Conjecture.

$$curv^{\perp}(X^n, B^N) < \sqrt{\frac{2n}{n+1}} \implies X =_{diffeo} S^n.$$

The "homeo-version" of this proven by Petrunin for $n = 2$. (See Pet ???and section ??? where we also explain the above and say more about Veronese maps and their generalizations.)

Exercise. Identify Veronese manifolds with the spaces of quadratic forms of rank one and trace one.

7 Hypersurfaces Inscribed in Convex Sets

Given a subset $V \subset \mathbb{R}^{n+1}$, let $ext_{+r}(V)$ denote the *r-neighbourhood of* V, that is the subset of points in $\mathbb{R}^{n+1}$ within distance $\leq r$ from V.

$$ext_{+r}(V) = \{y \in \mathbb{R}^n\}_{dist(y,V) \leq r} \subset \mathbb{R}^{n+1},$$

and let

$$int_{-r}(V) \subset V$$

be the complement of the interior of the r-exterior of the complement of $\mathbb{R}^{n+1} \setminus V$, that is equal to the set of points in V with distance $\geq r$ from the boundary of V,

$$int_{-r}(V) = \{v \in V\}_{dist(v,\partial V) \geq r} \subset V.$$

Clearly,

$$ext_{+r}(int_{-r}(V)) \subset V \text{ and } int_{-r}(ext_{+r}(V)) = V.$$

Let $R = R(V)$ denote the *in-radius of* V, that is the maximal distance from the boundary of V in V,

$$R = inrad(V) = \sup_{v \in V} dist(v, \partial V)$$

and let

$$cntr(V) = int_{-r}(V)$$

be the set of the centers of the R-balls in V, that is the subsets of $v \in V$ with $dist(v, \partial V) = R = inrad(V)$.

Let $V \subset \mathbb{R}^{n+1}$ be a *compact convex domain*, e.g. the $(n+1)$-cube $\square^{n+1} = [-1,1]^{n+1}$ or an $(n+1)$-simplex $\triangle^{n+1}$.

Then, clearly, the r-*interior* of V is convex and if $r = R = inrad(V)$ then $int_R(V)$ called *the central locus* in V,

$$int_R(V) = cntr(V)$$

is a *non-empty compact convex subset in* V *of dimension* $\leq n = dim(V) - 1$.

For instance, if V is a cube or a simplex, then $cntr(V)$ consists of a single point and $ext_{+r}(int_{-r}(V))$ is equal to the (unique maximal) ball inscribed into V.

(If V is a general $(n+1)$-dimensional rectangular solid then $int_{-r}(V)$ is a subsolid of certain dimension $0, 1, ..., n$.)

1.4.A. Exercises. (a) Let the boundary of V be $C^{1,1}$-*smooth*[12] (e.g. piecewise C^2-smooth) with curvature bounded by a constant c,

$$curv^{\perp}(\partial V \subset \mathbb{R}^{n+1}) \leq c.$$

Show that if $r \leq \frac{1}{c}$, then, the r-balls $B \subset \mathbb{R}^{n+1}$ tangent to ∂V either are fully contained in V or lie outside V, meeting W at a single contact point between the boundaries of B and V; consequently:

$$ext_{+R}(cntr(V)) = V \text{ for } R = inrad(W).$$

(b) *Let* $X \hookrightarrow \mathbb{R}^{n+1}$ *be a* C^2*-smooth compact immersed hypersurface* in $\mathbb{R}^{n+1}$ and let

$$W = conv(X)$$

be the *convex hull* of (the image of) $X \hookrightarrow \mathbb{R}^{n+1}$.

Show that the boundary of W is $C^{1,1}$-*smooth*[13] with curvature bounded by that of X,

$$curv^{\perp}(\partial W \subset \mathbb{R}^{n+1}) \leq curv^{\perp}(X \hookrightarrow \mathbb{R}^{n+1}).$$

(c) *Sphericity.* Let $V \subset \mathbb{R}^{n+1}$ be a convex bounded domain, e.g. a polytope, such as $(n+1)$-cube $\square^{n+1} = [-1,1]^{n+1}$ or an $(n+1)$-simplex $\triangle^{n+1}$, and let $X \overset{f}{\hookrightarrow} V$ be a C^2-smooth immersion, where X is a closed n-manifold.

Apply (a) and (b) to the convex hull $W = conv(X) \subset V$ of X and show that if

$$inrad(V) = R \leq \frac{1}{curv^{\perp}(X \hookrightarrow V)},$$

then, in fact,

$$inrad(V) = \frac{1}{curv^{\perp}(X \hookrightarrow V)}.$$

Firermore, if $cntr(V)$ consists of a single point $o \in V$, (e.g. $V = \square^{n+1}$ or $V = \triangle^{n+1}$), show that the image of the immersion f is contained the R-ball centred at o for $R = inradV$.

Consequently, (see 1.3.A)

the image of $X \overset{f}{\hookrightarrow} V$ *is equal to the* R*-sphere centered at* $o \in V$.

[12]Locally, the hypersurface $\partial V \subset \mathbb{R}^{n+1}$ is representable by the graph of a C^1-function with bounded measurable second derivatives.

[13]Locally, the hypersurface $\partial W \subset \mathbb{R}^{n+1}$ is representable by the graph of a C^1-function with bounded measurable second derivatives.

(d) *Stability.* Argue as in section 1.3 and, assuming as above that $cntr(V)$ consists of a *single point* $o \in V$, show that the (only) R-sphere in V is stable:

there exists an $\varepsilon = \varepsilon(V) > 0$, *such that all immersed closed hypersurfaces* $X \hookrightarrow V$ *with* $curv^{\perp}(X) \leq R + \varepsilon$ *are* δ*-close in the* C^1*-topology to to the* R*-sphere* $S^n_o(R)$, *where* $\delta \underset{\varepsilon \to 0}{\to} 0$.

More on Stability. Unlike 1.3.B, this ε is sensitive to dimension.

For instance if V is the regular unit simplex then $\varepsilon(\triangle^{n+1}) \sim \varepsilon_0/n$ and if it is the cube $\square^{n+1} = [-1,1]^{n+1}$, then $\varepsilon(\square^{n+1}) \sim \varepsilon_0/\sqrt{n}$.

On $dim(cntr(V) > 0$. If $int_{-r}(V)$ *has positive dimension*, then there are many non-spherical C^2-immersed (and even more $C^{1,1}$) hypersurfaces in V with curvatures $\leq \frac{1}{inrad V}$, see section???.

?????????????????????????????

On $\mathbf{dim(cntr(V)) > 0}$. Let $dim(cntr(V)) = k > 0$, let $Z \subset \mathbb{R}^{n+1}$ be an affine k-dimensional subspace which contains the (convex!) subset $cntr(V) \subset \mathbb{R}^{n+1}$ and let $B^{n+1}_Z(R) = ext_{+R}(Z) \subset \mathbb{R}^{n+1}$ be the R-neighbourhood of Z.

Unlike that case of $dim(Z) = 0$, there are many non-spherical C^2-immersed (and even more $C^{1,1}$) hypersurfaces X in $B^{n+1}_Z(1)$ with normal curvatures $curv^{\perp} \leq 1$

Examples. (a) Let $X_o \hookrightarrow Z$ be a smooth closed immersed submanifold with $curv^{\perp} \leq 1/2$.

Then the 1-encircling[14] $X_{o+1} = (X_o)_{+1} \hookrightarrow \mathbb{R}^{n+1}$ of the immersion $X_o \hookrightarrow Z$ in $\mathbb{R}^{n+1}$ is an smooth immersed hypersurface in $B^{n+1}_Z(1)$ with $curv^{\perp}(X_{o+1}) \leq 1$.

(b) Let $X_o \hookrightarrow Z$ be a smooth compact submanifold with a boundary, such that $curv^{\perp}(X_o) \leq 1/2$ and $curv^{\perp}(\partial X_o) \leq 1/2$.

Then the 1-encircling $X_{o+1} = (X_o)_{+1} \hookrightarrow \mathbb{R}^{n+1}$ of the immersion $X_o \hookrightarrow Z$ in $\mathbb{R}^{n+1}$ is a piecewise C^2 smooth C^1-immersed hypersurface in $B^{n+1}_Z(1)$ with $curv^{\perp}(X_{o+1}) \leq 1$.

(c) Let $X_\circ \hookrightarrow \mathbb{R}^2$ be the figure ∞ curve made of two unit circles (as in ??) and let $S^1 \times S^{n-1} = X_{\circ\circ} \hookrightarrow \mathbb{R}^{n+1}$ be obtained by rotating $X_\circ$ around an axes $A \subset \mathbb{R}^2 \subset \mathbb{R}^{n+1} = \mathbb{R}^{2+(n-1)}$.

If this axes is normal to the line between the centers of the circles, then the image of the immersion $X \hookrightarrow \mathbb{R}^{n+1}$ is contained in the unit tube $B^{n+1}_{\mathbb{R}^2}(1)$ and if $dist(A, X_\circ) \geq 1$ then $curv^{\perp}(X_{\circ\circ}) \leq 1$. This $X_{\circ\circ} \hookrightarrow \mathbb{R}^{n+1}$ is C^1-smooth and piecewise C^2 smooth as in X_{o+1} (b) but the geometry of $X_{\circ\circ}$ is significantly different from that of X_{o+1}.

These (a)(b)(c) well represent immersed hypersurfaces with curvatures one in the unit "tubes". $B^{n+1}_{\mathbb{R}^k}(1)$,especially for $k = 1$, where all immersions of closed n-manifolds to $B^{n+1}_{\mathbb{R}^k}(1)$ for $n \geq 2$ are embedding, which are 1-encirclings (boundaries of 1-neighbourhoods) of segments in the line $\mathbb{R}^1$ (see section ??).

8 Bowl Inequalities

Let $X \hookrightarrow \mathbb{R}^N$ be an immersed complete (e.g. closed) connected n-dimensional submanifold in the Euclidean N-space, let $x_0 \in X$, let $T = T_{x_0} \subset \mathbb{R}^N$ be the tangent space to X at x_0 (represented by an affine subspace in $\mathbb{R}^N$) and let $P_{x_0} : X \to T^n_0$ be the normal projection map.

[14] "R-Encircling" is a generalisation of "boundary of the R-neighbourhood" for embeddings, see section ???.

Let $U_{x_0} \subset X$ be the maximal connected neighbourhood of x_0, such that the normal projection $P = P_{x_0}$, from U_{x_0} to T_{x_0} is a one-to-one diffeomorphism onto a domain $V_{x_0} \subset T_{x_0} = \mathbb{R}^n$, which is *star convex* with respect to x_0.

Clearly such a U_{x_0} exists and unique. where,

Let $\underline{S} = S^n(R)$ an n-sphere of radius R, which is tangent to X at the point x_0, (such spheres $\underline{S} = \underline{S}_\nu$ are parametrised by the unit normal vectors $\nu \in T^\perp_{x_0}(X)$) let $\underline{P \to} T_{x_0}$, be the normal projection map and observe that the corresponding neighbourhood $\underline{U}_{x_0} \subset \underline{S}$ is the hemisphere $\underline{S}_+$ that is the ball $B_{x_0}(\frac{\pi}{2}R) \subset \underline{S} = S^n$ around x_0.

Let $d(x) = dist_T(P(x), x_0)$ and let $\underline{d}(s) = dist_T(\underline{P}(s), x_0) = R \sin \frac{1}{R} dist_{\underline{S}}(s, x_0)$ be the corresponding function for the sphere $\underline{S}$.

Let $h(x) = dist(x, y = P(x))$, $x \in X$, and let $\underline{h}(s) = R\cos \frac{1}{R} dist_{\underline{S}}(s, x_0)$ be the corresponding function for the sphere $\underline{S}$.

Remark. Both d–functions and both h-functions have their gradients *bounded by one*, in fact,

$$\|grad_{\underline{S}}(d(s))\|^2 + \|grad_{\underline{S}}(\underline{h}(s))\|^2 = 1 \text{ and } \|grad_X(d(x))\|^2 + \|grad_X(h(x))\|^2 \le 1,$$

The gradients of both d-functions have unit norms at x_0,[15], they don't vanish in the interiors of the domains U_{x_0} and $\underline{U}_{x_0}$ correspondingly; $grad(\underline{h})$ vanishes on the boundary. of $\underline{U}_{x_0}$ and $\underline{U}_{x_0}$ vanishes at at least 2 ponts at the boundary of U_{x_0}.

The gradients of the h-functions have *norms* < 1 in (the interiors of) domains U_{x_0} and $\underline{U}_{x_0}$ correspondingly, and these norms. are equal to one the boundaries of these domains.

In fact, $\underline{U}_{x_0}$ is the same as the maximal connected neighbourhood of x_0, where $\|grad_X(h)\| < 1$ and the P-image of which is star convex.

Hemisphere Comparison Inequalities. Let

$$curv^\perp(X) \le curv^\perp(\underline{S}) = 1/R.$$

Then:

The gradient of the h-function on X,

$$h : x \mapsto dist(x, P(x))$$

for $x \in U_{x_0}$ is bounded by that for the $\underline{h}$-function on $\underline{S}_+$

$$\|grad(h)\| \le \|grad(\underline{h})\| \text{ for } dist_X(x, x_0) \le dist_{\underline{S}}(s, x_0)\| \text{ and } s \in \underline{S}_+$$

Consequently, the domain $U_{x_0} \subset X$ *contains an open R-ball* centered at x_0.

$\bullet_d$ The gradient of the d-function on X in the radial direction is bounded from below by that for $\underline{d}$:

if $s \in \underline{S}_+$ and a unit vector $\tau \in T_x(X)$ which is tangent to a geodesic segment γ in X issuing from x_0 and termnating at x satisfy

$$length(\gamma) \le dist(s, x_0)$$

then

$$\langle grad(d), \tau\rangle \ge \|grad(\underline{d}(s))\|,$$

[15]These functions are non-differentiable at x_0 but the norms of their gradients continously extend to one at x_0.

Consequently, the P-images in T of the r-balls from $U_{x_0} \subset X$ centered at x_0, contain the $\underline{P}$-images of the corresponding spherical balls from $\underline{S}_+$,

$$P(B_{x_0}(r)) \supset B_{x_0}\left(R \cdot \sin \frac{1}{R} r\right) \subset T \text{ for all } r \leq \tfrac{\pi}{2} R,$$

$\bullet_h^{-1}$ The inverse function $h^{-1}(y)$, $y \in B_{x_0}(R) \subset T$, and the norm of its gradient are bounded by $\underline{h}^{-1}(y)$, and $\|grad(\underline{h}^{-1}(y))\|$ correspondingly.

Corollary. Let $\underline{B}^N(R) \subset \mathbb{R}^N$ be a ball, such that the boundary sphere $\underline{S}^{N-1}(R) = \partial \underline{B}^N(R)$ is tangent to X at x_0, i.e.

$$T_{x_0}(\underline{S}^{N-1}(R)) \supset T_{x_0}(X).$$

If $curv^\perp(X) \leq 1/R$, then the subset $U_{x_0} \subset X$ doesn't intersect the interior of this ball. Thus, U_{x_0} lies in the closure of the complement of the union of the R-balls tangent to X at x_0.

Spherical Bowl Theorem. Let $U_{x_0}(+r) \subset X$ be the r-neighbourhood of U_{x_0} in X. Then the gradient of the function $d(x) = dist_T(P(x), x_0)$ doesn't vanish in the interior of the complement $U_{x_0}(+R) \setminus U_{x_0}$ and the P-mage of the complement $U_{x_0}(+r) \setminus U_{x_0}$, $r \leq R$, doesn't intersect the interior of the ball $B_{x_0}(R - r) \subset T$.

Proof The bounds on the gradients of the functions h in the hemisphere comparison inequalities follow from the angular arc inequality. 1.1.E, while the bowl theorem follows from these inequalities applied to X at x_0 and at all ponts $x \in \partial U_{x_0}$.

If $dim(X) = 1$, then the bowl theorem, where the proof[16] becomes especially transparent [17] implies the following.

"Circular Arm" Inequality. Let a planar circular arc $A \subset \mathbb{R}^2$ (a segment of a circle) and a smooth spatial curve $X \hookrightarrow \mathbb{R}^N$ satisfy:

$$length(X) = length(A) = l \text{ and } curv^\perp(X) \leq curv^\perp(A).$$

Then the distance between the endpoints of X is *greater than or equal to that* in A, where the equality holds if and only if X is congruent to A.

Example Let $X \subset \mathbb{R}^N$ is a closed curve of length 2π. If $curv^\perp(X) \leq 1$, then the Euclidean distances between opposite points $x, x_{opp} \in X$ are ≥ 2, where an equality $dist(x_0, (x_0)opp = 2$ implies that X is circlular.

Exercise. Show that all closed curves of length 2π in the Euclidean space contain pairs of opposite points $x, x_{opp} \in X$,(i.e. with the X-distance π between them), such that $dist(x, x_{opp}) \leq 2$.

n-d Corollaries Then the geometry of such an X mainly (but not fully) determined by the behaviour of geodesic segments from X, which are *1-dimensional submanifolds* $\mathbb{R}^N$ – *curves with* $curv^\perp \leq 1$.

Remarks (a) Other proof Proof of the Circular Bowl Inequality, and ref to Schmid by Hopf

The following proposition, says that $f(U_0)$ lies at least as close to Y_0 in the C^1-metric as S_c^1 to Y_S.

[16]Hopf Schimd

[17]he abive Hopf Schimdt, oter proofs

5.C. C^1**-Flatness Theorem.** Let $curv^\perp(U_0) \le c = curv^\perp(S^1_c)$. Then

(i) *the domain* $V_0 \subset Y_0$ *contains the (open) unit ball* $B^n_{x_0}(l/2)$, *where* $l = length(S^1_c)$,

(ii) $dist(f(u), Y_0)) \le diam(S^1_c)$ *for all* $u \in U_0$.

Moreover, this distance function is bounded by the corresponding function for S^1_c in two ways:

$$dist_{S^1_c}(s, s_0) \ge dist_X(u, u_0) \implies dist_Y(f(u), Y_0)) \le dist(s, Y_S));$$

and

$$dist_Y(P_S(s), s_0) \ge dist_Y(P_0 \circ f(u), f(u_0)) \implies dist(f(u), Y_0)) \le dist(s, Y_S)).$$

(iii) The gradient of the distance function between $f(X)$ and Y_0 is also bounded by that for S^1_c in two ways;

$$dist_{S^1_c}(s, s_0) \ge dist_X(u, u_0) \implies \nabla_X dist(f(u), Y_0) \le \left|\frac{d}{ds} dist(s, Y_S)\right|,$$

and

$$dist_Y(y = P_S(s), s_0) \ge dist_Y(P_0 \circ f(u), f(u_0)) \implies \nabla_{Y_0}(dist(v, p_0^{-1}(v) \le \left|\frac{d}{dy} dist(y, P_S^{-1}(y)\right|.$$

(iv) the X-grad*ient* of the distance function $u \mapsto dist(f(u), Y_0$ *tends to one for* $u \to \partial U$, while the Y_0-gradient of the inverse function $v \mapsto dist(v.P_{-1}(v)$. *tends to infinity for* $v \to \partial V$.

Clearly such a U_0 exists and unique, where, this an essential example, if $X = S^n \subset \mathbb{R}^{n+1}$, then such a U_0 is the hemisphere around x_0.

==============

Let $f: X \hookrightarrow Y$ be an a $C^{1,1}$-smooth (e.g. C^2) immersion with

$$curv^\perp(X \overset{f}{\hookrightarrow} Y) \le c,$$

where the manifold Y is complete simply connected with constant curvature, (e.g. $Y = \mathbb{R}^N$ or $X = S^N(1)$) and where X is geodesically complete with respect to the induced Riemannian metric (e.g X is compact without boundary).

5.B. *Remarks.* (a) The $[2\sin]_{bow}$-inequality for infinitesimally close points $x_1, \underline{x}$ is equivalent to $curv^\perp \le 1$.

.

Let Y_S be the tangent (line) to the above circle $S^1_c \subset Y$ at some point $s_0 \in S^1_c$ and $P_S: S^1_c \to Y_S$ be the normal projection.

Let $f: X \hookrightarrow Y$ be an a $C^{1,1}$-smooth (e.g. C^2) immersion with

$$curv^\perp(X \overset{f}{\hookrightarrow} Y) \le c,$$

where the manifold Y is complete simply connected with constant curvature, (e.g. $Y = \mathbb{R}^N$ or $X = S^N(1)$) and where X is geodesically complete with respect to the induced Riemannian metric (e.g X is compact without boundary).

Then *the* Y*-distance between the ends of* γ *is bounded from below by the* Y*-distance between the ends* s_0 *and* s_1 *of* S, *where the equality holds if and only if* γ *is congruent to* S *in* Y.

For instance,

8.1 High Dimensional Applications of the Circular Bowl inequality

Basic geometry properties of immersed n-submanifolds X in Euclidean spaces with

$$curv^{\perp}(X \overset{f}{\hookrightarrow} \mathbb{R}^N) \leq c,$$

can be reduced to the case $n = 1$ applied to the geodesic segments from X, which are, by the definition of the normal curvature $curv^{\perp}(X)$, are *curves in* $\mathbb{R}^N$ *with* $curv^{\perp} \leq c$.

The circular bow inequality applied to geodesics in immersed n-submanifolds

$$X \overset{f}{\hookrightarrow} \mathbb{R}^N,\ dim(X) = 1, 2, ...n, ...$$

yields the following.

5.A. Geodesic Lower Expansion Bound.[18] Let $\gamma \hookrightarrow Y$ be a geodesic segment in X and let $S \subset Y$ be a planar arc with constant curvature $curv^{\perp}(S) = curv^{\perp}(S^1_c) = c$. [19]

Then *the Y-distance between the ends of γ is bounded from below by the Y-distance between the ends s_0 and s_1 of S, where the equality holds if and only if γ is congruent to S in Y.*

For instance,

1.4.A.$[2\sin]_{\mathbf{bow}}$-Inequality. Let $\gamma \hookrightarrow X$ be an (not necessary minimising) *geodesic segment*[20] between two points $x_0, x_1 \in X$. If the normal curvature of X is bounded by $1/R$ and if $length(\gamma) = l \leq 2\pi R$, then

Then the Euclidean distance between these points is bounded from below:

$$[2\sin]_{bow}, \qquad dist_Y\big(f(x_0), f(x_1)\big) \geq 2R \sin \frac{l}{2R}$$

and, the equality implies that the f-image of γ is a circular ark in a plane in $\mathbb{R}^N$.

Corollaries

?? If X is connected and the induced metric in X is *complete* (e.g, X is compact without boundary), then

$$[2\sin]_{dist} \qquad dist_Y\big(f(x_0), f(x_1)\big) \geq 2\sin\left(\frac{dist_X(x_0, x_1)}{2}\right)$$

for all $x_0, x_1 \in X$, such that $dist(x_0, x_1) \leq 2\pi$.

5.B. *Remark.* The $[2\sin]_{bow}$-inequality for infinitesimally close points x_0, x_1 is equivalent to the inequality $curv^{\perp} \leq R$.

(b) The $[2\sin]_{bow}$-inequality holds for immersions to (complete simply connected) manifolds Y with *non-positive* sectional curvatures and the full geodesic lower expansion bound also admits a generalisation to manifolds with non-constant curvatures.

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[18] See ??, ?? and references therein for the full Bow Lemma.

[19] If Y has zero or positive curvature, then this S is a part of a circle (or a straight line for $c = 0$) and if Y is a hyperbolic space, then S may also be a segment in an infinite planar curve of constant curvature, e.g. a *horocycle*.

[20] Recall "Geodesic" refers to the induced (inner) Riemannian metric in X,

.

5.C. 2π**-Injectivity** Let $TB_x(r)\subset T_x(X)$ gent space be the r-ball in the tangent space at a point $x\in X$ and let $\exp_x: TB_x(r)\to X$ be the exponential map. If $r<\pi$, then the composition of this map with our immersion $f: X\hookrightarrow\mathbb{R}^N$ is one-to one.

Here are two obvious sub-corollaries.

2π**-Geodesic Loop Inequality.** Geodesic loops γ in X have $length(\gamma)\leq 2\pi$.

5.C. 2π**-Diameter Inequality.** If the *intrinsic diameter*, i.e. the diameter with respect to to the induced Riemannian metric, of $X\overset{f}{\hookrightarrow}\mathbb{R}^N$, satisfies

$$diam_{int}(X)<2\pi,$$

then X is *embedded* to $\mathbb{R}^N$: the map f is one-to-one.

This inequality is sharp: the equality holds for $S^n_{Ver}(R_n)\to\mathbb{R}P^n_{Ver}\hookrightarrow B^{\frac{n(n+3}{2}}(1)$ by the above $[\frac{2}{curv^\perp}]$

$$diam_{int}(S^n_{Ver})=\pi R_n=\frac{2\pi}{curv^\perp(S^n_{Ver})}.$$

Question. Are Veronese the only ones with this property? (Compare with pet and also with section ???)

9 Bow, Arms Riemannian displcaement Controle

● **Circular Bow inequality.** Let a planar circular arc $A\subset\mathbb{R}^2$ (a segment of a circle) and a smooth spatial curve $X\hookrightarrow\mathbb{R}^N$ satisfy:

$$length(X)=length(A)=l \text{ and } curv^\perp(X)\leq curv^\perp(A).$$

Then the distance between the endpoints of X is *greater than or equal to that* in A, where the equality holds if and only if X is congruent to A.

Example Let $X\subset\mathbb{R}^N$ is a closed curve of length 2π. If $curv^\perp(X)\leq 1$, then the Euclidean distances between opposite points $x, x_{opp}\in X$ are ≥ 2, where an equality $dist(x_0,(x_0)opp=2$ implies that X is circlular.

Exercise. Show that all closed curves of length 2π in the Euclidean space contain pairs of opposite points $x, x_{opp}\in X$,(i.e. with the X-distance π between them), such that $dist(x,x_{opp})\leq 2$.

First Proof of ● . Parametrize the curves X and A by the ark length parameter $s\in[0,l]$, $l=length(A)=lenght(X)$, write $x(s)$ for X and $a(s)$ for A and let $x'(s)$ and $a'(s)$ denote the derivatives of these (vector) functions.

Let $\overrightarrow{[x(s_0),x(s_1)]}\subset\mathbb{R}^N$ be the oriented chord (straight segment) between the points $x(s_0),x(s_1)\in\mathbb{R}^N$ and let us use the same notation for the points on the curve $A\subset\mathbb{R}^2$.

Angular Bow Inequality. *If $l\leq\pi/curv^\perp(A)$, then the angles between the X-cord and the tangent vectors to X at the endpoints of X are bounded by the corresponding angles for A,*

$$[\angle_0]\qquad \angle\Big(x'(0),\overrightarrow{[x(0),x(l))]}\Big)\leq\angle\Big(a'(0),\overrightarrow{[a(0),a(l))]}\Big)$$

and

$$[\angle_l] \qquad \angle\left(x'(l), \overrightarrow{[x(0), x(l))]}\right) \leq \angle\left(a'(l), \overrightarrow{[a(0), a(l))]}\right),$$

where *equality* in either of two implies that X *is congruent to* A. [21]

Consequently, if $A_+ \subset \mathbb{R}^N$ is a circular arc of the same curvature as A between the points $x(0)$ and $x(l)$,[22] then the angles between the X-cord and the tangent vectors to X at the endpoints of X are bounded by the angles of the same chord $\overrightarrow{[x(0), x(l))]}$ with the tangent vectors to A_+ at the endpoints of A_+.

Proof. Let

$$\overrightarrow{\alpha}_X(s) = \angle\left(x'(s), \overrightarrow{[x(0), x(s))]}\right)$$

$$\overleftarrow{\alpha}_X(s) = \angle\left(x'(0), \overrightarrow{[x(0), x(s))]}\right)$$

and observe the following three inequalities.

(1) the derivative of $\overleftarrow{\alpha}$ is bounded by $\overrightarrow{\alpha}$ and the distance $r = r_X(s) = dist(x(0), x(s))$ as follows

$$|\overleftarrow{\alpha}'_X(s)| \leq \psi(\overrightarrow{\alpha}_X(s), r_X(s))$$

where the ψ is a smooth function monotone *increasing in* $\overrightarrow{\alpha}$ *for* $0 \leq \overrightarrow{\alpha} \leq \pi/2$ *and deceasing in* $r \geq 0$.

In fact, this inequality holds for

$$\psi(\overrightarrow{\alpha}, r) = \frac{\sin\overrightarrow{\alpha}}{r}$$

where it turns to equality for A in place of X,

$$\overleftarrow{\alpha}'_A(s) = \frac{\sin\overrightarrow{\alpha}_A(s)}{r_A(s)}, \ 0 \leq s \leq 2\pi/curv^\perp(A)$$

for

$$\overleftarrow{\alpha}_A(s) = \overrightarrow{\alpha}_A(s) = \frac{1}{2}s\cdot curv^\perp(A), \ r_A(s) = \frac{2}{curv^\perp(A)}\sin\tfrac{1}{2}s\cdot curv^\perp(A).$$

and

$$\overleftarrow{\alpha}'_A(s) = \overrightarrow{\alpha}'_A(s) = \frac{curv^\perp(A)}{2}.$$

(2) The *derivative of* $r_X(s)$ *is monotone decreasing* in $\overrightarrow{\alpha}_X(s)$.

In fact

$$r'_X(s) = \cos\overrightarrow{\alpha}_X(s).$$

(3) The derivative of the angle $\overrightarrow{\alpha}_X(s)$ is bounded by the curvature $c = c(s) = c_X(s) = curv^\perp(x(s))$ of X at $x(s) \in X$ as follows

$$\overrightarrow{\alpha}'_X(s) \leq \varphi(c_X(s), r_X(s), \overrightarrow{\alpha}_X(s)),$$

[21] Notice that $\angle(a'(0), \overrightarrow{[a(0), a(l))]} = \angle(a'(l), \overrightarrow{[a(0), a(l))]}) = \frac{l}{2}curv^\perp(A)$ and that the inequalities$[\angle_0]$ and $[\angle_l]$ follow one from another by reversing the direction of the s-parameter, but it is instructive to keep track of both angles.

[22] Such an ark exists only if $dist(x_0, x_1) \leq 2\sin\frac{\pi}{2\pi curv^\perp(A)}$

where φ is a smooth function, which is monotone *increasing in c.*

In fact, this inequality holds with

$$\varphi(s, r, \overrightarrow{\alpha}) = c - \frac{\sin \overrightarrow{\alpha}}{r},$$

which becomes equality for A instead of X for

$$\overrightarrow{\alpha}'_A(s) = curv^{\perp}(A) - \frac{\sin \overrightarrow{\alpha}_A(s)}{\frac{2}{curv^{\perp}(A)} \sin \frac{1}{2} s \cdot curv^{\perp}(A)} = curv^{\perp}(A)/2.$$

Remarks. (i) ADD Relation to Robotics

ADDADDADDADDADDADDADDADDADDADD

Conclusion of the proof of of the angular inequality

ADDADDADDADDADDADDADD

Derivation of Circular Inequality from the Angular one. Divide X by the point $x(l/2) \in X$ into two arcs of length $l/2$ and apply the angular bow inequality to these arcs. Thus we see that both distances

$$dist(x(0), x(l/2)) \text{ and } dist(x(l/2), x(l))$$

are *greater or equal* than $dist(a(0), a(l/2)) = dist(a(l/2), a(l))$ and that the angle between of the cords

$$\left(\overrightarrow{[x(0), x(l/2))]}\right) \text{ and } \left(\overrightarrow{[x(l/2), x(l))]}\right)$$

at the point $x(l/2)$ is *greater or equal* than the corresponding angle for A. Since $dist(a(0), a(l/2)) = dist(a(l/2), a(l))$, we conclude that

$$dist(x(0), x(l)) \geq dist(a(0), a(l)).$$

QED.

9.1 Riemannian Bow Inequalities.

The bow inequalities straightforwardly generalise to immersions to Riemannian manifolds Y with constant sectional curvatures κ – spheres and hyperbolic spaces and then extend in a comparison form to all $CAT(\kappa)$-spaces (see section???).

z *Exercise:* Derive circular bow inequality on the sphere $S^n \subset \mathbb{R}^{n+1}$ from that for $\mathbb{R}^{n+1}$

Hint: use planarity of geodesics in S^n

Exercise Prove the circular bow inequality in the hyperbolic space H^n

Hint: Aegue as in ??? with horofunction h instead of a linear function.

Examples (**1**) Let $h(y)$, $y \in \mathbb{R}^N$, be a linear function and let a curve $y(s)$ starts with zero h-growth, i.e. $\langle \tau_y(s_0), \mathsf{grad}_{y(s_0)}(h) \rangle = 0$. Then the fastest growth of $h(y(s))$ among curves $y(s)$ with $curv^{\perp}(y(s)) \leq 1$ is achieved by a circular arks of length $\pi/4$ followed by a straight ray in the direction of $\mathsf{grad}(h)$.

(**2**) Let $h(y)$ be the distance to the origin, $h(y) = ||y||$, let S be a half circular (of length π) ark and let

$$curv^{\perp}(y(s)) \leq curv^{\perp}(S) = 1.$$

(A) If $S \subset \mathbb{R}^N$ and the curve $y(s)$ start at zero,$s_0 = y(s_0) = 0$. Then

$$\|y(s)\| \geq \|s\| \text{ for all } s \in S.$$

Furthermore, the first point $s_\circ$ where the derivative of $h(y(s))$ vanishes, i.e. $\langle \tau_y(s_\circ), \mathsf{grad}_{y(s_\circ)}(h) \rangle = 0$, lies further from the origin then $s_1 \in S$,

$$\|s_\circ\| \geq \|s_1\|$$

and

$$lenght(y[s_0, s_\circ]) \geq lenght(S) = \pi.$$

(B) Let $S \subset \mathbb{R}^N$ be as above and $y(s)$ start within distance ≥ 2 from the origin normally to $\mathsf{grad}(h)$,

$$\|y(s_0)\| \geq 2 \text{ and } \langle \tau_y(s_0), \mathsf{grad}_{y_0}(h) \rangle = 0.$$

Then the function $h(y(s))$ decays slower than $\|s\|$ on S for s running from s_1 to s_0,

$$\|y(s)\| \geq \|\pi - s\|, s \in S.$$

The proofs of ??? are straight forward, where A and B together yield the $[2\sin]_{bow}$-inequality, where this argument generalises to several classes of Riemannian manifolds.

be monotone decreasing follow the "general direction" of the gradient of h, i.e.

$$\langle \tau(s), \mathsf{grad}_{\mathsf{s}}(h) \rangle \geq 0 \text{ for } s_0 \leq s \leq s_1$$

and let

$$\langle \tau_y(s_0), \mathsf{grad}_{y(s_0)}(h) \rangle \leq \langle (\tau(s_0), \mathsf{grad}_s(h) \rangle$$

and the tangent unit vector $\tau(s_0)$ to S is equal to $\tau_y(s_0)$ and compare the behaviours of the functions $h(s)$ and $h(y(s))$, where, due to umbilicity of $\mathcal{H}$ (hence of all level hypersufaces of the function $h(y)$), the function $h(s)$ doesn't depend on a specific position of S in $\mathbb{R}^N$, but only on the point $s_0 \in \mathbb{R}^N$ and the unit vector $\tau(s_0) \in \mathbb{R}^N$. (All such positions make a unit sphere S^{N-2} identified with the set of unit vectors normal to $\tau(s_0)$.

Specifically, we want decide when the inequality $curv(y(s) \leq curv(S, s)$ yields the inequality $h(y(s)) \geq h(s)$.

and let $D_0(y)$ be the differential of the distance function $y \mapsto dist(y, y_0)$. Let $\underline{y}(s)$ be a (planar) curve congruent to our $S \subset$Y (with constant curvature c now issuing from y_0, that is $\underline{y}(s_0) = y_0$.

Denote $\tau(s) = dy(s) \in T(Y)$ and $\underline{\tau}(s) = d\underline{y}(s) \in T(Y)$ denote the unit tangent vectors to $y(s)$ and to $\underline{y}(s)$ and let $[s_0, s_+] \subset S$ be the maximal segment where the value $D_0(\tau(s))$ is nonnegative and $[s_0, \underline{s}_+] \subset S$ the maximal segment, where $D_0(\underline{\tau}(s)) > 0$.

Let

$$curv^\perp(y(s)) \le c = curv^\perp(\underline{y}(s)).$$

Since

$$[d\tau/ds] \qquad curv^\perp(y(s)) = \left\| \frac{d\tau(s)}{ds} \right\| \text{ and } curv^\perp(\underline{y}(s)) = \left\| \frac{d\underline{\tau}(s)}{ds} \right\|$$

this inequality implies that the the function $D_0(\underline{\tau}(s))$ *decays slower* than $D_0(\underline{\tau}(s))$ on the segment $[s_0, s_+]$ and also implies the inequality $s_+ \ge \underline{s}_+$ Therefore,

$$D_0(\tau(s)) \ge D_0(\underline{\tau}(s), \ s \in [s_0, s_+].$$

and

$$dist(y(s_1), y_0) = \int_{s_0}^{s_1} D_0(\tau(s)) \ge \int_{s_0}^{s_1} D_0(\tau(s)) = dist(\underline{y}(s_1), y_0), \ s_1 \in [s_0, s_+].$$

9.2 Axel Schur's Bow Inequality and Cauchy's Arm lemma

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9.2.1 Displacement Lemmas

????Linear Displacement Lemma Let X and $\underline{X}$ be two curves parameterised by ark length $0 \le s \le l$ in the Euclidean N-space, written as $s \mapsto x(s) \in \mathbb{R}^N$ and $s \mapsto \underline{x}(s) \in \mathbb{R}^N$.

Let these curves be *tangent at* $s = 0$,

$$x(0) = \underline{x}(0) \text{ and } x'(0) = \underline{x}'(0)$$

and let $\underline{X}$ be a *locally convex curve* contained in the *plane* $\underline{P} \subset \mathbb{R}^N$ *generated by* $\tau = \underline{x}'(0)$ *and another unit vector* ν linearly independent from τ.

Let $h(x) =$ be the *linear function* on $\mathbb{R}^N$ with gradient ν, i.e.

$$h : x \mapsto h(x) = \langle x, \nu \rangle.$$

Let $\underline{X}$ be contained in the half space H_+ above the zero hyperplane $H_0 = \{h(x) = 0\} \subset \mathbb{R}^N$, i.e.

$$h(\underline{x}(s)) \ge 0$$

and let $\underline{X}$ be "downward oriented" with respect to h, i.e.

$$\langle x''(s), \rangle \nu \le 0 \ s \in [0, l],$$

which means that the curve $\underline{X}$ in the plane is the graph of of a *concave* function.

If the normal curvature of X *is bounded by that of* $\underline{X}$,

$$curv^\perp(x(s)) = \|x''(s)\| \le \|\underline{x}''(s)\| = curv^\perp(p(\underline{x}(s)),$$

then

$$[x \geq \frown] \qquad x(s) \geq \underline{x}(s) \text{ for all } s \in [0,l].$$

where the equality $x(l) = \underline{x}(l)$ implies that $X = \underline{X}$.

Furthermore, if the *normal projection of X to the plane $\underline{P} \supset \underline{X}$ meets $\underline{X}$ at a point $\underline{x}(l_0) \in \underline{P}$, $l_0 \leq l$,* then $x(s) = \underline{x}(s)$ for $s \in [0, l_0]$.

Proof. The angular variation of the tangent vectors to the first curve satisfies the following simple version of the angular arc inequality (1.1.E)

$$\angle(x_0'(s_0), x_1'(s_1)) \leq \int_{s_0}^{s_1} \|x_1''(s)\| ds$$

while the convexity of $\underline{X}$ and the inequality and the bound $length(\underline{G}[s_1, s_2]) \leq \pi$ imply the equality

$$\angle(\underline{x}(s_0), \underline{x}(s_1)) = \int_{s_0}^{s_1} \|\underline{x}''(s)\| ds.$$

This applies to $s_0 = 0$ and all $s = s_1 \leq l$ and show that

$$\langle grad(h), x_1'(s)\rangle \geq \langle grad(p), \underline{x}'(s)\rangle \text{ for all } 0 \leq s \leq l.$$

Then the proof follows by integration:

$$h(x_1(l)) - h(x_1(0)) = \int_0^l \langle grad(h), x_1'(s)\rangle ds \geq$$

$$\int_0^l \langle grad(h), \underline{x}'(s)\rangle ds = h(\underline{x}(l)) - h(\underline{x}(0)).$$

Convexity and Rigidity. Convexity of the curve $\underline{X}$ is necessary as well as sufficient for the validity of the displacement inequality.

In fact, given an arbitrary positive continuous function $c(s)$, $s \in l$ the there exist a unique up to congruence locally convex planar curve $\underline{x}(s)$, such that $curv^{\perp}\underline{x}(s) = c(s)$.

However, the inequality $curv^{\perp}(x(s)) = c(s)$ imposes non-trivial global constrains on the geometry of X only for "small" values of the integral $\int_0^l c(s)ds$, such as the circular bow inequality and more general *Axel Schur's bow Inequality* (see ???) for $\int_0^l c(s)ds$.

But if this integral is bounded by a "sufficiently large" constant C then the curves with $\int_0^l c(s)ds \leq C$ display much flexibility (see ???), where the critical C is somewhere (I am not certain exactly where) between 4π and 7π.

Proof of Circular Bow inequality.(Compare with ???) Let us parametrise X and S by the length parameter $s \in [-l, l]$ let $T \subset \mathbb{R}^N$ be the line tangent to X at the middle point $x(s=0)$, let $h : \mathbb{R}^N \to T_0$ be the normal projection regarded as a real valued (linear) function, where we identify TR, where $x(0) \in T$ serves for $0 \in \mathbb{R} = T$, and let us position S in $\mathbb{R}^N\mathbb{R}^2$ such that T is tangent to S at $s = 0$ as well.

Let $X_{\pm}$ and $S_{\pm}$ be the two halves of these curves corresponding to $s \gtrless 0$. and let us apply the lemma to the pair of curve $(X_+, \underline{X} = S_+)$ parametrised by $s \in [0,l]$ and the function h and also to the pair $(X_-, \underline{X}_- = S_-)$ parametrised by $-s$, $s \in [0,l]$ and the function $h_- = -h$.

Then we add the inequalities

$$h(x(l)) - h(x(0)) \geq h(\underline{x}(l)) - h(\underline{x}(0))$$

and

$$h_-(x_-(l)) - h_-(x_-(0)) \geq h_-(\underline{x}_-(l)) - h_-(\underline{x}_-(0))$$

and see that

$$h(x(l)) - h(x(-l)) \geq h(\underline{x}(l)) - h(\underline{x}(-l))$$

where, by the definition of h, the circular arc S represented by $\underline{x}(s)$ wth the ends $\underline{x}(\pm l))$ satisfies the equality

$$h(\underline{x}(l)) - h(\underline{x}(-l)) = dist(\underline{x}(-l), \underline{x}(-l)),$$

while the curve X satisfies the inequality

$$h(x(l)) - h(x(-l)) \leq dist(x(-l), \underline{x}(-l).$$

QED

move from the two ends of a curve to the center and evaluate the contraction rate

In general, the *Axel Schur's bow comparison inequality.*(See /////, and ???below)

applies to arcs S in arbitrary planar closed convex curves and to curves Y in $\mathbb{R}^N$, parametrised by the length parameter, which is identified with that of S, $S \ni s \mapsto y(s) \in Y \hookrightarrow \mathbb{R}^N$, where the curvature inequality for the corresponding points in the two curves,

$$curv(Y, y(s)) \leq curv(S, s), s \in S,$$

implies the distance inequality

$$dist(y(s_0), y(s_1)) \geq dist(s_0, s_1).$$

Here, Y may be only piecewise smooth (see ???), which includes as a special case the *Cauchy's Arm lemma*.

Arm Lemma. Let S be a *planar polygonal arc* that makes a part of *closed convex* curve and let Y be a polygonal curve in the Euclidean N-space obtained by "straitening" S:

Y composed of segments of *same lengths as* S and the angles between consecutive segments in Y are *bounded by the angles between the corresponding segments in* S. (see ???)

Then the distance between the ends of Y *is greater than or equal to the distance between the ends of* S.

(This lemma implies the general bow inequality by approximating general curves Y by piecewise linear ones.)

Proof of the Bow Inequality For Smooth Curves.[23] Let $S = [s_0, s_1]$, called an *arc*, be a segment of a smooth planar convex curve and let $y : s \mapsto y(s) \in \mathbb{R}^N$ be a curve parametrised by the arc-length identified with that in S.

[23]See ???and references therein for the proof general bow inequality.

Let $\tau_y(s) = dy(s)/ds$ be the (unit in the present case) tangent vector to $y(s)$ at s and let $h(y)$ $y \in \mathbb{R}^N$ be a function on $\mathbb{R}^N$, e.g. a linear function or the distance function $y \mapsto dist_{\mathbb{R}^N}(y, y(s_0))$.

Let the arc S be positioned in (some plane in) $\mathbb{R}^N$ and let us *compare the variation of the functions* $h(s)$ and $h(y(s))$, where¯these variations are the integrals of the *scalar products of the* grad*ient* of h with the (unit tangent) *vector functions* $s \mapsto \tau_y(s)$ and $s \mapsto \tau(s)$,

$$h(s_1) - h(s_0) = \int_{s_0}^{s_1} \langle \tau(s), \mathsf{grad}_s(h) \rangle ds \text{ and } h((y(s_1)) - h(y(s_0)) = \int_{s_0}^{s_1} \langle \tau_y(s), \mathsf{grad}_{y(s)}(h) \rangle ds.$$

Then evaluate these scalar products by integrating their derivatives, which satisfy

$$[d\tau/ds], \qquad \left\| \frac{d\tau_y(s)}{ds} \right\| = curv^{\perp}(y(s)) \text{ and } \left\| \frac{d\tau(s)}{ds} \right\| = curv^{\perp}(S, s)$$

and where the vector $\frac{d\tau(s)}{ds}$ lies in the plane of S, where it is normal to $\tau(s)$.

"Monotone" Lemma. Let the scalar product of $\langle \tau(s), \mathsf{grad}(h) \rangle$ be *monotone decreasing* in s, let $curv^{\perp}(y(s) \leq curv^{\perp}(S, s)$ and let

$$\langle \tau_y(s_0), \mathsf{grad}_{y(s_0)}(h) \rangle \geq \langle \tau(s_0), \mathsf{grad}_{s_0}(h) \rangle$$

for $s_0 \leq s \leq s_1$.

If $h(y)$ is a linear function on $\mathbb{R}^N$, then

$$h((y(s_1)) - h(y(s_0)) \geq h(s_1) - h(s_0).$$

Proof. Since the gradient of h is *constant* and the scalar product $\langle \tau(s), \mathsf{grad}_s(h) \rangle$ is *monotone decreasing*, the inequality $curv^{\perp}(y(s) \leq curv^{\perp}(S, s)$ and the relations $[d\tau/ds]$ imply that the derivatives of the two scalar products satisfy

$$\left| \frac{d\langle \tau_y(s), \mathsf{grad}_{y(s)}(h) \rangle}{ds} \right| \leq -\frac{d\langle \tau(s), \mathsf{grad}_{s_0}(h) \rangle}{ds}$$

By integrating this, we see that

$$\langle \tau_y(s), \mathsf{grad}_{y(s)}(h) \rangle \geq \langle \tau(s), \mathsf{grad}_s(h) \rangle,$$

and by integrating the second time we arrive at the required inequality $h((y(s_1)) - h(y(s_0)) \geq h(s_1) - h(s_0)$.

(2) Now, argue as in ??? and divide a general convex ark into two halves by a point $s_{1/2}$ in S, such that a unit tangent vector $\tau_{1/2}$ to S at $s_{1/2}$ is parallel to the difference $s_1 - s_0 \in \mathbb{R}^N$.

Let $h(y) = \langle y, \tau_{1/2} \rangle$, where $\tau_{1/2}$ is a unit tangent vector to S at $s_{1/2}$ observe that the "monotone" Lemma applies to $\pm h$ and the segments $[s_0, s_{1/2}]$ and $[s_1, s_{1/2}]$ and conclude the proof of the bow theorem by adding the "monotone" inequalities for the two segments:

$$|h(y(s_1) - h(y(s_0)| = |(h(y(s_{1/2})) - h(y(s_0)) + (h(y(s_1)) - y(s_{1/2})| \geq$$

$$|h(s_1) - h(s_0)| = dist(s_1, s_0).$$

5.B. Half Circle Lemma. Let $S = [s_0, s_1]$ be a convex ark in the plane, $S \subset \mathbb{R}^2$ and let $H^1 \subset \mathbb{R}^2$ be a line, which contains the point s_0 and which is normal to S at this point.

Let $y(s) = (y_1(s), y_2(s), ...y_n(s)) \in \mathbb{R}^N$, $s \in S$, be a smooth curve in $\mathbb{R}^N$ *isometrically* parametrized by S, i.e. $\|\frac{dy(s)}{ds}\| = 1$, and let $H^{n-1} \subset \mathbb{R}^N$ be a hyperplane, which contains the point $y(s_0)$ and which is normal to $y(s)$ at this point, i.e. normal to the vector $\frac{dy(s_0)}{ds}$.

If $curv(y(s)) \leq curv(S, s)$, $s \in [s_0, s_1]$ and if the total curvature of S is at most π, i.e. $\int_{s_0}^{s^1} curv^{\perp}(S, s) ds \leq \pi$, then

$$dist(y(s_1), H^{n-1}) \geq dist(s_1, H^1).$$

5.C. Furthermore, *extremality implies rigidity*:

if $dist(y(s_1), H^{n-1}) = dist(s_1, H^1)$, then the curve $y(s)$ is congruent to S.

Optimal Control Remark. Maximisation/minimisation of variations of functions, $h(y)$, by curves $y(s)$ with $curv^{\perp} \leq const$ is an instance of *an optimal control problem*[24] where solutions are often piecewise smooth rather than smooth (Optimal Control Systems by A. A. Fel'dbaum https://www.scribd.com/document/390018919/Optimal-Control-Systems-Feldbaum-pdf) `https://encyclopediaofmath.org/index.php?title=Pontryagin_maximum_principle` Add more remarks

observe that $\frac{1}{r_X(s)}$ is equal the curvature of the circle of radius $r_X(s)$ and that

Parametrize X and A be the length parameter $s \in [0, l]$, $l = length(A) = lenght(X)$, write X and A as $x(s)$ and $a(s)$ and derive ● from

Let $[x(0), x(l)] \subset \mathbb{R}^N$ and $[a(0), x(l)] \subset \mathbb{R}^2$ be straight segments (the chords) between the ends of the two curves,

There are several proofs of this theorem,[25] where the quickest one is by approximation of X and S by broken polygonal curves and applying the following

[24] Think of piloting a jet plane, where acceleration must be limited by a couple of G for your comfort.

[25] (see ?????)

theorem.

Cauchy-Legendre Arm's Lemma. Let $P \subset \mathbb{R}^2$ be a convex polygon in the plane with vertices $p_1, p_2, ...p_k$ and segments $s_i = [p_i, p_{i+1}]$, where $s_k = [p_k, p_1]$ and let $P' \subset \mathbb{R}^N$ be a polygonal curve composed of segments $s'_i = [p'_i, p'_{i+1}] \subset \mathbb{R}^N$, $i = 1, , 2, ...k-1$ for some points $p'_i \in \mathbb{R}^N$.

If the lengths of the segments s'_i (that are distances between p'_i and p'_{i+1}) are *equal to the lengths of* s_i and the *angles* between these segments at the vertices $p'_2, ..., p'_{k-1}$ are *greater* then those in P,

$$length(s'_i) = length(s'_i), i = 1, ...k-1, \text{ and } \angle_{p'_i}(s_{i-1}, s_i) \geq \angle_{p_i}(s_{i-1}, s_i), i = 2, ...k-1,$$

then the end points p_1 and p_k of P' are *further apart* then these of P',

$$dist(p'_1, p'_k) \geq dist(p_1, p_k),$$

where *the equality implies that* P' *is congruent to* P.

About the Proof. It is intuitively obvious and easy to prove (seventh grade school math) that increasing an angle between two edges in a convex polygonal curve P increases the dstance between the end points of P. But – this is counterintuitive – the resulting curve may become non-convex and one can't conclude the proof by induction on k.[26]

In any case, we need this lemma for *spacial* curves P', where there are at least two different proof, (see ??? and references therein) where the idea of one of them is well illustrated by the following observation going back to Euclid if not earlier.

Exercises (a) (b) Let X be a non-closed closed smooth spacial curve with curvature bounded by a positive function $c = c(s)$, $s \in [0, l = length X$

$$curv^{\perp}(x(s)) = \|x''(s)\| \leq c(s).$$

and $S = S_c$ be a (unique up to congruence) locally convex planar curve with $curv^{\perp}(S, s) = c(s)$.

Let the curve S be *globally convex*, i.e. it lies on the boundary of its convex hull, [27]

approximate X and S by polygonal curves and prove the following generalisation of the above $CIRCLE$.

●**Axel Schur's Bow Inequality.** The distance between the two ends of X is bounded from below by that in S. [28]

////////??????????????

5.B. *Remarks.* (a) The $[2\sin]_{bow}$-inequality for infinitesimally close points $x_1, \underline{x}$ is equivalent to $curv^{\perp} \leq 1$.

(b) The $[2\sin]_{bow}$-inequality holds for immersions to (complete simply connected) manifolds Y with *non-positive* sectional curvatures and the full geodesic lower expansion bound also admits a generalisation to manifolds with non-constant curvatures.

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[26]See Fig 3 in [Sabitov] and references to the contributions by Legendre, Cauchy and Steinitz.

[27]A sufficient condition for this is the inequality $\int_0^l c(s)ds \leq \pi$.

[28]See ????? for the history of this theorem.

10 Hypersurfaces in Balls and Spheres.

Let $N = n+1$, $n = dim(X)$, a let the image of an immersion $f : X \hookrightarrow \mathbb{R}^{n+1}$ (with $curv^{\perp}(X) \leq 1$ as earlier) be contained in the ball $B^{n+1}(2)$.

(a) If $n = 1$ and the *degree of the Gauss map* $S^1 = X \to S^1(1) \subset \mathbb{R}^2$ *equals zero*, then (this was stated in ???) the image $f(X) \subset B^2(1)$ equals the union of *two unit circles* which tangentially meet at the center of the disk $B^2(2)$.

(b) If $n \geq 2$, then either $f(X)$ is star convex, and the radial projection $X \to S^n(2)$ is a diffeomorphism, or $f(X)$ is equal to a unit n-sphere $S^n_{y_o}(1)$, where the center of this sphere is positioned half way from the boundary of the ball $B^{n+1}(2)$, i.e. $\|y_o\| = 1$.

(c) There exists an $\varepsilon > 0.01$, such that if $n \geq 2$ and the image $f(X)$ is contained in the ball $B^{n+1}(2\varepsilon))$ then $f(X) \subset B^{n+1}(2)$ is star convex with respect to some point in $B^{n+1}(2+\varepsilon)$.

Proof. If $f(X)$ is not star convex with respect to the center of the ball $B^{n+1}(2)$ then then some radial ray is tangent to $f(X)$ at some point $y_0 = f(x_0) \in f(X)$ and the half circle lemma implies that y_0 is equal to the center of $B^{n+1}(2)$ and the bow rigidity(see ???) implies that $f(x)$ equals a unit sphere passing through y_0.

This proves (b) while the bow stability argument (see???) yields an approximate unit sphere in $B^{n+1}(2+\varepsilon)$ and (c) follows as well.

Remarks (a)

Remark/Example. The boundary X_{+1} of the ρ-neighbourhood for $\rho = 1$ of a circular ark S with radius 2 has curvature bounded by 1. If such an S is slightly shorter than half circle, then, because of "shorter", X_{+1} can be fit to the ball of radius $3-\epsilon$ and X_{+1} and it is non-star convex because of "slghtly".

Question Do ε and ϵ ever meet or there is a definite gap between their possible values?

Encouraging Example with width $> \frac{\pi}{2}$. Let a proper compact Riemannian band $X$of dimension n admits an immersion to a complete n-dimensional Riemannian manifold X_+ with sectional curvatures $\kappa \geq 1$, such that the width of X with respect to the induced Riemannian metric is $> \frac{\pi}{2}$. Then

X contains a subband $X_- \subset X$ of width $d = width(X) > \frac{\pi}{2}$, which is homeomorphic to the spherical cylinder $S^{n-1} \times [0,1]$.

Acknowledgement. A similar result for $n = 3$ is proved in [Zhu(width) 2020], while our argument below follows that of Jian Ge from [Ge(linking) 2021], who sent me his preprint prior to publication.

Proof. Let, following a geometric idea from Ge's paper, X_- be the intersection of the d-neighbourhoods of the $\partial_{\mp}$-boundaries of X,

$$X_- = U_d(\partial_-) \cap U_d(\partial_-),$$

and observe that the $\partial_{\mp}$-boundaries of this X_- are *concave* for $\kappa \geq 1$ and $d > \frac{\pi}{2}$. Therefore, $\partial_{\mp}$ are diffeomorphic to S^{n-1} and the immersions

$$\partial_{\mp} \to X_+$$

extend to immersions of n-balls the *locally convex* boundaries of which are equal to $\partial_{\mp}$ (with their coorientations opposite to those in X_-). [29]

[29]Recall that *a closed immersed locally convex* hypersurface in a complete Riemannian manifold of dimension $n \geq 3$ with sectional curvatures > 0 bounds an immersed ball.

It follows, that if X_+ is simply connected, then it is homeomorphic to the n-sphere, the immersion $X_- \to X_+$ is one-to-one and the complement to X_- in X_+ is the union of two disjoint topological balls with convex boundaries; hence, X_- is homeomorphic to $S^{n-1} \times [0,1]$ for all X_+. QED.

(Probably, it is not hard to show that if X_+ is simply connected and X is an open band immersed to X_+ with $width = \frac{\pi}{2}$, then ether X is homeomorphic to $S^{n-1} \times (0,1)$, or X_+ is isometric to S^n and X is equal to the $\frac{\pi}{4}$- neighbourhood of $S^k \times S^{n-k-1} \subset S^n$.)

∞-Figure Corollary (compare 1.A)

(2+δ)-*Corollary* let $X \overset{f}{\hookrightarrow} B^{n+1}(1)$ be a smooth closed connected n-dimensional hypersurface in the unit $(n+1)$-ball and let

$$curv^{\perp}\big(X \hookrightarrow B^{n+1}(1)\big) \le 2 + \delta,$$

where $\delta > 0$ is an universal constant (probably $\delta > 0.2$. If $n \ge 2$, then either the radial projection $X \to S^n(1)$ is an immersion, hence f is a star convex embedding with respect to the origin $0 \in B^{n+1}(1)$, or f is a star convex embedding with respect to a point $x_0 \in B^{n+1}(1)$ with $\|x_0\| = 1/2$.

2. Let X be a C^1-smooth closed n-dimensional immersed hypersurface in $\mathbb{R}^{n+1}$ positioned between two parallel hyperplanes with distance 2 between them,

$$X \hookrightarrow R^n \times [-1,1] \subset \mathbb{R}^{n+1}$$

in the unit ball wit

Let $X \hookrightarrow \mathbb{R}^N$ be a compact (complete suffices) immersed n-submanifold, let $x_0 \in X$, and let $U_0 \subset X$ of x be the maximal connected neighbourhood such that the normal projection from U_0 the tangent space $T_0 = T_{x_0}(X) \subset \mathbb{R}^N$,

$$P_0 : U_0 \to T_0$$

is a one-to-one diffeomorphism onto a domain $V_0 \subset T_0 = \mathbb{R}^n$, which is star convex with respect to $x_0 \in T_0$.

11 Immersed Submanifolds in Balls, in Bands and in (k,R)-Tubes

Let an n-dimensional manifold X be immersed to the *k-tube* $B^N_{\mathbb{R}^k}(R)$ *of radius* R,

$$X \overset{f}{\hookrightarrow} B^N_{\mathbb{R}^k}(R) = B^N(R) \times \mathbb{R}^k \subset \mathbb{R}^{N+k}$$

(where $B^N(R) = B^N_0(R) \subset \mathbb{R}^N$ is the R-ball), let $p : X \to \mathbb{R}^k_{ax} = \{0\}\times = \mathbb{R}^k$ be the projection of $X \hookrightarrow B^N_{\mathbb{R}^k}(R)$ to the central axes of the tube, let

$$\mathcal{K} = \mathcal{K}(p) \subset T(X) \hookrightarrow T(B^N_{\mathbb{R}^k}(R))$$

be the *kernel of the differential* $dp : T(X) \to T(B^N_{\mathbb{R}^k}(R))$ and let

$$\Sigma = \Sigma(p) = \{x \in X\}_{rank(\mathcal{K}_x)>0} \subset X$$

be the support of $\mathcal{K}$.[30]

Let the induced Riemannian metric in X be *geodesically complete*, e.g. X is compact without boundary, and let

$$\gamma_\tau(l) \hookrightarrow X \hookrightarrow B^N_{\mathbb{R}^k}(R),\ \tau \in \mathcal{K}_x$$

be the geodesic segment of length l issuing from $x \in \Sigma$ in the τ-direction, where τ is a *non-zero vector* in the vector (sub)space $\mathcal{X}_x \subset T_x(X), X \in \Sigma$.

If

$$curv^\perp(X \hookrightarrow B^N_{\mathbb{R}^k}(R) \subset \mathbb{R}^{N+k}) \leq 1/R,$$

then the definition of $curv^\perp$ (section ???) and the half circle lemma applied to the curves $\gamma_{\pm\tau}(\frac{1}{2}\pi R)$ in the R-tube $B^N_{\mathbb{R}^k}(R)$ and to the hyperplane $H = H_{\perp\tau} \subset \mathbb{R}^{N+k} \supset B^N_{\mathbb{R}^k}(R)$, which contains $f(x) \in B^N_{\mathbb{R}^k}(R)$ and is normal to τ imply the following.

?? Either $\Sigma = \varnothing$, i.e. $p : X \to \mathbb{R}^k$ is an immersion, (in this case one may have $curv^\perp(X) < 1/R$) the curves $f(\gamma_\tau(\frac{1}{2}\pi R))$ and $f(\gamma_{necssarilymade-\tau}(\frac{1}{2}\pi R))$ are. quoters of planar circlers, both of which reach the boundary of the tube.

Thus, $f(X) \subset B^N_{\mathbb{R}^k}(R)$ intersect the boundary of $B^N_{\mathbb{R}^k}(R)$ at at least two points.

(a') C^2-*Remark.* If an immersion f is C^2-smooth, so is the πR-curve in the tube made of $f(\gamma_\tau(\frac{1}{2}\pi R))$ and $f(\gamma_{-\tau}(\frac{1}{2}\pi R))$. This necessarily make this curve a planar half circle.

But piecewise C^2-curves made of circular arc of same curvature $1/R$ are not always planar arks themselves. This, however can't happen to geodesic of n-dimensional piece-wise C^2-smooth X for $n \geq 2$ (see section???).

Thus

every point $x \in \Sigma \subset X^n$, $n \geq 2$, serves as the center of a geodesic R-hemisphere $(S^m_+)_x \subset X$ of dimension $m = rank(\mathcal{K})x$, such that the map f isometrically sends $(S^m_+)_x$ to an equatorial m-hemisphere in the $(N-1)$-sphere $S^{-1}_{p(x)}$, where the boundary of this hemisphere is contained in the boundary of the tube $B^N_{\mathbb{R}^k}(R)$.

Exercises. (a). Recall that the real projective spaces of dimension $n = 2^l$, admit no immersions to $\mathbb{R}^k$ for $k \leq 2n-2$, and show that they admit no immersions to the tubes $B^N_{\mathbb{R}^k}(R)$ with $curv^\perp(f) < 1/R$.

(b) Let let the f-mages all geodesic segments of length πR in $X^n \overset{f}{\hookrightarrow} B^N(R)$ issuing from a point $x_0 \in X$ have curvatures $\leq 1/r$ in the ball $B^N(R)$.

Show that the image $f(X) \subset B^N(R)$is equal to an equatorial n-sub-sphere in $S^{N_1}(R) = \partial B^N(R)$ and if X is connected and $dim(X) \geq 2$ then the immersion f is an embedding.

(c) Let a closed connected n-manifold X, $n \geq 2$, be immersed to a (cylindrical) $(1,R)$-tube

$$X \overset{f}{\to} B^N_{\mathbb{R}^1}(R) \subset \mathbb{R}^{N+1}.$$

Show that *the only critical points* $x \in X$ of the function $p : X \to \mathbb{R} = \mathbb{R}^1_{ax}$, i.e. where $rank(\mathcal{K}_x) = n$, are a maximum and a minimum ponts of p, and that the f-images of both of them in the tube are positioned on the axial line $\mathbb{R}^1_{ax} = \{0\} \times \mathbb{R}^1$,

[30]If $k < n$, then $\Sigma = X$ and $rank(\mathcal{K}_x(p)) = n-k$, for generic maps $p : X^n \to \mathbb{R}^k$ and generic points $x \in X$. If $k \geq n$, then ether p is an immersion, i.e. $\Sigma = \varnothing$, or $dim(\Sigma(p)) = 2n-k-1$ for generic p and and $rank(\mathcal{K}_x(p)) = 1$ at generic $x \in \Sigma$.

where they serve as the centers of n-hemispheres $(S^n_+)_{max}(R)$ and $(S^n_+)_{min}(R)$, both of radius R and where both are contained in the $f(X)\subset B^N_{\mathbb{R}^1}(R)$ and where the spherical $(S^{n-1}(R))$ boundaries of them are contained in the boundary of the tube.

Show that the $(n-1)$-hemispheres $(S^{n-1}_+)_{f(x)}(R)$ in the tube tangent at their centers $y=f(x)$ to the (topologically $(n-1)$-spherical) fibers of the map p for all non-critical points $x\in X$ *continuously depend on* x.

Show that the image of the immersion $f: X\to B^N_{\mathbb{R}^1}(R)$ equals the union of the two hemi-sperical cups $(S^n_+)_{max}(R)$ and $(S^n_+)_{min}(R)$ and a region between them *contained in the boundary of the tube.* (This is not so for n=1.)

(i) Let $n\geq 2$ and $N=n$and show that f is an embedding, the image of which is equal the $+R$-encircling of a segment in the central line $\mathbb{R}^1$ in $B_{\mathbb{R}^1}(R)^N$, that is a (convex) region between to half-R-spheres normal to this line, which is equal in the present case to the boundary of the convex hull of $f(X)\subset B_{\mathbb{R}^1}(R)^N$. (Unless the half-$R$-spheres have a common boundary, this region is only *piecewise* C^2.)

(ii) Let $n\geq 2$ and $N>n$. Show that f is an embedding into the $+R$-encircling of a central segment $[a,b]\subset\mathbb{R}^1$ in $B_{\mathbb{R}^1}(R)^N$, where this image contains two n-hemispheres of radius R and a cylindrical region between them which is fibered over $[a,b]$, where the fibers are equatorial $(n-1)$-subspheres in the $(N-1)$-spheres $S^{N-1}_y(R)\subset\partial B_{\mathbb{R}^1}(R)^N$, $y\in[a,b]$.

12 Veronese Revisited

Besides invariant tori, there are other submanifolds in the unit sphere S^{N-1}, which have small curvatures and which are transitively acted upon by subgroups in the orthogonal group $O(N)$.

The generalized Veronese maps are a *minimal equivariant isometric* immersions of spheres to spheres, with respect to certain homomorphisms (representations) between the orthogonal groups $O(m+1)\to O(m+1)$,

$$ver=ver_s=ver^m_s: S^m(R_s)\to S^m=S^{m_s}=S^{m_s}(1),$$

where

$$m_s=(2s+m-1)\frac{s+m-2)!}{s!(m-1!}<2^{s+m} \text{ and } R_s=R_s(m)=\sqrt{\frac{s(s+m-1)}{m}},$$

for example,

$$m_2=\frac{m(m+3)}{2}-1,\ R_2(m)=\sqrt{\frac{2(m+1)}{m}} \text{ and } R_2(1)=2,$$

(see [DW1971]If $s=2$ these, called *classical Veronese maps*, are defined by taking squares of linear functions (forms) $l=l(x)=\sum_i l_ix_i$ om $\mathbb{R}^{m+1}$,

$$Ver:\mathbb{R}^{m+1}\to\mathbb{R}^{M_m},\ M_m=\frac{(m+1)(m+2)}{2},$$

where tis $\mathbb{R}^{M_m}$ is represented by the space $\mathcal{Q}=\mathcal{Q}(\mathbb{R}^{m+1})$ of quadratic functions (forms) om $\mathbb{R}^{m+1}$,

$$Q=\sum_{i=1,j=1}^{m+1,m+1} q_{ij}x_ix_j.$$

The Veronese map, which is (obviously) equivariant for the natural action of the orthogonal group group $O(n+1)$ on $\mathcal{Q}$, where, observe, this action fixes the line $\mathcal{Q}_\circ$ spanned by the form $Q_\circ = \sum_i x^2$ as well as the complementary subspace $\mathcal{Q}_\diamond$ of the *traceless forms* Q, where the action of $O(n+1)$ is irreducible and, thus, it has a *unique, up to scaling* Euclidean/Hilbertian structure.

Then the normal projection[31] defines an equivariant map to the sphere in $\mathcal{Q}_\diamond$

$$ver : S^m \to S^{M_m-2}(r) \subset \mathcal{Q}_\diamond,$$

where the radius of this sphere, a priori, depends on the normalization of the $O(m+1)$-invariant metric in $\mathcal{Q}_\diamond$.

Since we want the map to be isometric, we either take $r = \frac{1}{R_2(m)} = \sqrt{\frac{m}{2(m+1)}}$ and keep $S^m = S^m(1)$ or if we let $r = 1$ and $S^m = S^m(R_2(m))$ for $R_2(m) = \sqrt{\frac{2(m+1)}{m}}$.

Also observe that the Veronese maps, which are not embeddings themselves, factor via embeddings of projective spaces to spheres

$$S^m \to \mathbb{R}P^m \subset S^{M_m-2} \subset \mathbb{R}^{M_m-1} = \mathcal{Q}_\diamond,\ M_m = \frac{(m+1)(m+2)}{2}.$$

Curvature of Veronese. Let is show that

CURvature of veronese by Petrunin formula

$$curv^\perp_{ver}\left(S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)\right) = \sqrt{\frac{R_2(1)}{R_2(m)} - 1} = \sqrt{\frac{m-1}{m+1}}.$$

Indeed, the Veronese map sends equatorial circles from $S^m(R_2(m))$ to planar circles of radii $R_2(m)/R_2(1)$, the curvatures of which in the ball B^{M_m-1} is $R_2(1)/R_2(m) = 2\sqrt{\frac{m}{m+1}}$ and the curvatures of these in the sphere,

$$curv^\perp(S^1 \subset S^{M_m-2}(1)) = \sqrt{curv(S^1 \subset B^{M_m-1}(1))^2 - 1} = \sqrt{\frac{4m}{m+1} - 1} = \sqrt{\frac{3m-1}{m+1}}$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)$ itself

$\sqrt{R_2(1)/R_2(m)} = \sqrt{\frac{2m}{m+1}}$, and the curvatures of these in the sphere,

$$curv^\perp(S^1 \subset S^{M_m-2}(1)) = \sqrt{curv(S^1 \subset B^{M_m-1}(1))^2 - 1},$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)$itself. QED.

It may be hard to prove (conjecture in section 1) that Veronese manifolds have the smallest possible curvatures among non-spherical m-manifold in the unit ball: *if a smooth compact m-manifold X admits a smooth immersion to the unit ball $B^N = B^N(1)$ with curvature $curv^\perp(X \hookrightarrow B^N) < \sqrt{\frac{2m}{m+1}}$, then X is diffeomorphic to S^m.*

It is more realistic to show that the Veronese have smallest curvatures among submanifolds $X \subset B^N$ *invariant under subgroups in $O(N)$, which transitively act on X.*

[31]The splitting $\mathcal{Q} = \mathcal{Q}_\circ \oplus \mathcal{Q}_\diamond$ is necessarily normal for all $O(m+1)$-invariant Euclidean metrics in $\mathcal{Q}$.

Remark. Manifolds X^m immersed to S^{m+1} with curvatures < 1 are diffeomorphic to S^n, see 5.5, but, apart from Veronese's, we can't rule out such X in S^N for $N \geq m+2$ [32] and, even less so, non-spherical X immersible with curvatures $< \sqrt{2}$ to $B^N(1)$, even for $N = m+1$.

It seems hard to decide this way or another, but it may be realistic to try to prove *sphericity of simply connected* manifolds immersed with curvatures < 1 to $S^N(1)$ for all N.

The curvatures of Veronese maps can be also evaluated with the *Gauss formula,* (teorema egregium), which also gives the following formula for curvatures of all ver_s:

$m = 2$ $1 - 2c^2 = 1/3$, $2c^2 = 2/3$ $c\sqrt{1/3}$

$C = \sqrt{1 + 1/3} = 2/\sqrt{3}$

From Veronese to Tori. The restriction of the map $ver_s : S^{2m-1}(R_s) \to S^{N_s}$ to the Clifford torus $\mathbb{T}^m \subset S^{2m-1}(R_s)$ obviously satisfies

$$curv^{\perp}_{ver_s}(\mathbb{T}^m) \leq A_{2m-1,s} + \frac{\sqrt{m}}{R_s} = \sqrt{3 - \frac{5}{2}m + \varepsilon(m,s)}$$

for

$$\varepsilon(m,s) = \frac{2}{4m^2} - \frac{4m-2}{s(s+2m-2)} + \frac{5(2m-1)}{2ms(s+2m-2)} - \frac{2m-1}{(ms(s+2m-2))^2}.$$

This, for $s >> m^2$, makes $\varepsilon(m,s) = O\frac{1}{m^2}$

Since $N_s < 2^{s+2m}$,

starting from $N = 2^{10m^3}$

$$curv^{\perp}_{ver_s}(\mathbb{T}^m) < \sqrt{3 - \frac{5}{2}m}.$$

where it should be noted that

the Veronese maps restricted to the Clifford tori are $\mathbb{T}^m$*-equivariant*

and that

this bound is *weaker than the optimal one* $\frac{||y||^2_{l_4}}{||y||^2} \geq \sqrt{3 - \frac{3}{m+2}} + \varepsilon$ from the previous section.

Remarks. (a) It is not hard to go to the (ultra)limit for $s \to \infty$ and thus obtain an

equivariant isometric immersion ver_∞ of the Euclidean space $\mathbb{R}^m$ to the unit sphere in the Hilbert space, such that

$$curv^{\perp}_{ver_\infty}(\mathbb{R}^m \hookrightarrow S^\infty) = \sqrt{\frac{(m-1)(2m+1)}{(m+1)^2}} = \sqrt{2 - \frac{5}{m+1} + \frac{2}{(m+1)^2}},$$

where equivariance is understood with respect to a certain unitary representation of the isometry group of $\mathbb{R}^m$.

Probably, one can show that this ver_∞ realizes the *minimum* of the curvatures among all equivariant maps $\mathbb{R}^m \to S^\infty$.

[32] Hermitian Veronese maps from the complex projective spaces $\mathbb{C}P^m$ to the spaces $\mathcal{H}_n$ of Hermitian forms on $\mathbb{C}^{m+1}$ are among the prime suspects in this regard.

(b) Instead of ver_s, one could achieve (essentially) the same result with a use of compositions of the classical Veronese maps, $ver : S^{m_i} \to S^{m_{i+1}}$, ${}_{i+1} = \frac{(m_i+1)(m_i+2)}{2} - 2$,

$$S^{m_1} \hookrightarrow S^{m_2} \hookrightarrow ... \hookrightarrow S^{m_i},$$

starting with $m_1 = 2m - 1$ and going up to $i = m$. (Actually, $i \sim \log m$ will do.)

12.1 Petrunins Veronese Rigidity Theorem

Large Simplex Property.(Compare with section 5 in pet.) Let the curvature of a complete [33] connected n-submanifold in an n-ball of radius r be bounded by one,

$$curv^{\perp}(X \hookrightarrow B^N(r)) \leq 1,$$

and let $x_0, ..., x_m \in X$ be $m+1$ points (e.g. m=n), such that

$$dist_X(x_i, x_j) = \pi, 0 \leq i < j \leq m.$$

Then

$$r \geq \sqrt{\frac{2m}{m+1}}.$$

In fact, the Euclidean distances between x_i are ≥ 2 by $[2\sin]_{bow}$inequality, the minimal ball which contains these point cant be smaller than the ball circumscribed about regular m-simplex with the edge length 2 by the Kirszbraun theorem.

Petrunin's two Balls Covering and the Sphere Theorem. Let the f-mage of X be contained in the ball of radius $r < 2/\sqrt{3}$ and let $x_-, x_+ \in X$ be two points joint by a geodesic segment of length π. Then the two geodesic balls $B_{x_\pm}(\pi) \subset X$ cover X.

It follows that X is homeomorphic to the sphere and, except for $n = 1$, the map $f : X \hookrightarrow \mathbb{R}^N$ is an embedding.

Proof. The above for $m = 2$ shows that the boundaries of these balls don't intersect and since these boundaries are connected for $n \geq 2$ the balls do cover X.

Petrunin's Veronese Planes Rigidity Theorem. If the image $f(X)\mathbb{R}^N$ is contained the ball $B^N(2/\sqrt{3})$ and is not homeomorphic to the sphere then f is an embedding and all geodesic segment in $f(X)$ are planar (contained in planes).

Consequently, X is either (congruent to) a *Veronese plane or its complex, quaternionic or Cayley numbers counterpart.*

Proof . Track the two balls covering argument in the extremal case with the bow rigidity at you hand or consult [Pet].

Embedding Remark. Petrunin requires that f is embedding, but this seems ? unneeded for his argument.

[33]"Complete" refers to the induced Riemannian metric .

13 Hilbert's Rational Spherical Designs and Optimal Tori

Let

$$E:\mathbb{R}^N \to B^{2N}(1) \subset \mathbb{R}^{2N}$$

be the composition of the Clifford embedding $\mathbb{T}^n \subset B^{2N}$ and the exponential (locally isometric covering) map

$$\mathbb{R}^N = T_0(\mathbb{T}^N) \overset{\exp}{\to} \mathbb{T}^N.$$

A simple computation shows (see ???) that the Euclidean curvature of E on the line $\bar{x} \subset \mathbb{R}^N$ generated by a non-zero vector $x \subset \mathbb{R}^n$ is

$$(\star) \qquad curv^{\perp}(\bar{x} \overset{E}{\hookrightarrow} \mathbb{R}^N) = \left(\frac{\|x\|_{L_4}}{\|x\|_{L_2}}\right)^2,$$

where $x = (x_1, ..., x_N)$ for the standard Euclidean (corresponding to the cyclic torical) coordinates x_i and

$$\|x\|_{L_p} = \sqrt[p]{\frac{\sum_1^N |x_i|^p}{N}}.$$

Let $P(n,4)$ be the linear space of homogeneous polynomials of degree 4 on $\mathbb{R}^n$, this has dimension $\binom{n+4}{n} = \frac{n(n-1)(n-2)(n-3)}{24}$, and let

$$V_4:\mathbb{R}^n \to P(n,4),\ V_4:(c_1,...,c_n) \mapsto (c_1x_1 + ...c_nx_n)^4$$

be the *4th degree Veronese map.*

Then A $(n-1)$-spherical N-multi-set, that is map from a set Σ of cardinality N to the unit sphere $S = S^{n-1} \subset \mathbb{R}^n$ written as $\sigma \overset{\mathcal{D}}{\mapsto} s(\sigma)$, is called is a called *a design of degree* 4 *and cardinality* N *in* $S = S^{n-1}$ if

the center of mass of the N*-multi-set* V_4D *in the image* $V_4(S^{n-1}) \subset P_(n,4)$ *is equal to the center of mass of* $V_4(S^{n-1})$ *itself with respect to the usual spherical measure* or, equivalently, if

$$\frac{1}{N}\sum_{\sigma\in\Sigma} l^4(\mathcal{D}(\sigma)) = \int_S l^4(s)ds$$

for all linear functions l on $S = S^{n-1}$, where ds is the normalised (i.e, of the full mass one) spherical measure.

Yet another way to characterise the design property of a muti-set D on S^{n-1} of cardinality N is via the tautological map

$$\mathbb{R}^n = \mathbb{R}_D \hookrightarrow \mathbb{R}^N$$

from the Euclidean n-space of linear functions $l(s)$ on S^{n-1} to the space $\mathbb{R}^N$ of (all) functions on Σ.

In these term $\mathcal{D}$ is a *a design (of degree* 4 *and cardinality* N in $S = S^{n-1}$) if and only if – this follows by the standard Γ-formulas for the $\int_S l^p(s)ds$-integrals, the L_2 and the L_4 norms on the non-zero vectors $x \in \mathbb{R}^N$ which are contained in $\mathbb{R}_{\mathcal{D}}$ satisfy:

$$\frac{\|x\|_{L_4}}{\|x\|_{L_2}} = \sqrt[4]{\frac{3n}{n+2}}$$

Thus, in view of $\bigstar$,

every Design $\mathcal{D}$ of degree 4 and cardinality N on S^{n-1} defines a homomorphism (which is a locally isometric immersion), call it $E_{\mathcal{D}}$, from $\mathbb{R}^n = \mathbb{R}_{\mathcal{D}}$ to the Clifford N-torus, such that the curvature of $E_{\mathcal{D}}$ in the ball $B^{2N}(1) \supset \mathbb{T}^N$ satisfies:

$$curv^{\perp}\big(\mathbb{R}^n \overset{E_{\mathcal{D}}}{\hookrightarrow} B^{2N}(1)\big) = \sqrt{\frac{3n}{n+2}}.$$

A design D is *rational* if all points in D are rational.

Hilbert's Lemma.[34] *If $N >> n$, then S^{n-1} contains a rational design of cardinality N.*

Proof. Use three simple facts.

(i) the center of mass $\mathbf{c_o} \in P(n,4) = \mathbb{R}^{\binom{n+4}{n}}$ lies in the *interior* of the convex hull of the image $V_4(S^{n-1}) \subset P(n,4)$

(ii) $\mathbf{c_o}$ is a *rational* point in $P(n,4)$,

(iii) rational points in S^{n-1} are dense

and proceed in four steps;

(1) Because of (i) and (iii) there exit finitely many *rational* points $s_i \in S^{n-1}$, $i = 1, ..., M$, such that the convex hull of these ponts contains $\mathbf{c'_o}$.

(2) Because of rationality of $\mathbf{c_o}$, there exist *rational* numbers $p_i \geq 0$, $p_1 + ... + p_M = 1$, such that $p_1 V_4(s_1) + ... p_M V_4(s_M) 1 = \mathbf{c_o}$.

(3) Let Q be the common denominator of these numbers and write them as $\frac{P_i}{Q}$ for integer P_i, $i = 1, ..., M$, where $P_1 + ... + P_M = Q$.

(4) Let D be the multi-set in S^{n-1}, which consists of the points s_i, each taken with multiplicity P_i.

Then the center of mass of $V_4 D$ is

$$\frac{1}{Q}\sum_i P_1 V_4(s_i) = \sum_i p_i V_4(s_i) = \mathbf{c_o}.$$

QED.

A. $2n^2$**-Designs**. The number N delivered. by the above proof is very big, a rough estimate is $N \leq$ but non-rational designs are known to exist for much smaller N.

For instance *If n is a power of 2, then there exists a design of cardinality $N = 2n^2 + 4n$.* [35]

homomorphism, (which is a locally isometric immersion) from the Euclidean n-space to the Clifford N-torus in the ball B^{2N} for $N = 8(n^2 + n)$, such that the

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[34]In his solution of the Waring problem, Hilbert uses this lemma (for all even degrees) in the form of an identity $\sum_{i=1}^N l(x_j)^{2d} = (\sum_{j=1}^N (x_j^2))^d$ for some linear form l_i with rational coefficients.

[35]This was stated and proved in a written message by Bo'az Klartag to me. Also, Bo'az pointed out to me that the Kerdock code used in [K1995] yields designs for $N = 4^k$ and $N = \frac{n(n+2)}{2}$. See ??? for references

normal Euclidean curvature of this immersion is

$$(\star\star)\qquad curv^\perp\big(\mathbb{R}^n \hookrightarrow B^{16(n^2+n)}(1)\big) = \sqrt{\frac{3n}{n+2}}$$

Since rational points are dense in the sphere, we conclude to the extence of subtori $\mathbb{T}^n_\varepsilon \subset \mathbb{T}^{8(n^2+n)}$, such that

$$(\star\ \star\ \star).\quad curv^\perp\big(\mathbb{T}^n_\varepsilon \hookrightarrow B^{16(n^2+n)}(1)\big) \le \sqrt{\frac{3n}{n+2}} + \varepsilon \text{ for all } n \text{ and all } \varepsilon > 0.$$

if $N >> n$ as in Hilbert's Lemma, then there exist n-subtori $\mathbb{T}^n \subset B^{2N}$, fsuch that

$$curv^\perp\big(\mathbb{T}^n \hookrightarrow B^{2N}\big) = \sqrt{\frac{3n}{n+2}}.$$

Example/Non-Example. Regular pentagons serve as designs of cardinality five and degree four on the circle; these are irrational and there is no apparent *simple* rational design on S^1.

14 Link with the Scalar Curvature via the Gauss Formula

The $curv^\perp$ problem came up (see ???) in the context of Riemannian geometry of manifolds X with *positive scalar curvatures*, where

the scalar curvature of an X at $x \in X$, denoted $Sc(X,x)$, is the sum of the values of the sectional curvatures κ at the $n(n-1)$ (ordered) orthonormal bivectors in $T_x(X)$, for $n = dim(X)$.[36]

For instance, scalar curvatures of surfaces are equal to twice their sectional (Gauss) curvatures.

Spheres Example. The n-spheres of radii R in the Euclidean space $\mathbb{R}^{n+1}$ (which have constant sectional curvatures $1/R^2$), satisfy:

$$Sc(S^n(R)) = n(n-1)/R^2 \text{ for all } n.$$

Additivity. It follows from the definition that the scalar curvature is additive under Riemannian products,

$$Sc(X_1 \times \underline{X}) = Sc(X_1) + Sc(\underline{X}).$$

For instance, the scalar curvature of the n-th power of the unit 2-sphere is

$$Sc(\underbrace{S^2 \times S^2 \times \ldots \times S^2}_{n}) = 2n = Sc(S^{2n}(R = \sqrt{2n-1})$$

[36]One knows that $Sc(X,x) > 0$ if and only if the volume of the ball $B_x(\varepsilon) \subset X$ is smaller than the volume of the ε-ball in $\mathbb{R}^n$, *provided $\varepsilon > 0$ is sufficiently small*: $\varepsilon \le \varepsilon(X,x) > 0$. Albeit looking explanatory, this is only an illusion of understanding the geometric meaning of the inequality $Sc(X) > 0$.

This also shows that the topology of manifolds with positive scalar curvatures of dimensions $n \geq 4$, can be arbitrary complicated[37] for

$$Sc(X \times S^2(\varepsilon)) \underset{\varepsilon \to 0}{\to} +\infty \text{ for } \textit{all compact} \text{ Riemannian manifolds } X.$$

Yet, there are limits to this complexity: there are compact manifolds of all dimensions, which admit no metrics with $Sc > 0$, called $\nexists$PSC, where the three basic examples are as follows.

Basic $\nexists$PSC Manifolds

A. Lichnerowicz Theorem. The (Kummer) surface defined by the equation $z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$ in the complex projective space $\mathbb{C}P^3$ and, more generally orientable spin manifolds with non vanishing $\hat{A}$ genus (dimensions of these are multiples of 4) admit no Riemannian metrics with $Sc > 0$.

Proved in 1963 with the first (1963) *Atiyah–Singer index theorem for the Dirac operator.*

B. Hitchin theorem: there exist manifolds Σ homeomorphic (but non-diffeomorphic!) to the spheres S^n for all $n = 8k+1, 8k+2$, $k = 1, 2, 3...$, which admit no metrics with $Sc > 0$.

Proved in 1974 with the second (1971) *Atiyah–Singer index theorem.*

C. Geroch Conjecture. n-Tori admit no metrics with $Sc > 0$.

Proposed in 1975, *proved* in 1979 by Schoen-Yau for $n \leq 7$ with via *minimal hipersurfaces* by induction on n and by Gromov-Lawson in 1980 for all n with the *index theorem for the Dirac operators twisted with almost flat bundles.* **D. Product Manifolds.** Products of the above manifolds, e.g. of tori by Hitchins spheres are also $\nexists$PSC.

This is proven with the *index theorem* for the (generalized) Dirac operators.

Sectional Curvature Remarks. Although the inequality $Sc > 0$ is much weaker then $sect.curv > 0$ (which is equivalent to geodesic triangles having the sums of the angles $> \pi$) *no alternative proofs* of non-existence of metrics with $sect.curv > 0$ on manifolds from **A** and **B** are available, while the $sect.curv > 0$ (and $Ricci > 0$) version of **C** follows by an elementary argument relying on the geometry of geodesics in X.

(The ancient Bonnet-Myers theorem says that $Ricci(X) \geq \kappa > 0 \implies diam(X) \leq \sqrt{1/\kappa}$, which rules out closed manifolds with infinite universal coverings, such as tori.)

Turning to Constant Sectional Curvature. If one requires the strongest possible condition of this kind, namely the sectional curvature to be constant as well as positive, then everything about X appears 100% transparent.

Indeed, one knows. that these metric are *locally spherical*; hence all simply connected n-manifold X with $sect.curv(X) = \kappa > 0$ admit *locally isometric immersions to* $S^n(R)$ *for* $R = \sqrt{1/\kappa}$.

Consequently,

the universal coverings of closed (compact without boundaries) manifold X with $sect.curv(X) = \kappa$ are isometric to $S^n(R)$. This is the end of the story.

Yet, this may be hard to believe, there are *non-trivial links between geometry and topology* of manifolds X with *constant sectional curvatures* if these X have

[37]Three manifolds with $Sc > 0$ are not too simple either : *connected sums lens spaces* and copies of $S^1 \times S^2$ admit metrics with $Sc > 0$ by a theorem by Schoen and Yao.

non-empty boundaries, where the available proofs of these properties rely on the scalar curvature inequality $Sc(X) \geq n(n-1)/R^2$ and where one doesn't know how to exploit to full power of the condition $sect.curv = const = 1/R$. (see ???)

14.1 Gauss Formula and Petrunin's Curvature

Let $X \subset Y$ be a smooth n-dimensional submanifold in a Riemannian N-manifold, e.g. in $Y = \mathbb{R}^N$ and let II= II(X,x)II=II(τ_1, τ_2) be the *second fundamental form* (corresponding to the *shape operator*) of X at $x \in X$, where $\tau_1, \tau_2 \in T_x(X)$ are tangent vectors to X and the form II takes values in the normal space $T_x^{\perp}(X)$ and where II(τ, τ) is equal to the second derivative of the geodesic in X issuing from x with the velocity τ.

The normal curvature of $X \subset Y$ at $x \in X$, in these terms is

$$curv_x^{\perp} = \sup_{\|\tau\|=1} \|\mathrm{II}(\tau, \tau)\|.$$

The l_2-norm of II at x is

$$\|\mathrm{II}\|_{\mathrm{l}_2}^2 = \sum_{\mathrm{i}_1, \mathrm{i}_2 = 1, \ldots \mathrm{m}} \|\mathrm{II}(\tau_{\mathrm{i}_1}, \tau_{\mathrm{i}_2})\|^2,$$

where $\{\tau_i\}$, $i = 1, ..., n = dim(X)$, is a frame of orthonormal vectors in the tangent space $T_x(X)$.

We shall need the simple inequality

$$\|\mathrm{II}\|_{l_2}^2 \leq kn \cdot curv^{\perp}(X)^2,$$

which is useful for $k < n$. One can also show that $\|\mathrm{II}\|_{l_2}^2 \leq n^2 \cdot curv^{\perp}(X)^2$, for all k but the following inequality. will serve us better.

Petrunin curvature $\Pi = \Pi_{\mathsf{x}}(X \subset Y)$ is the average of

$$\|\mathsf{II}(\tau, \tau)\|^2$$

over the unit vectors $\tau \in S_x^{m-1} \subset T_x(X)$, where clearly,

$$\frac{(curv_x^{\perp})^2}{n-1} \leq \Pi_x \leq (curv_x^{\perp})^2$$

and where the equality $\frac{(curv_x^{\perp}))^2}{n-1} = \Pi_x$ holds if the form II has rank one and $\Pi_x = (curv_x^{\perp})^2$ if $||\mathrm{II}||_{l_2}^2 = ||mean.curv(X, x)||^2$.

For instance, if $codim(X) = 1$, the latter means that all principal curvatures X at x are mutually equal.

More interestingly [Pet2023])

$$\Pi = \tfrac{2}{n(n+2)}\left(||\mathrm{II}||_{l_2}^2 + \tfrac{1}{2}\|mean.curv^{\perp}\|^2\right)$$

or

$$\|mean, curv\|^2 - \|\mathrm{II}\|_{l_2}^2 = \frac{3}{2}mean, curv^2 - \frac{n(n+2)}{2}\Pi,$$

which is proven with the *same* Γ-function formula for the integrals of polynomials of degree four on S^{n-1}, which goes along with spherical designs and used for construction of immersions $\mathbb{T}^n \to \mathbb{R}^N$ with $curv^{\perp} = \sqrt{3n/(n+2)} + \varepsilon$.

(One wanders if there is a geometric reason for this, e.g. a "Riemannian curvature averaging formula" of some kind.)

For instance, if $n = dim(X) = 2$, $N = dim(Y) = 3$ and α_1 and α_2 dnote the principal curvatures of X at x, then

$$curv^\perp(X,x) = \max(|\alpha_1|, |\alpha_2|),$$

$$||\mathrm{II}||^2_{l_2} = \alpha_1^2 + \alpha_2^2,$$

$$\|mean.curv^\perp\| = |\alpha_1 + \alpha_2|$$

and

$$\sqcap = \frac{1}{4}(\alpha_1^2 + \alpha_2^2) + \frac{1}{8}(\alpha_1 + \alpha_2)^2 = \frac{3}{8}(\alpha_1^2 + \alpha_2^2) + \frac{1}{4}\alpha_1\alpha_2;$$

if $X = S^2 \subset Y = \mathbb{R}^3$, where $\alpha_1 = \alpha_2 = 1$, this makes $\sqcap = 1$ as well.

Gauss Formula. Let Y have constant sectional curvature κ and let $Sc_{|n} = Sc_{|n}(Y) = nk(k-1)$. Then the scalar curvature of X satisfies:

$$Sc(X,x) = Sc_{|n} + ||mean.curv^\perp(X,x)||^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2},$$

where by Petrunin's formula

$$Sc(X,x) = Sc_{|n} + \frac{3}{2}\|mean.curv(X,x)\|^2 - \frac{n(n+2)}{2}\cdot\sqcap,$$

Hence, the inequality $Sc_{|m}(Y) \geq \sigma_n$ implies that

$$Sc(X) \geq \sigma_n - ||\ \mathrm{II}(X,x)||^2.$$

Therefore

$$[kn] \qquad Sc(X) \geq \sigma_n - kn \cdot curv^\perp(X)^2$$

for $k \leq n$ and

$$Sc(X) \geq \sigma_n - n^2 curv^\perp(X)^2.$$

for all k, where Petrunin's formula yields better, in fact optimal, inequality for $k >> n$

$$Sc(X) \geq \sigma_n - \frac{n(n+2)}{2}\sqcap \geq \sigma_n - \frac{n(n+2)}{2}curv^\perp(X)^2.$$

It follows that *if the manifold X is $\nexists$PSC, i.e. it admits no metric with $Sc > 0$, then*

$$curv^\perp(X) \geq \sqrt{\sqcap} \geq \sqrt{\frac{2\sigma_n}{n(n+2)}} \text{ for all } k \text{ and } N = n+k = dim(Y),\ Y \hookleftarrow X,$$

and

$$curv^\perp(X) \geq \sqrt{\frac{\sigma_n}{kn}} \text{ for } k < n/2.$$

Examples and Corollaries.

Let X be an n-dimensional $\nexists$PSC manifold, e.g. the n-torus $\mathbb{T}^n$, Hitchin's exotic n-sphere Σ^n or a product $\Sigma^m \times \mathbb{T}^{n-m}$.

($*_{S^{n+k}}$) Then immersions from X to the unit sphere satisfy

[**A**] $$curv^\perp\big(X \hookrightarrow S^{n+k}(1)\big) \geq \sqrt{\frac{n-1}{k}}$$

and

[**B**] $$curv^\perp\big(X \hookrightarrow S^{n+k}(1)\big) \geq \sqrt{\Pi} \geq \sqrt{\frac{2n-2}{n+2}}.$$

Inequality **[A]** is better than **[B]** roughly for $k \leq n/2$, while Petrunin's **[B]** takes over for larger N, where it is, as we known (see sections ???) , optimal for $k >> n^2$.

14.2 Petrunin's $\sqrt{3}$ Extremality Theorem

The above doesn't directly apply to immersions to the Euclidean balls, since these have $Sc_{|n} = 0$, where the Gauss and Petrunin formulas for the induced metric g, reduce to

[**a**] $$Sc(g) = ||mean.curv^\perp||^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2}$$

and

[**b**] $$Sc(g) = \frac{3}{2}\|mean.curv\|^2 - \frac{n(n+2)}{2}\Pi.$$

Yet, inequality **[A],** applied to the image of $X \hookrightarrow B^N(1)$ in S^N under the radial projection of of the unit ball in tangent hyperplane $B^N \subset \mathbb{R}^N = T_s(S^N) \subset \mathbb{R}^{n+1} \supset S^N$ to S^N shows that

[$\frac{\mathbf{1}}{\mathbf{8}-\varepsilon}$] $$curv^\perp\big(X \hookrightarrow B^{n+k}(1)\big) \geq \sqrt{\frac{n-1}{(8-\varepsilon_{n,k})k}}.$$

for some (moderately small) $\varepsilon_{n,k} > 0$.

This is crude, but in the Π-case Petrunin proves the sharp $curv^\perp$-inequality

[**B$\star$**] $$curv^\perp\big(X \hookrightarrow B^N(1)\big) \geq \sqrt{\frac{3n}{n+2}}.$$

for all n-dimensional $\nexists$PSC manifolds X, all n and N.

This is done by showing that *if*

$$curv^\perp\big(X \overset{f}{\hookrightarrow} B^N(1)\big) < \sqrt{\frac{3n}{n+2}},$$

then a conformal change of the induced metric g on X has positive scalar curvature. Namely, if $n \geq 3$[38]*, then*

$$Sc(u^{\frac{4}{n-2}}g) > 0 \textit{ for } u(x) = \exp -l\tfrac{1}{2}\|f(x)\|^2 \textit{ and } l = \tfrac{3}{4}\cdot\tfrac{n-2}{n-1}\cdot n.$$

[38]If $n = 2$ then the average value of Π is $\geq \sqrt{\frac{3}{2}}$, see ???

Remark. One might think, that Petrunin's argument with the Gauss formula

$$Sc(g) = \|mean.curv\|^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2} \geq \|\mathrm{mean.curv}\|^2 - \mathrm{k}(\mathrm{curv}^{\perp})^2$$

rather than Petrunin's

$$Sc(g) = \frac{3}{2}\|mean.curv\|^2 - \frac{n(n+2)}{2}\sqcap \geq \frac{3}{2}\|mean.curv\|^2 - \frac{n(n+2)}{2}(curv^{\perp})^2$$

would improve the above inequality $[\frac{1}{8-\varepsilon}]$.

In fact, if one uses Petrunin's formula for the Laplace operator $\Delta = \Delta_g$ applied to the above function $u(x)$ on X:

$$-\frac{\Delta u}{u} = lrc \cdot |H| + (ln - l^2 r^2 s^2),$$

where $H = mean.curv\big(X \overset{f}{\hookrightarrow} B^{n+k}(1)\big)$, $r = r(x) = \|f(x)\|$, and $c = c(x)$, $s = s(x)$ are function (cos and sin of certain angles), which are bounded in the absolute values by one, $|c|, |s| \leq 1$, one arrives at the following version of $[\frac{1}{8-\varepsilon}]$. :

$$curv^{\perp}\big(X \hookrightarrow B^{n+k}(1)\big) \geq \sqrt{\frac{n}{k(8 + (4/(n-2)))}}$$

This is no better $[\frac{1}{8-\varepsilon}]$. but can be slightly improved with the inequalities $c^2 + s^2 \leq 1$ and $r^2 + s^2 \leq 1$ proved in [Pet] under the assumption $curv^{\perp} \leq 2$.

14.3 Lower Bounds on $curv^{\perp}(X \hookrightarrow Y)$ for Manifolds Y with $Sc_n \geq \sigma_n$.

Let us define the n-dimensional scalar curvature $Sc_n(Y)$ for general Riemannian manifolds Y of dimension $N \geq n$, that is a function on the tangent n-planes $T^n_y \subset T(Y)$ in Y, which is eual to the sum of the sectional curvatures κ of Y on the bivectors in T^n_y at y.

Equivalently, $Sc_n(Y, T_y)$ is the scalar curvature of the submanifold $\exp(T_y) \subset Y$ at y, that is is the germ of the image of the exponential map from T_y to Y.

Then the Gauss' and Petrunin's formulas for the scalar curvature of $X \hookrightarrow Y$ remains as they were for manifolds Y with constant sectionl curvatres

$$Sc(X,x) = Sc_{|m}(Y, T_x(X)) + ||mean.curv^{\perp}(X,x)||^2 - \|\mathrm{II}\|^2_{\mathrm{l}_2},$$

and

$$||mean.curv^{\perp}(X,x)||^2 - ||\ \mathrm{II}(X,x)||^2_{l_2} = ||\tfrac{3}{2} mean.curv^{\perp}(X,x)||^2 - \tfrac{n(n+2)}{2}\sqcap.$$

Thus, the above inequalities **[A]** and **[B]** concerning immersions of n-manifolds X to the unit sphere S^{n+k} generalize to immersions to $(n+k)$-dimensional manifolds Y, such that $Sc_n(Y) \geq n(n-1)$:

[A$_{\mathbf{Y}}$]
$$curv^{\perp}\big(X \hookrightarrow Y\big) \geq \sqrt{\frac{n-1}{k}}$$

and

$$[\mathbf{B_Y}] \qquad curv^\perp(X \hookrightarrow Y) \geq \sqrt{\Pi} \geq \sqrt{\frac{2n-2}{n+2}}.$$

Example. Let $Y = S^{n+k_0}(R)(1) \times H^l_{-1}$, where the sphere $S^{n+k_0}(R)$ has constant curvature $+1/\rho^2$ and H^l_{-1} is the hyperbolic space with the sectioanal curvature -1 and let $n \geq l+2$. Then

$$Sc_n(Y) \geq \frac{1}{\rho^2}(n-l)(n-l-1) - l(l-1)$$

and the two above inequalities hold with $k = k_0 + l$, if

$$\rho^2 \leq \frac{(n-l)(n-l-1)}{n(n-1) + l(l-1)}.$$

For instance, if $l = 2$, and $n \geq 4$ one needs $\rho^2 \leq \frac{1}{7}$. for this purpose.

Notice in conclusion, that neither
the above inequalities $[\frac{1}{8-\varepsilon}]$ and Petrunin's $[\mathbf{B}\star]$ for immersion to unit balls nor such inequalities from the previous sections based on the $\frac{2p}{n}$inequalities admit (not at lest obvious) counterparts for these Y.

15 Second Link with the scalar Curvature: Width Inequalities for Riemannian Bands

D. Example: Torical $\frac{2\pi}{n}$-Inequality. Let V be a Riemannian manifold homeomeorphic to the product of the n-torus by the unit interval $V = \mathbb{T}^n \times [-1,+1]$, such that $Sc(V) \geq \sigma > 0$. Then the distance between the two components of the boundary of V is bounded as follows:

$$dist(\mathbb{T}^n \times \{-1\}, \mathbb{T}^n \times \{+1\}) \leq 2\pi\sqrt{\frac{n}{\sigma(n+1)}}.$$

(See ??? below for a few words about the proof.)

E. Corollary: No Wide Torical Bands in the Spheres. *If a Riemannian $(n+1)$-manifold V homeomorphic to $\mathbb{T}^n \times [-1,+1]$ admits a locally isometric immersion to the $(n+1)$-sphere of radius R then*

$$dist(\mathbb{T}^n \times \{-1\}, \mathbb{T}^n \times \{+1\}) \leq \frac{2\pi R}{n+1}.$$

F. Large Normal Curvature Sub-corollary. *Let*

$$f : \mathbb{T}^n \hookrightarrow B^{n+1}(1)$$

be a smooth immersion from the n-torus to the unit Euclidean $(n+1)$-ball $B^{n+1} \subset \mathbb{R}^{n+1}$. Then the curvature of f is bounded from below by:

$$curv^\perp(\mathbb{T}^n \overset{f}{\hookrightarrow} B^{n+1}(1)) \geq \frac{n+1}{\pi} - 1.$$

Proof of **D** $\implies$ **E.** Let

$$E_f : \mathbb{T}^N \times \mathbb{R}^1 \to \mathbb{R}^{n+1} \supset B^{n+1}(1)$$

be the *normal exponential map*, i.e. such that the restriction $E_f|\mathbb{T}^N \times \{0\} = f$ and where E_f isometrically sends the lines $\{t\} \times \mathbb{R}^1$, $t \in \mathbb{T}^n$, to the straight lines in $\mathbb{R}^{n+1}$ normal to the immersed torus $f(\mathbb{T}^n) \subset \mathbb{R}^{n+1}$ at the points $f(t) \in f(\mathbb{T}^n)$.

If $curv(f) < c$, then, (this is the same as it is for circles of radii $1/c$ in the plane) the map E_f is an *immersion* on $\mathbb{T}^N \times [-r, r] \subset T^N \times \mathbb{R}^1$ for $r = 1/c$, while the image of $f(\mathbb{T}^n)$ is contained in the ball $B^{n+1}(1+r)$.

Let

$$\mathbb{R}^{n+2} \subset S^{n+1}_+(1+r) \overset{p}{\to} \mathbb{R}^{n+1} \supset B^{n+1}(1+r)$$

be the normal projection from the hemisphere, compose E_f on $\mathbb{T}^N \times [-r, r]$ with the inverse map to p and let

$$\tilde{E} : p^{-1} \circ E_f : \mathbb{T}^N \times [-r, r] \to S^{n+1}_+(1+r).$$

Since the projection p is *distance decreasing*, the spherical distance between the two components of the boundary of $T^N \times [-r, r]$ with respect to the Riemannian metric $\tilde{g}$ in $\mathbb{T}^N \times [-r, r]$ induced by $\tilde{E}$ from the spherical metric in $S^{n+1}_+(1+r)$ V is bounded from below by $2r$. Then **D** applied to

$$(\mathbb{T}^N \times [-r, r], \tilde{g}) \overset{\tilde{E}}{\to} S^{n+1}_+(1+r) \subset S^{n+1}(1+r)$$

shows that

$$\tilde{d} = dist_{\tilde{g}}(\mathbb{T}^n \times \{-r\}, \mathbb{T}^n \times \{+r\}) \le \frac{2\pi(1+r)}{n+1}$$

and since $\tilde{d} > 2r = 2/c$ the inequality $c \ge \frac{n+1}{\pi} - 1$ follows. QED.

Exercise. Generalise the large normal curvature sub-corollary to immersions of tori to products of balls:

$$curv^\perp\big(\mathbb{T}^{n+k} \overset{f}{\hookrightarrow} B^{n+1}(1) \times B^k(R)\big) \ge \frac{n+1}{\pi} - 1.$$

for all $k = 0, 1, 2, ...$ and all $R \ge 0$.

On Low Dimensions. The inequality $curv^\perp\big((\mathbb{T}^n \hookrightarrow B^{n+1}(1)\big) \ge \frac{n+1}{\pi} - 1$ may be asymptotically optimal for $n \to \infty$ but its performance for small n is poor.

For instance, if $n \le 5$ then $\frac{n+1}{\pi} - 1 < 1$ and our inequality is weaker than $curv^\perp(X^n \hookrightarrow B^{n+k}(1) \ge 1$, which follows for all closed n-manifolds X and all n, k by the obvious "maximal principle" argument.

Furthermore, since

$$curv^\perp\big((X^n \hookrightarrow B^{n+1}(1)\big) > 2$$

for all *non-spherical* X (this is elementary, see section ...), our $\big(\ge \frac{n+1}{\pi} - 1\big)$-bound is of any interest only for $n \ge 9$.

$\mathbb{T}^\rtimes$-*Remark.* In section ???, we introduce the notion of $\mathbb{T}^\rtimes$-*stabilized scalar curvature,* $Sc^\rtimes(X)$, improve the inequalities **E** and **F** and will see, for example, that

$$curv^\perp\big(\mathbb{T}^n \hookrightarrow B^{n+1}(1)\big) > 2.5 \text{ for } n \ge 7.$$

Codimension two Remark. The inequality **E** applied to the unit tangent bundles of immersed n-tori with codimensions 2,[39] shows (see $[1+2c]$-Example

[39]If the Euler class of such an immersion is non-zero one needs a mild generalisation of **E**.

in ???)

$$curv^\perp\big(\mathbb{T}^{n+1}\hookrightarrow B^{n+2}(1)\big)\le 1+2curv^\perp\big(\mathbb{T}^n\hookrightarrow B^{n+2}\big)$$

and

$$curv^\perp\big(\mathbb{T}^n\hookrightarrow B^{n+2}\big)\ge\frac{1}{2}curv^\perp\big(\mathbb{T}^{n+1}\hookrightarrow B^{n+2}(1)\big)-\frac{1}{2}\ge\frac{n+2}{2\pi}-1.$$

This has any merit only for $n\ge 11$, where $\frac{n+2}{2\pi}-1>1$, and it becomes better than Petrunin's inequality only for $n\ge 15$, where $\frac{n+2}{2\pi}-1>\sqrt{\frac{3n}{n+2}}$.

(The improvement with the $\mathbb{T}^{\rtimes}$-remark doesn't significantly change the picture.)

G. Conjecture. Immersed n-tori in the unit $(n+k)$-ball satisfy

$$curv^\perp\big(\mathbb{T}^n\hookrightarrow B^{n+k}(1)\big)\ge\frac{n}{k}.$$

This, by no means (not even conjecturally) optimal, inequality is motivated only by its simple form.

(????)**Immersions with curvatures $\sim n^\alpha$.** It not impossible (but unlikely) that all immersion of n-tori to unit balls satisfy

$$curv^\perp\big(\mathbb{T}^n\hookrightarrow B^{n+k}(1)\big)\ge\frac{cn^\alpha}{k}$$

for some small $c>0$, $\alpha>1$, e.g. $c=0.001$ and $\alpha=\frac{3}{2}$, where the exponent $\alpha=\frac{3}{2}$ is maximal possible.

Indeed, n-tori embed to $B^{n+n}(1)$ with curvatures $n^{\frac{1}{2}}$ and also there exit codimension one embedding of n-tori with curvatures about $n^{\frac{3}{2}}$,

$$curv^\perp\big(\mathbb{T}^n\subset B^{n+1}(1)\big)<6n^{\frac{3}{2}}.$$

In fact, arguing as in $bullet_1$, ... in ??? one construct $X_m=S^{n_1}\times\ldots\times S^{n_m}\subset B^{n_1+\ldots n_m+1}(1)$ by induction on m as boundaries of ρ_m-neighbourhoods of

$$X_{m-1}=S^{n_1}\times\ldots\times S^{n_{m-1}}\subset B^{n_1+\ldots n_{m-1}+1}(1-\rho_m)\subset B^{n_1+\ldots n_m+1}(1),$$

where the curvatures of these embeddings grow *exponentially* with m, roughly as 2^{m-1}.

Thus one embeds X_m to the ball $B^{n_1+\ldots n_m+1}(1)$ with the curvature growing *polynomially* in $n=dim(X_m)$ (rather than in m):

$$curv^\perp\big(X_m\subset B^{n+1}(1)\big)\le const_\mu n^{\frac{\mu+2}{\mu+1}},\ n=dim(X_m)=n_1+\ldots n_m,\ \mu=\min_i n_i.$$

For all we know, if all n_i are equal to a single n_o, then all immersions of $(S^{n_o})^m$ immersions to the unit (mn_o+1)-ball satisfy

$$curv^\perp\big((S^{n_o})^m\hookrightarrow B^{mn_o+1}(1)\big)\le const_{n_o}(mn_o)^{\frac{\mu+2}{\mu+1}}.$$

15.1 On Three Proofs of $\frac{2\pi}{n}$-Inequalities

All three proofs apply to manifolds V, where their boundaries are decomposed into two disjoint parts $\partial V = \partial_- \sqcup \partial_+$, and show that

$$dist(\partial_-, \partial_+) > 2\pi\sqrt{\frac{n}{\sigma(n+1)}} \text{ for } \sigma = \inf_{x\in X} Sc(X,x).$$

under certain topological assumptions on V specific to each proof.

1. The first proof applies to suitably *enlargeable* manifolds (see ???) V, e.g. to $V = X \times [-1,1]$, where X admits a metric with $sect.curv \le 0$.

This proceeds by induction on n with minimal hypersurfaces with boundaries as in §12 from GL, where the original Schoen-Yau argument was augmented with Fischer-Colbrie&Schoen warped product symmetrization idea.

If $dim(V) > 7$, the proof encounters a technical difficulty where minimal hypersurfaces may have singularities, but this was resolved by a partial regularity theorem of Schoen and Yau ???[40]

2. The second proof whenever applies, delivers a hypersurface (μ-*bubble*) $X \subset V$ which separates ∂_-. from ∂_+ and which admits a metric with positive scalar curvature. This shows, in particular that in the following three cases,

V can't be diffeomorphic to $X \times [-1,1]$, where X admits no metric with $Sc > 0$,

(i) X is a *spin manifold* , e.g. as in the above **A** and **B**.

(ii) X is as in the original Schoen-Yau paper ??? or a manifold as in [GH]

(iii) X is an aspherical manifold of dimension| ≤ 5 or a closely related manifold (see ???,???)

(These (i), (ii) and (iii) cover all *known* classes of manifolds, except for dimension 4, which admit no metrics with $Sc > 0$.)

This second proof also encounter the singularity problem for $dim(V) > 7$, where it is more serious the]an in the first proof, since the Schoen-Yau partial regularity theorem is not sufficient in this case.

However if $dim(V) = 8$ then a requred desingularisation follows by a version of Nathan Smale argument (see ????) and if $n = 9, 10$, then the desingularisation from ????? most probably apply in the present case.

3. The third proof, relies on the generalized Callias-Dirac operators technique (see ?????), needs V to be a spin manifold.

This proof applies, in particular, to V diffeomorphic to $X \times [-1,1]$, where X admits no metric with $Sc > 0$, ad where non-existence of such a metric follows via the index theorem for a generalized Dirac operator, as for instance, for X from the above **A** and **B**. 1mm

As far as the curvature of immersion is concerned, this is most useful for the Hitchin's spheres Σ^n for $n = 8l+1, 8l+2$ and which admit immersions to $\mathbb{R}^{n+1}$ by Hirsch theorem[41] and all immersions Σ^n to the unit $(n+1)$ ball satisfy the same inequality as tori

$$curv^\perp\big(\Sigma^n \hookrightarrow B^{n+1}(1)\big) \ge \frac{n+1}{\pi} - 1$$

[40]The proof is difficult...

[41]Lichnerowicz's manifolds, which have non-zero $\hat{A}$-genus admit no Euclidean immersions with codimenension one and two.

and, by a similar argument,

$$curv^{\perp}\big(\Sigma^n \hookrightarrow B^{n+2}(1)\big) \geq \frac{n+2}{\pi} - 2.$$

These inequalities can be improved for small n the same way as in the above (b) for tori, but unlike conjecture **G** for tori, there is *no reason to expect* that immersions of Σ^n to the unit balls B^{n+k}, $k \geq 3$, satisfy $curv^{\perp} \geq const_k n$. (We say more about it in section???)

Question. Do *all* Milnor's spheres Σ^n, including those, which carry metrics with $Sc > 0$, develop large normal curvatures when immerssed to the balls $B^{n+1}(1)$?

On Generalized Geroch's conjecture?????

15.2 $\mathbb{T}^{\rtimes}$-Stabilized Scalar Curvature.

Given a compact Riemannian manifold X, let

$$Sc^{\rtimes}(X) = 4\lambda_1^{\rtimes}(X),$$

where $\lambda_1^{\rtimes}(X)$ is the lowest eigenvalue of the operator $-\Delta + \frac{1}{4}Sc$ on X with the Dirichlet (vanishing on the boundary) condition.[42]

It is easy to see that $Sc^{\rtimes}$ is additive for Riemannian products

$$Sc^{\rtimes}(X_1 \times \underline{X}) = Sc^{\rtimes}(X) + Sc^{\rtimes}(\underline{X}).$$

and, more relevantly,

$Sc^{\rtimes}(X)$ is *decreasing* under equidimensional locally isometric immersions:

if X immerses to Y then $Sc^{\rtimes}(X) \geq Sc^{\rtimes}(X)$.

About $-\Delta + \beta \cdot Sc$**.** The two above relations remain valid for the first eigenvalues of the operators

$$f(x) \mapsto -\Delta f(x) + \beta \cdot Sc(X,x) \cdot f(x)$$

for all $\beta \geq 0$, but $\beta = 1/4$ is essential for the $\frac{2\pi}{\sqrt{Sc^{\rtimes}}}$-inequality below.

Besides 1/4, a significant value is $\beta = \frac{1}{4}\frac{n-2}{n-1}$, where positivity of the operator $-\Delta_X + \beta \cdot \frac{1}{4}\frac{n-2}{n-1}Sc(X)$ for $n \geq 3$ on X implies that X admits a metric with positive scalar curvature (as in the proof of the Petrunin's inequality in section ???).

Since $\frac{1}{4}\frac{n-2}{n-1} < \frac{1}{4}$ the inequality $Sc^{\rtimes} > 0$ also implies the existence of a metric with positive scalar curvature on X.

This shows that the conditions *$\nexists PSC$ and $\nexists PSC^{\rtimes}$ are equivalent.*

But unlike how it is with the effects of the positive signs of $Sc(X)$ and of $Sc^{\rtimes}(X)$ on the *topology* of X, the $Sc(X)$ and $S^{\rtimes}(X)$ plays different roles in the geometry of X.

Let V be a Riemannian manifold homeomorphic to the product $X \times [-1, +1]$, where X is a *basic $\nexists PSC$ n-manifold*, i.e. where the underlying reason for non-existence of a metric with $Sc > 0$ on X is *of the same kind as what is presented*

[42] See ??? for justification of this definition/notation and for the proofs of the properties of this $Sc^{\rtimes}$-curvature used in this paper.

in section ???.[43] For instance X isdiffeomorphic to the product of the *torus by Hitchin's sphere.*

$\frac{2\pi}{\sqrt{Sc^{\rtimes}}}$**-Inequality**. Let V be a Riemannian manifold homeomorphic to the product $X \times [-1,+1]$, where X is a *basic* $\nexists$*PSC n-manifold*, i.e. where the underlying reason for non-existence of a metric with $Sc > 0$ on X is *of the same kind as what is presented in section ???.*[44] For instance X isdiffeomorphic to the product of the *torus by Hitchin's sphere.*

Then the distance between the two boundary components of V is *bounded as follows:*

$$dist(X\times\{-1\}, X\times\{+1\}) \le 2\pi\sqrt{\frac{n}{Sc^{\rtimes}(V)(n+1)}}.$$

Examples of Evaluation of $Sc^{\rtimes}$. The rectangular solids satisfy

$$Sc^{\rtimes}\left(\bigtimes_1^n[-a_i,b_i]\right) = 4\sum_1^n\lambda_1[a_i,b_i] = \sum_1^n\frac{4\pi^2}{(b_i-a_i)^2},$$

the unit hemispheres satisfy:

$$Sc^{\rtimes}(S^n_+) = n(n-1)+4n = n(n+3),$$

the unit balls satisfy

$$Sc^{\rtimes}(B^n) = 4j^2_\nu,$$

for the first zero of the Bessel function J_ν, $\nu = \frac{n}{2}-1$, where $j_{-1/2} = \frac{\pi}{2}$, $j_0 = 2.4042...$, $j_{1/2} = \pi$ and if $\nu > 1/2$, then

$$\nu + \frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}} < j_\nu < \nu + \frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}} + \frac{3}{20}\frac{2^{\frac{2}{3}}a^2}{\nu^{\frac{1}{2}}}$$

where $a = \left(\frac{9\pi}{8}\right)^{\frac{2}{3}}(1+\varepsilon) \approx 2.32$ with $\varepsilon < 0.13\left(\frac{8}{2.847\pi}\right)^2 < 0.1$.

Corollary. Let X be a basic $\nexists$PSC$^{\rtimes}$ manifold of dimension $n-1$, e.g. $X = \mathbb{T}^{n-1}$, and $f : X \to B^n(r)$ be a smooth immersion. Then the focal radius and thetisfy normal curvature of f sa then the focal radii of immersions $X \hookrightarrow B^n(r)$ satisfy:

$$[foc.rad]_{j_\nu} \qquad rad^{\perp}\big(X \hookrightarrow B^n(r)\big) \le \frac{\pi r}{2j_\nu}\sqrt{\frac{n}{n+1}}$$

and

$$[curv^{\perp}]_{j_\nu} \qquad curv^{\perp}\big(X \hookrightarrow B^n(r)\big) \ge \left(\frac{2j_\nu}{\pi r}\sqrt{\frac{n+1}{n}}\right) - r$$

where

$$\frac{2j_\nu}{\pi r} \ge \frac{n-1/2+3.68(n/2-1)^{1/3}}{\pi r}$$

This implies, in particular, the *low curvature bounds from the* $\mathbb{T}^{\rtimes}$*-remark in section ???.*

[43] Conjecturally, all $\nexists$PSC manifolds will do, at least for $n \neq 4$

[44] Conjecturally, all $\nexists$PSC manifolds will do, at least for $n \neq 4$

Also this can be used along with the following.

Mean Curvature/Ricci $4j_\nu^2$–Inequality. Let Y be a compact connected Riemannian n-manifold with a non-empty boundary, such that *the Ricci curvature of* Y *is nonnegative*, e.g. Y is a bounded Euclidean domain, and *the mean curvature of the boundary of* W *is bounded from below* by that of the unit ball,

$$mean.curv(\partial Y) \geq n-1 = mean.curv(\partial B^n).$$

Then

$$Sc^{\rtimes}(Y) \geq Sc^{\rtimes}(B^n) = 4j_\nu^2.$$

Thus, the above inequalities

$[foc.rad]_{j_\nu}$ and $[curv \perp]_{j_\nu}$ remain valid for immersions $X \hookrightarrow Y_r$ for *all compact connected Riemannian n-manifolds* Y_r *with non-empty boundaries, such that* $Ricci(Y_r) \geq 0$ *and* $mean.curv(\partial Y_t) \geq \frac{n-1}{r}$.

Remark/Question. Let $V \subset \mathbb{R}^n$ be a bounded domain with two boundary components, let $d(V)$ be the distance between these componets and let $\lambda_1(V)$ the first eigenvalue of the Dirchlet problem in V.

The above shows that

topology of V *may impose a non-trivial bound on the product* $d^2(V)\lambda_1(V)$.

What are other cases of a similar role of the topology of a $V \subset \mathbb{R}^n$ on metric invariants of V?

15.3 Curvatures of Regular Homotopies of Immersions

Due to the Atiyah-Singer index theorem for families of Dirac operators, the index theoretic obstructions to $Sc > 0$ apply to families of metrics with $Sc > 0$, which imply the following (see Hit)

3.5.A. The spheres S^{n-1}, $n = 8k+1, 8k+2$, $k = 1, 2, ...$ admit (Smale/Milnor)

$$\text{diffeomorphisms } \mu : S^{n-1} \to S^{n-1},$$

such that the usual spherical metric g_o ($sect.curv(g_o) = 1$) and the induced metric $g_o^* = \mu^*(g_o)$ (also $sect.curv(g_o^*) = 1$) *can't be joined by a* C^2*-continuous homotopy* g_t*, such that* $Sc(g_t) > 0$.

(The diffeomorphism μ establishes an *isometry* of (S^{n-1}, g_o^*) with the usual sphere (S^{n-1}, g_o), where Milnor's theorem doesn't allow a homotopy g_t between g_o and g_o^*, such that the metrics g_t have *constant sectional* curvatures.)

3.5.B. $O(\sqrt{n})$***-Curvature Corollary.*** Let $f_o : S^{n-1} \to S^n(1)$, be the standard equatorial embedding of the sphere and let $f_t : S^{n-1} \to S^n(1)$, $t \in [0,1]$, be a C^2-continuous regular homotopy, (a family of C^2-immersions[45]) between f_o and $f_o^* = f_o \circ \mu : S^{n-1} \to S^n(1)$. Then *there exists* $t_0 \in [0,1]$, such that the normal curvature of the immersion f_{t_0} satisfies :

$$curv^\perp\left(S^{n-1} \overset{f_{t_0}}{\hookrightarrow} S^n(1)\right) \geq \sqrt{n-2}.$$

Indeed, if $curv^\perp\left(S^{n-1} \overset{f_{t_0}}{\hookrightarrow} S^n(1)\right) < \sqrt{n-2}$. for all t then, by 3.3.???, the f_t-induced metrics g_t on S^{n-1} would have $Sc > 0$ in contradiction with 3.5.A.

[45]Such a family does exist by the Smale immersion theorem.

3.5.C. $O(n)$**-Curvature Conjectural Corollary.** Let $f_o : S^{n-1} \to B^n(1) \subset \mathbb{R}^n$ be the standard embedding of the sphere and let $f_t : S^{n-1} \to B^n(1)$, $t \in [0,1]$, be a C^2-continuous regular homotopy, (a family of C^2-immersions[46]) between f_o and $f_o^* = f_o \circ \mu : S^{n-1} \to B^n(1)$. Then *there exists* $t_0 \in [0,1]$, such that the normal curvature of the immersion f_{t_0} satisfies :

$$curv^{\perp}\big(S^{n-1} \overset{f_{t_0}}{\hookrightarrow} B^n(1)\big) \geq j_\nu/\pi > \frac{n+1}{\pi} - 1.$$

To show this one needs an index theorem for families of Callias operators on Riemannian bands.

3.5.D. Higher Homotopy Remark. There is a body of results on higher homotopy groups of the space $\mathcal{G}_{Sc>0}(S^n)$ of metrics g with $Sc(g) > 0$ on S^n, but it is unclear what to do with (the homotopy structure of) the map from the space of immersions $S^n \to B^{n+k}(1)$ (and/or $S^n \to S^{n+k}(1)$) with sufficiently small curvatures to $\mathcal{G}_{Sc>0}(S^n)$.

Not only Hitchin's spheres but all ∄PSC manifolds X of dimension $n \geq 5$ contain hypersurfaces $H \subset X$, which support pairs of Riemannian metrics g_0 and g_1, such that $Sc(g_i) > 0$, $i = 0,1$, and where these metrics *can't be joined by a* C^2*-continuous homotopies* g_t*, such that* $Sc(g_t) > 0$*,* $0 \leq t \leq 1$.

To see that, let $\psi : X \to \mathbb{R}$ be a Morse function and let $Z = \psi^{-1}(r_0) \subset X$, for some $r_0 \in \mathbb{R}$ be a level of ψ, such that all critical point $x \in X$ of ψ with indices $\leq m$ lie below Z, i.e. $\psi(x)(x) < r_0$.

Then Z serves as the common boundary of the regions $X_0 \subset X$ and $X_1 \subset X$, where

$$X_0 = \{x \in X\}_{\psi(x) \leq r_0} \text{ and } X_1 = \{x \in X\}_{psi(x) \geq r_0}.$$

Since X_0 represents a regular neighbourhood of a (ψ-cellular) m-skeleton of X the manifold X_0 carries a natural Riemannian metric g_0 with $Sc(g_0) > 0$, provided $n - m \geq 3$ and since X_1 represents a regular neighbourhood of a $n-m-1$-skeleton of X there is another "natural"metric g_1 on Z with $Sc(g_1) > 0$ for $m \leq 2$. (see ????)

Also on knows (see ???) that if g_0 and g_1 lie in the same connected component of $\mathcal{G}_{Sc>0}(Z)$, then X admits a metric with $Sc > 0$.

Similarly, if g_0 and g_1 lie in the same connected component of $\mathcal{G}_{Sc^{\rtimes}>0}(Z)$, then X admits a metric with $Sc^{\rtimes} > 0$.

3.5..??? Higher Homotopy Problem. Is there a developement of this construction in the spirit of *3.5.D. Higher Homotopy Remark*, e.g. something about the fundamental group of the space $\mathcal{G}_{Sc^{\rtimes}>0}(Z')$ for some hypersurface $Z' \subset Z$?

3.5.C. Toral Example/Question. Let $X = \mathbb{T}^n$ and $2 \leq m \leq n-3$. Then, one can show that Z admits an immersion $f_0 : Z \to B^n(1)$ with

$$curv^{\perp}\big(Z \overset{f_0}{\hookrightarrow} S^n(1)\big) \leq c_m.$$

It follows that if $n >> m$, then the induced metric g_{f_0}. on Z has $Sc > 0$; moreover, one can find an f_0 such that g_{f_0} is homotopic to g_0 in $\mathcal{G}_{Sc>0}(Z)$.

When does Z also admits a similar immersion f_1 to S^n with a sufficiently small curvature and a homotopy between g_{f_1} and g_1?

When do manifolds like Z admit pairs of regularly homotopic immersion $f_0, f_1 : Z \hookrightarrow B^n(1)$ with curvatures $\leq c$, yet not regularly homotopic by immersions with curvatures $\leq C$ for some costants c and $C >> c$?

[46]Such a family does exist by the Smale immersion theorem.

16 Overtwisted Immesrdions.

17 1.A. Unknowledge Conjectures/Problems

[47]: $curv^{\perp} > 3$. Since (disjoint unions of) products of spheres are the

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Immersions of manifolds with boundaries: 2 versions

with control of the curvature of the boundary and/or wide base around boundary or with bounary of X in (and normal to ?) the boundary of $Y \supset X$

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extension of immersions with small curvature from the boundary ∂X to X

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rho-regularisition of wide bands

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immersions and thickening of polyhedra (skeletons of triangulations)

=============

immersion with small curvature problem for (the boundary of) the complement of the 2-skeleton of the torus

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versions of the the Sc-obstructions to small curvature to mean.curv obstruction to ???

immersions of $\mathbb{T}^n s S^k \to B^{n_k+1}$

conjecture: $curv^{\perp}(T^n \to B^{n^{1+\alpha}} \sim dn^{\beta}$???? Heat flow and the mean curvature flow on immersions with small curvatures

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2. Let X be a C^1-smooth closed n-dimensional immersed hypersurface in $\mathbb{R}^{n+1}$ positioned between two parallel hyperplanes with distance 2 between them,

$$X \hookrightarrow R^n \times [-1,1] \subset \mathbb{R}^{n+1}$$

in the unit ball wit

Let $X \hookrightarrow \mathbb{R}^N$ be a compact (complete suffices) immersed n-submanifold, let $x_0 \in X$, and let $U_0 \subset X$ of x be the maximal connected neighbourhood such that the normal projection from U_0 the tangent space $T_0 = T_{x_0}(X) \subset \mathbb{R}^N$,

$$P_0 : U_0 \to T_0$$

is a one-to-one diffeomorphism onto a domain $V_0 \subset T_0 = \mathbb{R}^n$, which is star convex with respect to $x_0 \in T_0$.

Clearly such a U_0 exists and unique, where, this an essential example, if $X = S^n \subset \mathbb{R}^{n+1}$, then such a U_0 is the hemisphere around x_0.

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hypersurfaces (and their stability) with $curv^{\perp} \leq 1$ in cylinders $B^k \times \mathbb{R}^{n-k}$

18 References

N. Nadirashvili, Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces. Invent. Math. 126 (1996), 457-465. MR 98d:53014

[47]Probably, majority of conjectures in mathematics are based on the absence of evidence to the contrary.

E. R. Rozendorn, The construction of a bounded, complete ‘ surface of non-positive curvature, Uspekhi Mat. Nauk, 1961, Volume 16, Issue 2, 149–156

https://arxiv.org › pdf › 0804ArxivCauchy's Arm Lemma on a Growing Sphere - arXiv.org We propose a variant of Cauchy's Lemma, proving that when a convex chain on one sphere is redrawn (with the same lengths and angles) on a larger sphere, the distance between its endpoints increases.

Curves of Finite Total CurvaturearXiv:math/0606007v2 [math.GT] 24 Oct 2007 John M. Sullivan

An Extension of Cauchy's Arm Lemma 2000 with Application to Curve Development Joseph O'Rourke´ Dept. Comput. Sci., Smith College Northampton, MA 01063, USA orourke@cs.smith.edu Abstract. Cauchy's "Arm Lemma" may be generalized to permit nonconvex "openings" of a planar convex chain. Although this (and further extensions) were known, no proofs have appeared in the literature. Here two induction proofs are offered. The extension can then be employed to establish that a curve that is the intersection of a plane with a convex polyhedron "develops" without self-intersecti

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