Formulas, Connections, Differential Operators, Clifford Algebras, Spinors, Scalar Curvature

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Contents

1 Preliminaries

1.1 Analytic Techniques

The logic of most (all?) arguments concerning the global geometry of manifolds X with *scalar curvatures bounded from below* is, in general terms, as follows.

Firstly, one uses (or proves) the *existence theorems for solutions* Φ *of certain partial differential equations*, where the existence of these Φ and their properties depend on global, topological and/or geometric assumptions $\mathcal{A}$ on X, which are, a priori, unrelated to the scalar curvature.

Secondly, one concocts some *algebraic-differential expressions* $\mathcal{E}(\Phi, Sc(X))$, where the crucial role is played by certain *algebraic formulae* and issuing inequalities satisfied by $\mathcal{E}(\Phi, Sc(X))$ under assumptions $\mathcal{A}$.

Then one arrives at *a contradiction*, by showing that

if $Sc(X) \geq \sigma$*, then the implied properties, e.g. the sign, of* $\mathcal{E}(\Phi, Sc(X))$ *are* *opposite to those satisfied under assumption(s) $\mathcal{A}$.*

1.1.1 Spin Manifolds, Dirac Operators $\mathcal{D}$, Atiyah-Singer Index Theorem and S-L-W-(B) Formula

[I] Historically the first Φ in this story were *harmonic spinors* on a Riemannian manifold $X = (X, g)$, that are solutions s of $\mathcal{D}(s) = 0$, where $\mathcal{D} = \mathcal{D}_g$ is the *(Atiyah-Singer)-Dirac* on X.[1]

[I_{yes}]. The existence of *non-zero harmonic* spinors s on certain smooth manifolds X follows from *non-vanishing of the index of* $\mathcal{D}$, where this index, which is *independent of* g, identifies, by the the *Atiyah-Singer theorem of 1963* , with a certain (smooth) topological invariant, denoted $\hat{\alpha}(X)$ (see section 3.1).

Then the relevant formula involving $Sc(X)$ is the following algebraic identity between the squared *Dirac operator* and the (coarse) *Bochner-Laplace operator* $\nabla^*\nabla$ also denoted ∇^2,

[I_{no}]. ***Schroedinger-Lichnerowicz- Weitzenboeck-(Bochner) Formula***[2]

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc,$$

shows that if $Sc > 0$, then $\mathcal{D}^2 s = 0$ implies that $s = 0$, since

$$0 = \int \langle \mathcal{D}^2 s, s\rangle = \int \langle \nabla^2 s, s\rangle + \frac{Sc}{4}||s||^2 = \int ||\nabla s||^2 + \frac{Sc}{4}||s||^2,$$

where the latter identity follows by integration by parts (Green's formula).

By confronting these *yes* and *no*, André Lichnerowicz[3] showed in 1963 that

$$Sc(g) > 0 \Rightarrow \hat{\alpha}(X) = 0.$$

and proved the following.

Non-Existence Theorem Number One: **Topological Obstruction to $Sc > 0$ for $n = 4k$.** *There exists smooth closed $4k$-dimensional manifolds X, for all $k = 1, 2, \ldots,$ which admit no metrics with $Sc > 0$.*

A decade later, empowered by a general Atiyah-Singer index theorem, Nigel Hitchin extended Lichnerowitz' result to manifolds of dimensions $n = 8k + 1$ and $8k + 2$ and showed, in particular, that

the class of manifolds X with $\hat{\alpha}(X) \neq 0$, that support non-zero g-harmonic spinors all metrics g on X by the Atiyah-Singer theorem, hence no g with $Sc(g) > 0$ by **S-L-W-B** formula, includes certain *homotopy spheres.* [4] [5]

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[1] All you have to know at this stage about $\mathcal{D}$ is that $\mathcal{D}$ is a certain first order differential on sections of some bundle over X associated with the tangent bundle $T(X)$. Basics on $\mathcal{D}$ are presented in [Min-Oo(K-Area) 2002] and, comprehensively, in [Lawson&Michelsohn(spin geometry) 1989]. Also see sections 3.2.3,**??**.

[2] All natural selfadjoint geometric second order operators differ from the Bochner Laplacians by *zero order* terms, i.e. (curvature related) endomorphisms of the corresponding vector bundles, but it is remarkable that this in the case of $\mathcal{D}^2$ reduces to *multiplication by a scalar function*, which happens to be equal to $\frac{1}{4}Sc_X(x)$. From a certain perspective, the existence of such an with a wonderful combination of properties is the most amazing aspect of the Atiyah-Singer index theory.

[3] See [Lichnerowitz(spineurs harmoniques) 1963]

[4] See [AS(index) 1971], [Hitchin(spinors)1974].

[5] Prior to 1963, one didn't even know if therere were *simply connected* manifold that would admit *no metric with positive sectional curvature* was known. But Lichnerowicz' theorem,

1.1.2 Inductive Descent with Minimal Hypersurfaces and Conformal Metrics

[II] Another class of solutions Φ of geometric PDE, that are essential for understanding scalar curvature and that are quite different from harmonic spinors, are *solutions to the Plateau problem.*

More specifically, these are *smooth* stable minimal hypersurfaces $Y \subset X$ that represent non-zero integer homology classes from $H_{n-1}(X)$, $n = dim(X)$.

The existence of minimal Y, possibly singular ones, was established by Herbert Federer and Wendell Fleming in 1960, while the smoothness of these Y, that is crucial for our applications, was proven by Federer in 1970 who relied on *regularity* of volume minimizing cones of dimensions ≤ 6 proved by Jim Simons in 1968.

The relevance of these minimal Y of codimension 1 to the scalar curvature problems was discovered by Schoen and Yau who proved in 1979 that

$\bigstar^{codim1}_{\text{mini}}$ if $Sc(X) > 0$ and $Y \subset X$ is a smooth stable minimal hypersurface, then *Y admits a Riemannian metric h with $Sc(h) > 0$.*[6]

In fact, if $dim(Y) = n - 1 = 2$, the stability of Y, that is *positivity* of the second variation of the area of Y, implies that (see sections 2.5, 2.4.1)

$$\int_Y (Sc(Y,y) - Sc(X,y))dy \geq 0$$

where the scalar curvature $Sc(Y)$ refers to the metric h_0 in Y induced from the Riemannian metric g of X.

Therefore, positivity of $Sc(X)$ implies positivity of the Euler characteristic of Y, for

$$4\pi\chi(Y) = \int_Y Sc(Y,y)dy \geq \int_Y Sc(X,y)dy > 0.$$

If $m = n - 1 \geq 3$, then h is obtained by a *conformal modification* of the metric h_0 on Y,

$$h_0 \mapsto h = (f^2)^{\frac{2}{m-2}} h_0,$$

where, as in the 1975 "conformal paper" by Jerry Kazdan and Frank Warner $f = f(y)$ is the first eigenfunction of the *conformal Laplacian* L on $Y = (Y, h_0)$, that is

$$L_{conf}(f) = -\Delta(f) + \frac{m-2}{4(m-1)} f,$$

where derivation of positivity of the L from positivity of the second variation of $vol_{n-1}(Y)$ relies on the *Gauss formula* suitably rewritten for this purpose by Schoen and Yau and where the issuing positivity of $Sc(f^{\frac{4}{m-2}} h_0)$ follows, as

saying, in fact, that

if X is spin, then $Sc(X) > 0 \Rightarrow \hat{A}[X] = 0$

delivered lots of *simply connected manifolds* X that admitted *no metrics with positive scalar curvatures, (see section 3.1)*.

Most of these X have *large Betti numbers*, that, as we know nowadays, is *incompatible* with $sect.curv(X) \geq 0$, but one still doesn't know if there are homotopy spheres not covered by Hitchin's theorem which admit no metrics with positive sectional curvatures.

[6]See [SY(structure) 1979]: *On the structure of manifolds with positive scalar curvature.*

in [Kazdan-Warner(conformal)],[7] by a simple (for those who knows how to do this kind of things) computation. [8]

Consecutively applied implication $Sc(X,g) > 0 \Rightarrow Sc(Y,h) > 0$ delivers a descending chain of closed oriented submanifolds

$$X \supset Y = Y_1 \supset Y_2 \supset \dots \supset Y_i \dots \supset Y_{n-2}$$

of dimensions $n-i$ which support Riemannian metrics h_i with $Sc(h_i) > 0$; thus, all connected components of Y_{n-2} must be a spherical.

Thus, Schoen and Yau inductively define a topological class of manifolds ($\mathcal{C}$ in their terms) and prove, in particular, the following.

Non-Existence Theorem Number Two Accompanied by Rigidity Theorem. *Let a compact oriented manifold X of dimension n dominate (a non-zero multiple of the fundamental class of) the n-torus, i.e, X admits a map of non-zero degree to the n-torus $\mathbb{T}^n$,*

$$f : X \to \mathbb{T}^n.$$

If $n \leq 7$,[9] X admits no metric with $Sc > 0$. then X support no metric g with $Sc(g) > 0$.

Moreover, the inequality $Sc(g) \geq 0$ for a metric g on X, implies that g is Riemannian flat and the universal covering of (X, g) is isometric to the Euclidean space $\mathbb{R}^n$.

(The submanifolds Y_i in this case are taken in the homology classes of transversal f-pullbacks of subtori in $\mathbb{T}^n \supset \mathbb{T}^{n-1} \supset \dots \supset \mathbb{T}^{n-i} \supset \dots \supset \mathbb{T}^2$.)

Remark. The authors of [SY(structure) 1979] say in their paper that it was motivated by problems in general relativity communicated to one of the authors by Stephen Hawking, [10] but I as haven't studied this field I can't judge how much of the current development in geometry of the scalar curvature is rooted in ideas originated in physics.

1.1.3 Twisted Dirac Operators, Large Manifolds and Dirac with Potentials

The index theorem also applies to Dirac operators $\mathcal{D}_{\otimes L}$ that act on spinors with values in Hermitian vector bundles $L \to X$, called *L-twisted spinors*, where *non-vanishing* of the index of $\mathcal{D}_{\otimes L}$ and, thus the existence of non-zero L-twisted harmonic spinors, is *ensured* for bundles L with *sufficiently large top dimensional*

[7] There is more to this paper, than the implication $L_{conf} > 0 \rightsquigarrow \exists g$ with $Sc(g) > 0$ on X. For instance, Kazdan and Warner prove

the existence of metrics g on connected manifolds X, $dim(X) \geq 3$, with prescribed scalar curvatures $Sc(g,x) = \sigma(x)$, for smooth functions $\sigma(x)$, which are negative somewhere on X and

the existence of metrics with $Sc = 0$ on manifolds X, which admits metrics with $Sc \geq 0$.

[8] This computation, probably, going back at least hundred years, was brought from the field of infinitesimal geometry to the context of non-linear PDE and *global analysis* by Hidehiko Yamabe in his 1960-paper *On a deformation of Riemannian structures on compact manifolds.*

[9] The dimension restriction was removed in [Lohkamp(smoothing) 2018] and in [SY(singularities) 2017].

[10] It is shown in [Hawking (black holes) 1972], by an argument elaborating on ideas from [Penrose(gravitational collapse) 1965] and resembling those in [SY(structure) 1979], that surface of the event horizon has *spherical topology.* (See [Bengtsson(trapped surfaces) 2011] for more about it.)

Chern numbers, essentially regardless of the topology of the underlying manifold X itself.

On the other hand, the *twisted* S-L-W-(B) formula, which now reads

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \frac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$

shows that such spinors don't exist if the *g-norm* of the curvature of L is *small* compare with the scalar curvature of $X = (X, g)$. Since this norm is inverse proportional to the size of g, large Riemannian manifolds admit *topologically complicated* bundles L with *small* curvatures, which, by the above, shows, as it was observed in [GL(spin) 1980], that, similarly how it is with the sectional and Ricci curvatures,

scalar curvatures of large manifolds must be small.

This delivers confirmation of the main $[Sc \not> 0]$ conjecture from the previous section for certain compact manifolds X, with large fundamental groups, e.g. for X, which support metrics with *non-positive sectional curvatures*:

Spin-non-Domination theorem of $\kappa \leq 0$ by $Sc > 0$. Non-torsion homology classes of complete manifolds $\underline{X}$, with non-positive sectional curvatures can't be dominated by compact (and also by complete) orientable spin manifolds with $Sc > 0$. [11]

In standard terms,

If a compact orientable spin Riemannian manifold X has $Sc > 0$ and $\underline{X}$ is complete with sect.curv$(\underline{X}) \leq 0$, and if

$$f : X \to \underline{X}$$

is a continuous map, then the image of the fundamental class $[X] \in H_n(X)$ is torsion*: some* non-zero *multiple $i \cdot f_*[X] \in H_n(\underline{X})$* vanishes.[12]

For instance,

if $\underline{X}$ is compact of dimension $n = dim(X)$, then all continuous maps $f : X \to \underline{X}$ have *zero* degrees.

Homotopy Invariance of Obstructions to $Sc > 0$ that Issues from $\otimes$ in $\mathcal{D}$. *Non-vanishing of topological invariants* delivered by the twist in $\mathcal{D}_{\otimes L}$ that prevent the existence of metrics with $Sc > 0$ are *stable under toplogical domination* that is, recall, a map $X \to \underline{X}$ of degree ± 1 between orientable manifolds, such that

if such an invariant doesn't vanish for $\underline{X}$, then it doesn't vanish for X either.

(An instance of such an invariant is the $\smile$-product homomorphism $\bigwedge^n H^1(X) \to H^n(X)$, $n = dim(X)$ behind the Schoen-Yau $[Sc > 0]$-non-existence theorem in section 1.1.2 for manifolds mapped to the n-tori)

This is similar to what happens to invariants issuing by the geometric measure theory but very much unlike to those coming from the untwisted index theorem, namely to non-vanishing of $\hat{\alpha}(X)$: the connected sum of two copies of an X with opposite orientations satisfies: $\hat{\alpha}(X \# (-X)) = 0$.

In fact, if X is simply connected of dimension $n \geq 5$, then $\hat{\alpha}(X \# (-X))$ *does admit a metric with $Sc > 0$.* [13]

[11] See [GL(spin) 1980] and sections 3.1 and **??** for more specific statements and proofs.

[12] It's unclear if $f_*[X] \in H_n(\underline{X}$ can be non-zero, yet (odd?) torsion.

[13] I am uncertain about $n = 4$.

Dirac with Potentials. The contribution of the connection of L to the Dirac operator can be seen as a vector potential added to $\mathcal{D}$ twisted with a the trivial bundle of rank = $rank(L)$.

Besides, this there are other kinds of – zero order terms – that can significantly influence geometric effects of $\mathcal{D}$.

As far as the scalar curvature is concerned, the first (to the best of my knowledge) potential of this kind (*Cartan connection*) was introduced by Min-Oo in his proof of the positive mass theorem for hyperbolic spaces, [Min-Oo(hyperbolic) 1989], and, recently, applications of *Callias-type* potentials in the work by Checcini, Zeidler and Zhang have significantly extended the range of the Dirac-theoretic applications to the scalar curvature problems.[14] – –

1.1.4 Stable μ-Bubbles

In general, μ-bubbles $Y \subset X$, are solutions of the "non-homogeneous Plateau equation"

$$mean.curv(Y, y) = \mu(y)$$

for a given function $\mu(x)$ on X.

What we deal with in this paper are *stable μ-bubbles* that are *local minima* of the functional

$$Y \mapsto vol_{n-1}(Y) - \mu(Y_<)$$

where μ is a Borel measure on X and $Y_< \subset X$ is a region in X with boundary $\partial Y_< = Y$ (see section **??**).

Often our measure is "continuous", i.e. representable as $\mu(x)dx$, for a continuous function $\mu(x)$ on X,and all basic existence and regularity properties of minimal hypersurfaces automatically extend to μ-bubbles in this case.

And what is especially useful for our purposes, is that the Schoen-Yau form of the the second variation formula neatly extends to μ-bubbles with continuous (and some discontinuous) $\mu \neq 0$.

Example/non-Example. The unit sphere $S^{n-1} \subset \mathbb{R}^n$ (with the mean curvature $n-1$) around the origin is a *stable μ-bubble* for the measure $\mu(x) = (n-1)||x||^{-1}dx$ in $\mathbb{R}^n$ and the same sphere also is the μ-bubble for $\mu(x) = (n-1)dx$; but this μ-bubble is an *unstable* one.

A significant gain achieved with μ-bubbles compared with the "plain" minimal hypersurfaces is due to the *flexibility in the choice of μ*, which can be adapted to the geometry of X, similarly to how one uses *twisted* Dirac operators $\mathcal{D}_{\otimes L}$ on X with " adaptable" unitary bundles $L \to X$.

For example, one obtains this way the following version of Schoen-Yau theorem ★ from section 1.1.2.

$☆^{codim1}_{bbl}$ Let X be a complete Riemannian n-manifold with *uniformly positive* scalar curvature, i.e, $Sc(X) \geq \sigma > 0$. If $n \leq 7$, then

[14] Exposition of Dirac operators with potentials, especially of their recent applications to manifolds with boundaries, are, regretfully, missing from our lectures. The reader has to turn to the original papers by Checcini, Zeidler, Zhang and [Guo-Xie-Yu(quantitative K-theory) 2020]. Also we say very little about the mass/energy theorems for hyperbolic spaces extending that in [Min-Oo(hyperbolic) 1989]; we refer for this subject matter to [Chrusciel-Herzlich [asymptotically hyperbolic) 2003], [Chrusciel-Delay(hyperbolic positive energy) 2019], [Huang-Jang-Martin(hyperbolic mass rigidity) 2019] and [Jang-Miao(hyperbolic mass) 2021] where one can find further references.

X can be exhausted by compact domains with smooth boundaries,

$$V_1 \subset V_2 \subset \ldots \subset V_i \subset \ldots X, \ \bigcup_i V_i = X,$$

where the boundaries ∂V_i, for all $i = 1, 2, \ldots$, admit metrics with positive scalar curvatures.

(Here, as in section 1.1.2, this needs additional analytical work to be extended to $n \geq 7$.)

1.1.5 Warped FCS-Symmetrization of Stable Minimal Hypersurfaces and μ-Bubbles.

Positivity of the conformal Laplacian $-\Delta + \frac{m-2}{4(m-1)}Sc$ doesn't fully reflect the positivity of the second variation of the volume $vol_{n-1}(Y)$, where the former actually yields positivity of the $-\Delta + \frac{1}{2}Sc$, which is, a priori, smaller then $-\Delta + \frac{m-2}{4(m-1)}Sc$, since $-\Delta \geq 0$ and $\frac{1}{2} > \frac{m-2}{4(m-1)}$.

Remarkably, positivity of the $-\Delta + \frac{1}{2}Sc$ on $Y = (Y, h_0)$ *neatly implies* positivity of the scalar curvature of the (warped product) metric $h^{\rtimes} = h_0(y) + \phi^2(y)dt^2$ for the first eigenfunction ϕ of $-\Delta + \frac{1}{2}Sc$, where this metric is defined on the products of Y with the real line $\mathbb{R}$ and with the unit circle $S^1(1) = \mathbb{T} = \mathbb{R})/\mathbb{Z}$, and where the resulting Riemannian manifolds are denoted

$$\bar{Y}^{\rtimes} = Y \rtimes \mathbb{R} = (Y \times \mathbb{R}, h^{\rtimes}) \text{ and } Y^{\rtimes} = Y \rtimes \mathbb{T} = \bar{Y}^{\rtimes}/\mathbb{Z}.$$

In fact, if $(-\Delta + \frac{1}{2}Sc)(\phi) = \lambda\phi$ with $\lambda \geq 0$, then

$$Sc(h^{\rtimes}(y,t)) = Sc(h_0, y) - \frac{2}{\phi}\Delta\phi(y) = \frac{2}{\phi}\left(-\Delta + \frac{1}{2}Sc(h_0,)\right)(\phi) = \lambda > 0m$$

see sections **??**.

The operation

$$Y \rightsquigarrow Y^{\rtimes}$$

is applied in the present case to stable minimal hypersurfaces $Y \subset X$, where the resulting passage $X \rightsquigarrow Y^{\rtimes}$ can be regarded as *symmetrisation* of X (or rather of infinitesimal neighbourhood of $Y \subset X$), because

the metric $h^{\rtimes}$ is invariant under the natural action of $\mathbb{T}$ on $Y^{\rtimes}$ and

$$Y^{\rtimes}/\mathbb{R} = Y \subset X$$

.

This $h^{\rtimes} = h_0(y) + \phi^2(y)dt^2$ defined with the first eigenfunction ϕ of the $-\Delta + \frac{1}{2}Sc$ on Y was introduced by Doris **F**ischer-**C**olbrie and Rick **S**choen[15] who used it for

classification of complete stable minimal surfaces in 3-manifolds X with $Sc(X) \geq 0$, including $X = \mathbb{R}^3$.

Then $h^{\rtimes}$ was used in [GL(complete) 1983], where, with an incorporation of Schoen-Yau's inductive descent, this allowed higher dimensional applications of the following kind.

[15] *The structure of complete stable minimal surfaces Y in 3-manifolds of non-negative scalar curvature.*

Given a Riemannian metric g on a product manifold $X = X_0 \times \mathbb{T}^k$, a consecutive symmetrization

$$X = X_0 \rightsquigarrow X_1 = Y_1^{\rtimes}/\mathbb{Z} \rightsquigarrow X_2 = Y_2^{\rtimes}/\mathbb{Z} \rightsquigarrow ...$$

delivers a $\mathbb{T}^k$-*invariant* metric $\bar{g}$ on $\bar{X}_k = Y_{-k} \times \mathbb{T}^k$, where $Y_{-k} \subset X$ is a submanifold of codimension k which is *homologous to* $X_0 = X_0 \times t_0 \subset X$ and such that the ($\mathbb{T}^k$-invariant) scalar curvature $Sc(\bar{g})$ on $\bar{X}_k$ is *bounded from below* by $Sc(g)$ on $Y_{-k} = \bar{X}_k/\mathbb{T}^k \subset X$.

Thus, for instance, one obtains a somewhat different proof of the Schoen-Yau theorem for $n \leq 7$:

no metric g on $X = \mathbb{T}^n$ can have $Sc(g) > 0$, because all $\mathbb{T}^n$-*invariant* metrics on $\mathbb{T}^n$ are Riemannian flat.

Non-Compact Case. An apparent bonus of this argument is its applicability to *non-compact complete manifolds.*

Example: Non-domination of $\mathbb{T}^n$ by $Sc > 0$. The n-torus admits no domination by *complete* manifolds X with $Sc(X) > 0$.[16]

For instance, if a closed subset in the torus $Y \subset \mathbb{T}^n$ is contained in a topological ball $B \subset \mathbb{T}^n$, then

the complement $T^n \setminus Y$ admits no complete metric with $Sc > 0$.

The main role of the above $\mathbb{T}^k$-symmetrization, however, is *not for the proof of topological non-existence theorems* of metrics with $Sc > 0$ on closed or non-compact complete manifolds, but for the *geometric study of such metrics* on, possibly *non-compact and non-complete,* manifolds X.

In fact, this symmetrization applies to stable minimal hypersurfaces $Y \subset X$ *with prescribed as well as free boundaries*, say with $\partial Y \subset \partial X$ and also to *stable μ-bubbles.* [17]

1.1.6 Averaged Curvature of Levels of Harmonic Maps

Recently, Daniel Stern [Stern(harmonic) 2019] found a version of the 3d Schoen-Yau argument for the levels of *non-constant harmonic maps $f : X \to \mathbb{T}^1$*, where, instead of the second variation formula for $area(Y)$, one uses

the Bochner identity, which expresses the Laplace of the norm of the gradient of f in terms of the Hessian of f and the Ricci curvature,

$$\frac{1}{2}\Delta|\nabla f|^2) = |Hess(f)|^2 + Ricci_X(\nabla f, \nabla f).$$

[16]Here, as at other similar occasions, singularities of minimal hypersurfaces and of μ-bubbles create complications for $n = dim(X) \geq 8$.

In the present case, if X is spin, this non-domination property follows by a Dirac operator argument from section 6 in [GL(complete) 1983].

If $n = 8$ the perturbation argument from [Smale(generic regularity) 2003] takes care of things.

If $n = 9$ one can still apply Dirac operators to non-spin manifolds, exploiting the fact that singularities of hypersurfaces are at most 1-dimensional, while the obstruction to spin (the second Stiefel-Whitney class) is 2-dimensional, see section 5.3 in [G(billiards) 2014].

If $n \geq 8$ the recent desingularization results presented in [Lohkamp(smoothing) 2018] and in [SY(singularities) 2017] apply to all X.

[17]See section 12 in [GL(complete) 1983], [G(inequalities) 2018] and sections 3.6, **??**).

Thus, Stern proved that the average Euler characteristics of these levels $Y_t = f^{-1}(t), t \in \mathbb{T}^1$ satisfies:

Harmonic Map Inequality.

$$4\pi \int_{\mathbb{T}^1} \chi(Y_t)dt \geq \int_{\mathbb{T}^1} dt \int_{Y_t} (|df(y,t)|^{-2}|Hessf(y,t)|^2 + Sc(X,(y,t)))dy.$$

This shows that

$$4\pi \int_{\mathbb{T}^1} \chi(Y_t)dt \geq \int_{\mathbb{T}^1} dt \int_{Y_t} Sc(X,(y,t))dy.$$

and implies, among other things, that

if the universal covering of a compact 3-manifolds with positive scalar curvatures is connected at infinity, then the one-dimensional cohomology $H^1(X;\mathbb{Z})$ vanishes.[18]

Indeed, if $H^1(X;\mathbb{Z}) \neq 0$, then X admits a non-constant harmonic map to the circle $\mathbb{T}^1$, where non-singular levels $Y_t \subset \mathbb{X}$ *can't contain spherical components*, because lifts of such a component to the universal covering of X would bound balls on which (the lift of) f would be constant by the maximum principle for harmonic functions. [19]

Vague Questions. Is there an algebraic link between S-L-W-(B) and the above Bochner formula that would connected Dirac operators with harmonic maps?

Do *Dirac harmonic* and/or similar maps bear a relevance to the scalar curvature problem?

1.1.7 Seiberg-Witten Equation

The third kind of Φ are solutions to the 4-dimensional *Seiberg-Witten equation* of 1994, that is the Dirac equation coupled with a certain non-linear equation and where the relevant formula is essentially the same as in [I].

Using these, Claude LeBrun[20] established a non-trivial (as well as sharp)

Fundamental 4D lower bound *on $\int_X Sc(X,x)^2dx$ for Riemannian manifolds X diffeomorphic to algebraic surfaces of general type.*

1.1.8 Hamilton-Ricci Flow

Hamilton1

The *Hamilton Ricci flow* $\Phi = g(t)$ of Riemannian metrics on a manifold X, that is defined by a *parabolic* system of equations, also delivers a geometric information on the scalar curvature, where the main algebraic identity for $Sc(t) = Sc(g(t))$ reads

$$\frac{dSc(t)}{dt} = \Delta_{g(t)}Sc(t) + 2Ricci(t)^2 \geq \Delta_{g(t)}Sc(t) + \frac{2}{3}Sc(t)^2,$$

[18] It is known that compact 3-dimensional manifolds with $Sc > 0$ are connected sums of space forms and $S^2 \times S^1$, see [GL(complete) 1983] and [Genoux(3d classification) 2013].

[19] In this respect, the surfaces Y_t are radically different from minimal surfaces and μ-bubbles which tend to localize around narrow necks in X, e.g. in "thin" connected sums $\mathbb{T}^3\#S^3$ described in section **??**.

[20] [LeBrun(Yamabe) 1999]: *Kodaira Dimension and the Yamabe Problem.*

which implies by the maximum principle that the minimum of the scalar curvature grows with time as follows:

$$Sc_{\min}(t) \geq \frac{Sc_{\min}(0)}{1 - \frac{2tSc_{min}(0)}{3}}.$$

If $X = (X, \underline{g})$ is a closed 3-manifold of constant sectional curvature -1, then, using the Ricci flow, Grisha Pereleman proved

Sharp 3D Hyperbolic Lower Volume Bound. All Riemannian metrics g on X with $Sc(g) \geq -6 = Sc(\underline{g})$ satisfy

$$Vol(X, g) \geq Vol(X, \underline{g}).$$

(See Proposition 93.9 in [Kleiner-Lott(on Perelman's) 2008].)

And, more recently, Richard Balmer, Paula Burkhardt-Guim and Man-Chun Lee, Aaron Naber and Robin Neumayer applied the Ricci flow for regularization of of (limits of) metrics with $Sc \geq \sigma$.[21]

(The logic of the Ricci flow, at least on the surface of things, is quite different from how it goes in the above three cases that rely on *elliptic* equations:

the quantities Φ in the former result from geometric or topological *complexities* of underlying manifolds X, that is necessary for the very existence of these Φ, while the Ricci flow, as a road roller, leaves a uniform terrain behind itself as it crawls along erasing complexity.)

Question. Do 3D-results obtained with the Ricci flow generalize to n-manifolds which have $Sc \geq \sigma$ and which come with *free isometric actions of the tori* $\mathbb{T}^{n-3}$?

For instance, let X^3 be a 3-dimensional Riemannin manifold which admits a hyperbolic metric $\underline{g}$ with sectional curvature -1 and let $X = X^3 \rtimes \mathbb{T}^1$ be a warped product (with $\mathbb{T}^1$-invariant metric), such that $Sc(X) \geq -6$.

Is the volume of $X^3 = X/\mathbb{T}^1$ is bounded from below by that of $(X^3, \underline{g})$?

(It is not even clear if the inequality $Sc(X^3 \rtimes \mathbb{T}^1) \geq -6$ imposes *any lower bound* on the Riemannin metric g of X^3. Namely,

Can such a $g = g_\varepsilon$ satisfy $g \leq \varepsilon \underline{g}$ for a given $\varepsilon > 0$?[22])

1.1.9 Modifications of Riemannian Metrics by a Single Function

Riemannian metrics g on an n-manifold X are given locally by $\frac{n(n-1}{2}$ functions $g_{ij}(x)$, where the scalar curvature $Sc(g)$ is a (messy) non-linear function of these g_{ij} and their first and second derivatives.

There are several constructions of Riemannian metrics on X and of modifications of a given metric g_0 on X by means of a *single* function $\phi(x)$, where the the scalar curvature of the resulting metric $g(\phi) = g(\phi, g_0)$ is expressed by a "nice" non-linear second order differential applied to ϕ.

The simplest and most studied case of this is the conformal transformation $g \mapsto \varphi^2 g$, where for $n \geq 3$ the scalar curvature of this metric is given by the (Yamabe?) equation

$$Sc(\varphi^2 g_0) = -\frac{4(n-1)}{n-2}\varphi^{\frac{n+2}{2}}\Delta\varphi^{\frac{n-2}{2}} + \varphi^2 Sc(g_0),$$

[21] See [Bamler(Ricci flow proof) 2016], [Burkhart-Guim(regularizing Ricci flow) 2019], [Lee-Naber-Neumayer](convergence) 2019] and section **??**.

[22] An elementary proof of such a bound on g is suggested in [G(foliated) 1991].

where $\Delta = \Delta_{g_0}$ is the Laplace on functions $\phi = \phi(x)$ on the Riemannian manifold (X, g_0).

We present some properties of this equation, due to Jerry Kazdan and Frank Warner, in section 2.6, which are used in the proof of Schoen-Yau's *non-existence* theorem for metrics with $Sc > 0$ on tori in sections 1.1.2, 2.7.

Also we briefly discuss in 2.6 similar transformations of metrics, where the scaling takes place only in some preferred directions, e.g. in a single direction, where the scalar curvature satisfies a non-linear parabolic (Bartnik-Shi-Tamm) equation, special *solutions* of which used for the proofs of *non-extension* theorems for metrics with $Sc > 0$, see section **??**.

Finally, recall Kähler metrics defined with single functions via the $\partial\bar{\partial}$, where, as we mention in section **??**, Yau's solution of the Calabi conjecture delivers "interestingly thick" metrics with $Sc > 0$ on complex algebraic manifolds.

2 Curvature Formulas for Manifolds and Submanifolds.

We enlist in this section several classical formulas of Riemannian geometry and indicate their (more or less) immediate applications.

2.1 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) Riemannian Variation Formula. Let h_t, $t \in [0, \varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0, \varepsilon]$.

Notice that an arbitrary Riemannian metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0, \varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = 2A^*,$$

where

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1, ..., n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y,y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

SIGN CONVENTION. The first derivative of h changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ of a hypersurface $Y \subset X$ depends on the *coorientation* of Y in X, where our convention is such that

the boundaries of *convex* domains have *positive (semi)definite* second fundamental forms A^*, also denoted II_Y, hence, *positive* mean curvatures, with respect to *the outward* normal vector fields.[23]

(2.1.B) First Variation Formula. This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t{}^{24}$$

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for C^1-smooth functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t{}^{25}$$

Now comes the first formula with the Riemannian curvature in it.

2.2 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannian manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y,\tau) = \kappa(X,\tau) + \wedge^2 A^*(\tau),$$

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

[23]At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.

[24]This come with the *minus* sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal(2019].

[25]This remains true for Lipschitz functions but if ψ is (badly) non-differentiable, e.g. it is equal to the characteristic function of a domain $U \subset Y$, then the derivative $\partial_{\psi\nu} vol_{n-1}(Y_t)$ may become (much) larger than this integral.

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y,y) = Sc(X,y) + \sum_{i\neq j} \alpha_i^*(y)\alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

• $\alpha_i^*(y)$, $i = 1, ..., n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;

• α^*-sum is taken over all ordered pairs (i,j) with $j \neq i$;

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν, τ_i) for ν being a unit (defined up to ±-sign) normal vector to Y;

• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu,\nu).$$

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i\neq j} \alpha_i\alpha_j = \left(\sum_i \alpha_i\right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

(Sc ≥ −2Ric) $$Sc(Y) \geq -2Ricci(\nu,\nu).$$

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n\max_i(c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape* that is the symmetric on $T(Y)$ associated with A^* via the Riemannian scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau,\tau) = \langle A(\tau), \tau\rangle_g \text{ for all } \tau \in T(Y).$$

2.3 Variation of the Curvature of Equidistant Hypersurfaces and Weyl's Tube Formula

(2.3.A) Second Main Formula of Riemannian Geometry.[26] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape operators $A_t = A(Y_t)$ satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where B_t is the symmetric associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $\left(\tau, \nu = \frac{d}{dt}\right)$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry. [27]

Tracing this formula yields

(2.3.B) Hermann Weyl's Tube Formula.

$$trace\left(\frac{dA_t}{dt}\right) = -||A^*||^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -||A^*||^2 - Ricci(\nu, \nu),$$

where

$$||A^*||^2 = ||A||^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where $Ricci$ is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu, \nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i \neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

[26] The first main formula is *Gauss' Theorema Egregium.*

[27] Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darmos(Gravitation einsteinienne) 1927] with the reference to Darboux' textbook of 1897.

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $\|A^*\|^2 = \sum_i (\alpha_i^*)^2$.

2.4 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called *umbilic* if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $Y = (Y, h)$ be a smooth Riemannian (n-1)-manifold and $\varphi = \varphi(t) > 0$, $t \in [0, \varepsilon]$ be a smooth positive function. Let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0, \varepsilon]$.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'(t)}{\varphi(t)}$, $i = 1, ..., n-1$ for

$$A_t^* = \frac{\varphi'(t)}{\varphi(t)} h_t \text{ for } \varphi' = \frac{d\varphi(t)}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}.$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2 h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

(c) What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[28]

Cylindrical Extension Exercise. Let Y be a smooth manifold, $X = Y \times \mathbb{R}_+$, let g_0 be a Riemannian metric in a neighbourhood of the boundary $Y = Y \times \{0\} = \partial X$, let h denote the Riemannian metric in Y induced from g_0 and let Y has *constant mean curvature* in X with respect to g_0.

Let X' be a (convex if you wish) ball in the standard (i.e complete simply connected) space with constant sectional curvature and of the same dimension n as X, let $Y' = \partial X'$ be its boundary sphere, let, let $Sc(h) > 0$ and let the mean and the scalar curvatures of Y and Y' are related by the following (comparison) inequality.

[<] $$\frac{|mean.curv_{g_0}(Y)|^2}{Sc(h,y)} < \frac{|mean.curv(Y')|^2}{Sc(Y')} \text{ for all } y \in Y.$$

Show that

if Y is compact, there exists a smooth positive function $\varphi(t)$, $0 \le t < \infty$, which is constant at infinity and such that the the warped product metric $g = \varphi^2 h + dt^2$ has

the same Bartnik data *as g_0, i.e.*

$$g|Y = h_0 \text{ and } mean.curv_g(Y) = mean.curv_{g_0}(Y),$$

Then show that

one *can't make $Sc(g) \ge Sc(X')$* in general, if [<] is relaxed to the corresponding *non-strict* inequality, where an example is provided by the Bartnik data of $Y' \in X'$ itself.[29]

Vague Question. What are "simple natural" Riemannian metrics g on $X = Y \times \mathbb{R}_+$ with given Bartnik data $(Sc(Y), mean, curv(Y))$, where $Y \subset X$ is allowed *variable* mean curvature, and what are possibilities for lower bound on the scalar curvatures of such g granted $|mean.curv(Y,y)|^2/Sc(Y,y) < C$, e..g. for $C = |mean.curv(Y')|^2/Sc(Y')$ for Y' being a sphere in a space of constant curvature.

---

[28]See §12 in [GL(complete) 1983].

[29]It follows from [Brendle-Marques(balls in S^n)N 2011] that the the cylinder $S^{n-1} \times \mathbb{R}_+$ admits a complete Riemannian metric g cylindrical at infinity which has $Sc(g) > n(n-1)$, and which has the same Bartnik data as the boundary sphere X'_0 in the hemisphere X' in the unit n-sphere. But the non-deformation result from [Brendle-Marques(balls in S^n) 2011], suggests that this might be impossible for the Bartnik data of *small* balls in the round sphere.

2.4.1 Higher Warped Products

Let Y and S be Riemannian manifolds with the metrics denoted dy^2 (which now play the role of the above dt^2) and ds^2 (instead of h), let $\varphi > 0$ be a smooth function on Y, and let

$$g = \varphi^2(y)ds^2 + dy^2$$

be the corresponding warped metric on $Y \times S$,

Then

$(\star\star)$

$$Sc(g)(y,s) = Sc(Y)(y) + \frac{1}{\varphi(y)^2}Sc(S)(s) - \frac{m(m-1)}{\varphi^2(y)}||\nabla\varphi(y)||^2 - \frac{2m}{\varphi(y)}\Delta\varphi(y),$$

where $m = dim(S)$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace on Y.

To prove this, apply the above c $(\star)$ to $l \times S$ for naturally parametrised geodesics $l \subset Y$ passing trough y and then average over the space of these l, that is the unit tangent sphere of Y at y.

The most relevant example here is where S is the real line $\mathbb{R}$ or the circle S^1 also denoted $\mathbb{T}^1$ and where $(\star)$ reduces to

$$(\star\star)_1 \qquad Sc(g)(y,s) = Sc(Y)(y) - \frac{2}{\varphi}\Delta\varphi(y).$$[30]

For instance, if the $L = -\Delta + \frac{1}{2}Sc$ on Y is strictly positive, that is the lowest eigenvalue λ is strictly positive and if φ equals to the corresponding eigenfunction of L, then

$$-\Delta\varphi = \lambda\cdot\varphi - \frac{1}{2}Sc\cdot\varphi$$

and

$$Sc(g) = 2\lambda > 0,$$

The basic feature of the metrics $\varphi^2(y)ds^2 + dy^2$ on $Y \times \mathbb{R}$ is that they are $\mathbb{R}$-*invariant*, where the quotients $(Y \times \mathbb{R})/\mathbb{Z} = Y \times \mathbb{T}^1$ carry the corresponding $\mathbb{T}^1$-*invariant* metrics, while *the* $\mathbb{R}$*-quotients are isometric to* Y.

Besides $\mathbb{R}$-invariance, a characteristic feature of warped product metrics is *integrability* of the tangent hyperplane field normal to the $\mathbb{R}$-orbits, where $Y \times \{0\} \subset Y \times \mathbb{R}$, being normal to these orbits, serves as an integral variety for this field.

Also notice that $Y = Y \times \{0\} \subset Y \times \mathbb{R}$ is totally geodesic with respect to the metric $\varphi^2(y)ds^2 + dy^2$, while the ($\mathbb{R}$-invariant) *curvature* (vector field) *of the* $\mathbb{R}$*-orbits* is equal to the *gradient field* $\nabla\varphi$ extended from Y to $Y \times \mathbb{R}$. coordinates

In what follows, we emphasize $\mathbb{R}$-invariance and interchangeably speak of $\mathbb{R}$-invariant metrics on $Y \times \mathbb{R}$ and metrics warped with factors φ^2 over Y.

Gauss-Bonnet $g^{\rtimes}$*-Exercise.* Let the above S be the Euclidean space $\mathbb{R}^N$ (make it $\mathbb{T}^n$ if you wish to keep compactness) with coordinates $t_1, ..., t_N$, let

$$\Phi(y) = (\varphi_1(y), ..., \varphi_i(y), ..., \varphi_N(y))$$

be an N-tuple of smooth positive function on a Riemannian mnanifold $Y = (Y, g)$ and define the (iterated t warped product) metric $g^{\rtimes} = g^{\rtimes}_{\Phi}$ on $Y \times S$ as follows:

$$g^{\rtimes} = g(y) + \varphi_1^2(y)dt_1^2 + \varphi_2^2(y)dt_2^2 + ... + \varphi_N^2(y)dt_N^2$$

[30] The roles of Y and $S = \mathbb{R}$ and notationally reversed here with respect to those in $(\star)$

Show that the scalar curvature of this metric, which, being $\mathbb{R}^N$-invariant, is regarded as a function on Y, satisfies:

$$Sc(g^{\rtimes}, y) = Sc(g) - 2\sum_{i=1}^{N} \Delta_g \log\varphi_i - \sum_{i=1}^{N} (\nabla_g \log\varphi_i)^2 - \left(\sum_{i=1}^{N} \nabla_g \log\varphi_i\right)^2,$$

thus

$$\int_Y Sc(g^{\rtimes}, y)dy \leq \int_Y Sc(g, y)dy,$$

and, following [Zhu(rigidity) 2019], obtain the following

"Warped" Gauss-Bonnet Inequality for Closed Surfaces Y:

$$\int_Y Sc(g^{\rtimes}, y)dy \leq 4\pi\chi(Y)$$

for the (iterated) warped product metrics $g^{\rtimes} = g^{\rtimes}_{\phi}$ for all positive N-tuples of Φ of positive functions on Y. [31]

2.5 Second Variation Formula

The Weyl formula also yields the following formula for the *second derivative* of the $(n-1)$-volume of a cooriented hypersurface $Y \subset X$ under a normal deformation of Y in X, where the scalar curvature of X plays an essential role.

The deformations we have in mind are by vector fields directed by geodesic normal to Y, where in the simplest case the norm of his field equals one.

In this case we have an equidistant motion $Y \mapsto Y_t$ as earlier and the second derivative of $vol_{n-1}(Y_t)$, denoted here $Vol = Vol_t$, is expressed in terms of of the shape $A_t = A(Y_t)$ of Y_t and the Ricci curvature of X, where, recall $trace(A_t) = mean.curv(Y_t)$ and

$$\partial_\nu Vol = \int_Y mean.curv(Y)dy$$

by the first variation formula.

Then, by Leibniz' rule,

$$\partial_\nu^2 Vol = \partial_\nu \int_Y trace(A(y))dy = \int_Y trace^2(A(y))dy + \int_Y trace(\partial_\nu A(y))dy,$$

and where, by Weyl's formula,

$$trace(\partial_\nu A) = -trace(A^2) - Ricci(\nu, \nu)$$

for the normal unit field ν.

Thus,

$$\partial_\nu^2 Vol = \int_Y (mean.curv)^2 - trace(A^2) - Ricci(\nu, \nu),$$

which, combining this with the above expression

$$Ricci(\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + ||A^*||^2\right),$$

[31] See [Zhu() 2019] and sections **??**, **??** for applications and generalizations.

shows that

$$\partial_\nu^2 Vol = \int \frac{1}{2}\left(Sc(Y) - Sc(X) + mean.curv^2 - ||A^*||^2\right).$$

In particular, if $Sc(X) \geq 0$ and Y is minimal, then,

$$(\textstyle\int \mathrm{Sc} \geq 2\partial^2 \mathrm{Vol}) \qquad \int_Y Sc(Y,y)dy \geq 2\partial_\nu^2 Vol$$

(compare with the $(Sc \geq -2Ric)$ in 2.2).

Warning. Unless Y is minimal and despite the notation ∂_ν^2, this derivative depends on how the normal filed on $Y \subset X$ is extended to a vector filed on (a neighbourhood of Y in) X.

Illuminative Exercise. Check up this formula for concentric spheres of radii t in the spaces with constant sectional curvatures that are S^n, $\mathbb{R}^n$ and $\mathbf{H}^n$.

Now, let us allow a non-constant geodesic field normal to Y, call it $\psi\nu$, where $\psi(y)$ is a smooth function on Y and write down the full second variation formula as follows:

$$\partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y ||d\psi(y)||^2 dy + R(y)\psi^2(y)dy$$

for

$$[\circ\circ] \qquad R(y) = \frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - ||A^*(Y)||^2\right),$$

where $M(y)$ stands for the mean curvature of Y at $y \in Y$ and $||A^*(Y)||^2 = \sum_i(\alpha^*)^2$, $i = 1,...,n-1$.

Notice, that the "new" term $\int_Y ||d\psi(y)||^2 dy$ depends only on the normal field itself, while the R-term depends on the extension of $\psi\nu$ to X, unless

Y is minimal, where $[\circ\circ]$ *reduces to*

$$[**] \qquad \partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y ||d\psi||^2 + \frac{1}{2}\left(Sc(Y) - Sc(X) - ||A^*||^2\right)\psi^2.$$

Furthermore, if Y is volume minimizing in its neighbourhood, then $\partial_{\psi\nu}^2 vol_{n-1}(Y) \geq 0$; therefore,

$$[\star\star] \qquad \int_Y (||d\psi||^2 + \frac{1}{2}(Sc(Y))\psi^2 \geq \frac{1}{2}\int_Y (Sc(X,y) + ||A^*(Y)||^2)\psi^2 dy$$

for all non-zero functions $\psi = \psi(y)$.

Then, if we recall that

$$\int_Y ||d\psi||^2 dy = \int_Y \langle -\Delta\psi, \psi\rangle dy,$$

we will see that $[\star\star]$ says that

the $\psi \mapsto -\Delta\psi + \frac{1}{2}Sc(Y)\psi$ *is greater than*[32] $\psi \mapsto \frac{1}{2}(Sc(X,y) + ||A^*(Y)||^2)\psi$.

Consequently,

if $Sc(X) > 0$, then the $-\Delta + \frac{1}{2}Sc(Y)$ on Y is positive.

[32] $A \geq B$ for selfadjoint operators signifies that $A - B$ is positive semidefinite.

Justification of the $||d\psi||^2$ *Term.* Let $X = Y \times \mathbb{R}$ with the product metric and let $Y = Y_0 = Y \times \{0\}$ and $Y_{\varepsilon\psi} \subset X$ be the graph of the function $\varepsilon\psi$ on Y. Then

$$vol_{n-1}(Y_{\varepsilon\psi}) = \int_Y \sqrt{1+\varepsilon^2||d\psi||^2}dy = vol_{n-1}(Y) + \frac{1}{2}\int_Y \varepsilon^2||d\psi||^2 + o(\varepsilon^2)$$

by the Pythagorean theorem
and

$$\frac{d^2 vol_{n-1}(Y_{\varepsilon\psi})}{d^2\varepsilon} = ||d\psi||^2 + o(1).$$

by the binomial formula.

This proves $[\circ\circ]$ for product manifolds and the general case follows by *linearity/naturality/functoriality* of the formula $[\circ\circ]$.

Naturality Problem. All "true formulas" in the Riemannian geometry should be derived with minimal, if any, amount of calculation – only on the basis of their "naturality" and/or of their validity in simple examples, where these formulas are obvious.

Unfortunately, this "naturality principle" is absent from the textbooks on differential geometry, but, I guess, it may be found in some algebraic articles (books?).

Exercise. Derive the second main formula 2.3.A by pure thought from its manifestations in the examples in the above *illuminative exercise.*[33]

2.6 Conformal Laplacian and the Scalar Curvature of Conformally and non-Conformally Scaled Riemannian Metrics

Let (X_0, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and let $\varphi = \varphi(x)$ be a smooth positive function on X.

Then, by a straightforward calculation,[34]

◐ $$Sc(\varphi^2 g_0) = \gamma_n^{-1}\varphi^{-\frac{n+2}{2}} L(\varphi^{\frac{n-2}{2}}),$$

where L is the *conformal Laplace* on (X_0, g_0)

$$L(f(x)) = -\Delta f(x) + \gamma_n Sc(g_0, x) f(x)$$

for the ordinary Laplace (Beltrami) $\Delta f = \Delta_{g_0} f = \sum_i \partial_{ii} f$ and $\gamma_n = \frac{n-2}{4(n-1)}$.

Thus, we conclude to the following.

Kazdan-Warner Conformal Change Theorem. [35] Let $X = (X, g_0)$ be a closed Riemannian manifold, such the the conformal Laplace L is positive.

Then X *admits a Riemannian metric* g *(conformal to* g_0*) for which* $Sc(g) >$ 0.

Proof. Since L is positive, its first eigenfunction, say $f(x)$ is positive[36] and since $L(f) = \lambda f$, $\lambda > 0$,

[33] I haven't myself solved this exercise.
[34] There must be a better argument.
[35] [Kazdan-Warner(conformal) 1975]: *Scalar curvature and conformal deformation of Riemannian structure.*
[36] We explain this in section 2.9.

$$Sc\left(f^{\frac{4}{n-2}}g_0\right) = \gamma_n^{-1}L(f)f^{-\frac{n+2}{n-2}} = \gamma_n^{-1}f^{\frac{2n}{n-2}} > 0.$$

Example: Schwarzschild metric. If (X_0, g_0) is the Euclidean 3-space, and $f = f(x)$ is positive function, then

the sign of $Sc(f^4 g_0)$ is equal to that of $-\Delta f$.

In particular, since the function $\frac{1}{r} = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}$, is harmonic, *the Schwarzschild metric $g_{Sw} = \left(1 + \frac{m}{2r}\right)^4 g_0$ has zero scalar curvature.*

If $m > 0$, then this metric is defined for all $r > 0$ and it is invariant under the involution $r \mapsto \frac{m^2}{r}$.

If $m = 0$, this the flat Euclidian metric.

If $m < 0$, then this metric is defined only for $r > m$ with a singularity ar $r = m$.

Non-Conformal Scaling. Let $X = (X, g)$ be a smooth n-manifold, and let $\mathbb{R}_x^\times \subset GL_x(n)$, $x \in X$, be a smooth family of diagnosable (semisimple) 1-parameter subgroups in the linear groups $GL_x(n) = GL_n$ that act in the tangent spaces $T_x(X)$.

Then the the multiplicative group of functions $\phi : X \to \mathbb{R}^\times$ acts on the tangent bundle $T(X)$ by

$$\tau \mapsto = \phi(x)(\tau) \text{ for } \phi(x) \in \mathbb{R}^\times = \mathbb{R}_x^\times \subset GL_x = GL(T_x(X))$$

and, thus on the space of Riemannin metrics g on X.

The main instance of such an action is where the tangent bundle is orthogonally split, $T(X) = T_1 \oplus T_2$, and ϕ acts by scaling on the subbundle T_2.

It is an not hard to write down a formula for the scalar curvature of $g_1 + \phi^2 g_2$, but it is unclear what, in general, would be a workable criterion for solvability of the inequality $Sc(g_\varphi) > 0$ in φ, e.g. in the case where $X = X_1 \times X_2$ and the subbundles T_1 and T_2 are equal to the tangent bundles of submanifolds $X_1 \times x_2 \subset X$, $x_2 \in X_2$, and $x_1 \times X_2 \subset X$, $x_1 \in X_1$.

Yet, in the case of $rank(T_2) = 1$, this equation introduced, I believe, by Robert Bartnik in [Bartnik(prescribed scalar) 1993] was successfully applied to extension of metrics with $Sc > 0$ (see section **??**)[37]

2.7 Schoen-Yau's Non-Existence Results for $Sc > 0$ on SYS Manifolds via Minimal (Hyper)Surfaces and Quasisymplectic $[Sc \not> 0]$–Theorem

Let X be a three dimensional Riemannian manifold with $Sc(X) > 0$ and $Y \subset X$ be an orientable cooriented surface with minimal area in its integer homology class.

Then the inequality $(\int Sc \geq 2\partial^2 V)$ from section 2.5, which says in the present case that

$$\int_Y Sc(Y, y)dy > 2\partial_\nu^2 area(Y),$$

implies that

Y must be a topological sphere.

[37] Other special cases of this are (implicitly) present in the geometry of Riemannin warped product, in the process of *smoothing corners with $Sc \geq \sigma$* and in the *transversal blow up* of foliations with $Sc > 0$.

In fact, minimality of Y makes $\partial^2_\nu area(Y) \geq 0$, hence $\int_Y Sc(Y,y)dy > 0$, and the sphericity of Y follows by the Gauss-Bonnet theorem.

And since all integer homology classes in closed orientable Riemannian 3-manifolds admit area minimizing representatives by the geometric measure theory developed by Federer, Fleming and Almgren, we arrive at the following conclusion.

$\bigstar_3$ **Schoen-Yau 3d-Theorem**. *All integer 2D homology classes in closed Riemannian 3-manifolds with $Sc > 0$ are spherical.*

For instance, the 3-torus admits no metric with $Sc > 0$.

The above argument appears in Schoen-Yau's 15-page paper [SY(incompressible) 1979], most of which is occupied by an independent proof of the existence and regularity of minimal Y.

In fact, the existence of minimal surfaces and their regularity needed for the above argument has been known since late (early?) 60s[38] but, what was, probably, missing prior to the Schoen-Yau paper was the innocuously looking corollary of Gauss' formula in 2.2,

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu)$$

and the issuing inequality

$$Sc(Y) > -2Ricci(\nu,\nu)$$

for minimal Y in manifolds X with $Sc(X) > 0$.

For example, Burago and Toponogov, come close to the above argument, where, they bound from below the injectivity radius of Riemannian 3-manifolds X with $sect.curv(X) \leq 1$ and $Ricci(X) \geq \rho > 0$ by

$$inj.rad(X) \geq 6e^{-\frac{6}{\rho}},$$

where this is done by carefully analysing minimal surfaces $Y \subset X$ bounded by, a priori very short, closed geodesics in X, and where an essential step in the proof is the lower bound on the first eigenvalue of the Laplace on Y by $\sqrt{Ricci(X)}$.[39]

Area Exercises. Let X be homeomorphic to $Y \times S^1$, where Y is a closed orientable surface with the Euler number χ.

(a) Let $\chi > 0$, $Sc(X) \geq 2$ and show that there exists a surface $Y_o \subset X$ homologous to $Y \times \{s_0\}$, such that $area(Y_o) \leq 4\pi$.[40]

(b) Let $\chi < 0$, $Sc(X) \geq -2$ and show that all surfaces $Y_* \in X$ homologous to $Y \times \{s_0\}$ have $area(Y_*) \geq -2\pi\chi$.

(c) Show that (a) remains valid for complete manifolds X homeomorphic to $Y \times \mathbb{R}$.[41]

$\bigstar^{codim1}$ **Schoen-Yau Codimension 1 Descent Theorem,** [SY(structure) 1979]. Let X be a compact orientable n-manifold with $Sc > 0$.

[38] Regularity of volume minimizing hypersurfaces in manifolds X of dimension $n \leq 7$, as we mentioned earlier, was proved by Herbert Federer in [Fed(singular) 1970], by reducing the general case of the problem to that of minimal cones resolved by Jim Simons in [Simons(minimal) 1968].

[39] [BurTop(curvature bounded above)1973], *On 3-dimensional Riemannian spaces with curvature bounded above.*

[40] See [Zhu(rigidity) 2019] for a higher dimensional version of this inequality.

[41] I haven't solved this exercise.

If $n \le 7$, then all integer homology classes $h \in H_{n-1}(X)$ are representable by compact oriented $(n-1)$-submanifolds Y in X, which admit metrics with $Sc > 0$.

Proof. Let Y be a volume minimizing hypersurface representing h, the existence and regularity of which is guaranteed by a Federer 1970-theorem[42] and recall that by $[\star\star]$ in 2.5 the $-\Delta + \frac{1}{2}Sc(Y)$ is positive. Hence, the conformal Laplace $-\Delta + \gamma_n Sc(Y)$ is also positive for $\gamma_n = \frac{n-2}{4n-1} \le \frac{1}{2}$ and the proof follows by Kazdan-Warner conformal change theorem.

$\bigstar_{\mathbb{T}^n}$ **Mapping to the Torus Corollary.** If a closed orientable n-manifold X admits a map to the torus $\mathbb{T}^n$ with *non-zero degree*, then X admits *no metric with* $Sc > 0$.

Indeed, if a closed submanifold Y^{n-1} is *non-homologous to zero* in this X then it (obviously) admits a map to $\mathbb{T}^{n-1}$ with non-zero degree. Thus, the above allows an inductive reduction of the problem to the case of $n = 2$, where the Gauss-Bonnet theorem applies.

SYS-Manifolds. Schoen and Yau say in [SY(structure) 1979] that their codimension 1 descent theorem delivers a topological obstruction to $Sc > 0$ on a class of manifolds, which is, even in the spin case, [43] is not covered by the twisted Dirac operators methods.

This claim was confirmed by Thomas Schick, who defined, in homotopy theoretic terms, integer homology classes in aspherical spaces, say $h \in H_n(\underline{X})$ and who proved using the codimension one descent theorem that these h for $n \le 7$ can't be dominated by compact orientable n-manifolds with $Sc > 0$.

In more geometric terms, the n-manifolds X, to which Schick's argument applies, we call them *Schoen-Yau-Schick*, can be described d as follows.

A closed orientable n-manifold is *Schoen-Yau-Schick* if it admits a smooth map $f : X \to \mathbb{T}^{n-2}$, such that the homology class of the pullback of a generic point,

$$h = [f^{-1}(t)] \in H_2(X)$$

is *non-spherical,* i.e. it is not in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.

Then Schick's corollary to Schoen-Yau's theorem reads.

$\bigstar_{SYS}$ **Non-existence Theorem for SYS Manifolds.** *Schoen-Yau-Schick manifolds of dimensions $n \le 7$ admit no metrics with $Sc > 0$.*

(b) *Exercises.* (b_1) Construct examples of SYS manifolds of dimension $n \ge 4$, where all maps $X \to \mathbb{T}^n$ have zero degrees.

Hint: apply surgery to $\mathbb{T}^n$.

(b_2) Show that if the first homology group $H_1(X)$ of a SYS-manifold has no torsion, then a finite covering of X admits a map with degree one to the torus $\mathbb{T}^n$.

(c) The limitation $n \le 7$ of the above argument is due a presence of singularities of minimal subvarieties in X for $dim(X) \ge 8$.

[42][Federer(singular) 1970]: *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension.*

[43]A smooth connected n-manifolds X is *spin* if the frame bundle over X admits a double cover extending the natural double cover of a fiber, where such a fiber is equal to the linear group, (each of the two connected components of) which admits a a unique non-trivial double cover $\tilde{G}L(n) \to GL(n)$.

If $n = 8$, these singularities were proven to be unstable by Nathan Smale; this improves $n \leq 7$ to $n \leq 8$ in $\bigstar_{SYS}$

More recently, as we mentioned earlier, the dimension restriction was removed for all n by Lohkamp and by Schoen-Yau; the arguments in both papers are difficult and I have not mastered them.[44]

Although the Dirac operator arguments don't apply to SYS-manifolds, they do deliver topological obstructions to $Sc > 0$, which, according to the present state of knowledge, lie beyond the range of the minimal surface techniques. Here is an instance of this.

$\otimes_{\wedge^k \tilde{\omega}}$ **Quasisymplectic Non-Existence Theorem.** Let X be a compact $\otimes_{\wedge^k \tilde{\omega}}$-*manifold* of dimension $n = 2k$, i.e. X is orientable and it carries a *closed 2-form* ω (e.g. a symplectic one), such that $\int_X \omega^k \neq 0$, and such that the lift $\tilde{\omega}$ of ω to the universal covering $\tilde{X}$ is *exact*, e.g. $\tilde{X}$ is contractible.[45]

Then X *admits no metric with* $Sc > 0$.

This applies, for instance, to *even dimensional tori*, to *aspherical* 4-*manifolds with* $H^2(X, \mathbb{R}) \neq 0$ and to *products* of such manifolds[46] but *not* to general SYS-manifolds.

Idea of the Proof. Assume without loss of generality that ω serves as the curvature form of a complex line bundle$L \to X$and let $\tilde{L} \to \tilde{X}$ be the lift of L to the universal covering $\tilde{X} \to X$.

Since the curvature $\tilde{\omega}$ of $\tilde{L}$, is exact the bundle $\tilde{L}$is topologically trivial, hence it can be represented by k-th tensorial power of another line bundle,

$$L = (L^{\frac{1}{k}})^{\otimes k},$$

where the curvature of $L^{\frac{1}{k}}$ is $\frac{1}{k}\tilde{\omega}$. By Atiyah's L_2-*index theorem*, there are *non-zero harmonic* L_2-*spinors* on $\tilde{X}$ *twisted with* $L^{\frac{1}{k}}$ for infinitely many k, but the twisted Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula applied to large k doesn't allow such spinors for $Sc(\tilde{X} \geq \sigma > 0$.[47]

Exercise. Show that if X is $\otimes_{\wedge^k \tilde{\omega}}$, then the classifying map $X \to \mathsf{B}(\Pi)$, where $\mathsf{B}(\Pi) = K(\Pi, 1)$ is the classifying space for the group $\Pi = \pi_1(X)$, sends the fundamental homology class $[X]$ to a *non-torsion* class in $H_n(\mathsf{B}(\Pi))$.

Problem. *Is there a unified approach that would apply to* SYS*-manifolds and to the above* $\otimes_{\wedge^k \tilde{\omega}}$*-manifolds* X*, e.g. symplectic ones with contractible universal coverings?*

For instance,

do products of SYS *and* $\otimes_{\wedge^k \tilde{\omega}}$*-manifolds ever carry metrics with positive scalar curvatures?*

[44] See [Smale(generic regularity) 2003], SY(singularities) 2017], [Lohkamp(smoothing) 2018] and section 3.6.1.

[45] It's enough to have $\tilde{X}$ spin.

[46] Recently, Chodosh and Li proved that

compact aspherical manifolds of dimensions 4 and 5 admit no metrics with positive scalar curvatures. (See [Chodosh-Li(bubbles) 2020], [G(aspherical) 2020] and section**??**)

But this remains problematic for products of pairs of aspherical 4-manifolds.

[47] Atiyah's theorem from [Atiyah(L2) 1976] needs a slight adjustment here, since the action of the fundamental group $\Gamma = \pi_1(X)$ on $\tilde{X}$ doesn't lift to $L^{\frac{1}{k}}$; yet the fundamental group of the (total space) of the unit circle bundle of L does naturally act on $L^{\frac{1}{k}}$. Also, there is no difficulty in extending Lichnerowicz' vanishing argument to the L_2 case, see §$9\frac{1}{8}$ in [G(positive) 1996].

2.8 Warped $\mathbb{T}^{\rtimes}$-Stabilization and Sc-Normalization

Many geometric properties of Riemannian manifolds $X = (X, g)$ implied by the inequality $Sc(g) \geq \sigma$ follow (possibly in a weaker form) from the same inequality for a larger manifold, say X^*, that, topologically, is the product of X with the a torus, $X^* = X \times \mathbb{T}^N$ for some $N = 1, 2, \ldots$, where the Riemannian metric g^* on X^* is invariant under the action of $\mathbb{T}^N$ and where $X^*/\mathbb{T}^N$ is isometric to X.

Surface Examples. Let $X = (X, g)$ be a closed surface and g^* be a $\mathbb{T}^N$-invariant metric on $X \times \mathbb{T}^N$, such that

$$(X \times \mathbb{T}^N, g^*)/\mathbb{T}^N = (X, g).$$

(a) ***Sharp Equivariant Area Inequality.*** If $Sc(g^*) \geq \sigma > 0$, then a special case a theorem by Jintian Zhu,[48] says that

the area of X is bounded the same way as it is for $Sc(g) \geq \sigma$,

$$area(X) \leq \frac{8\pi}{\sigma}.$$

Moreover,

the equality holds only if X^* is the isometric *product $X \times \mathbb{T}^N$.*

(b) (Weakened) $\mathbb{T}^*$***-Stable 2d Bonnet-Myers Diameter Inequality.*** If $Sc(g^*) \geq \sigma$, then

$$[BMD] \qquad diam(X) \leq 2\pi\sqrt{\frac{N+1}{(N+2)\sigma}} < \frac{2\pi}{\sqrt{\sigma}}.$$

Proof. Given two points $x_1, x_2 \in X$, take two small ε-circles Y_{-1} and Y_{+1} around them, let $X_\varepsilon \subset X$ be the band between them and and apply (the relatively elementary $\mathbb{T}^N$-invariant case of) the $\frac{2\pi}{n}$-*Inequality* from section 3.5.[49]

Non Trivial Torus Bundles. The inequality [BMD] is valid for (all) Riemannian $(N+2)$-manifolds X^* with free isometric $\mathbb{T}^N$-actions:

if $Sc(X^*) \geq \sigma > 0$, then $diam(X^*/\mathbb{T}^N) \leq 2\pi\sqrt{(N+1)/(N+2)\sigma}$.

In fact, the above proof applies, since, *topologically*, the part of X^* that lies over the band $X_\varepsilon \subset X$ *is* the product, $X_\varepsilon \times \mathbb{T}^N$.

It is *unclear*, however, if the areas of $X^*/\mathbb{T}^N$ are bounded in terms of $Sc(X^*)$ for all such X^*.

And, as we shall see later, possible non-triviality of torus bundles create complications for other problems with scalar curvature.

General Question. The above examples suggests that quotients X of manifolds X^* with $Sc(X^*) \geq \sigma$ under free isometric actions of tori have *similar* geometric properties to those of manifolds which have $Sc \geq \sigma$ themselves. But it is unclear how far this similarity goes.

Example. let X be a closed surface and $X^{\rtimes} = X \rtimes \mathbb{T}^1$ be a warped product as described below.

Does the inequality $Sc(X^{\rtimes}) \geq 2$ yield an upper bound on *all of geometry* of X?

[48] See [Zhu(rigidity) 2019] and **??**, **??** for related inequalities.

[49] Also see §2 in [G(inequalities) 2018] and the proof of theorem 10.2 in [GL(complete) 1983].

For instance,

is there a bound on the number of unit discs needed to cover X?

(If $Sc(X) \geq 2$, then X admits a *distance decreasing* homeomorphism from the unit sphere S^2, that can be constructed using the family of boundary curves of concentric discs with center at some point in X.)

Warped Products. As far as geometric applications are concerned, the relevant X^* are (iterated) *warped products*, we denote them $X^{\rtimes}$ and call *warped* $\mathbb{T}^N$*-extensions of* X, that are characterized by the existence of *isometric* sections $X \to X^{\rtimes}$ for $X^{\rtimes} \to X = X^{\rtimes}/\mathbb{T}^N$.

Clearly, metrics $g^{\rtimes}$ on these $X^{\rtimes}$ are

$$g^{\rtimes} = g + \varphi_1^2(x)dt_1^2 + \varphi_2^2(x)dt_2^2 + ... + \varphi_N^2(x)dt_N^2$$

for some positive functions φ_i on X.

Among these we distinguish $O(N)$*-invariant* warped extensions, where the $\mathbb{Z}^N$ covering manifolds $\tilde{X}^{\rtimes} = X \times \mathbb{R}^N$, where

$$\tilde{X}^{\rtimes}/\mathbb{Z}^N = X^{\rtimes},$$

are invariant under the action of the orthogonal group $O(N)$. Thus, $\tilde{X}^{\rtimes}$ are acted upon by the full isometry group of $\mathbb{R}^N$, that is $\mathbb{R}^N \rtimes O(N)$.

Equivalently, the metric in such an $X^{\rtimes}$ is a "simple" warped product: $g^{\rtimes} = g + \varphi^2 d\|\bar{t}\|^2$ for $\bar{t} = (t_1, t_2, ..., t_N)$, the scalar curvature of which, as we know, 2.4 is

$$Sc(g^{\rtimes})(x, \bar{t}) = Sc(X)(x) - \frac{2N}{\varphi(x)}\Delta_g\varphi(x) - \frac{N(N-1)}{\varphi^2(x)}$$

and which is most simple (and useful) for $N = 1$, where

$$[\rtimes_\varphi] \qquad Sc(g^{\rtimes})(x, \bar{t}) = Sc(X)(x) - \frac{2}{\varphi(x)}\Delta_g\varphi(x).$$

for the Laplace (Beltrami) Δ_g on $X = (X, g)$.

$[\rtimes_\varphi]^N$**-Symmetrization Theorem.** Let $X = (X, g)$ be a closed oriented Riemannian manifold of dimension $n = m + N$ and let

$$X \supset X_{-1} \supset ... \supset X_{-i} \supset ... \supset X_{-N},$$

be a descending chain of closed oriented submanifolds, where each $X_{-i} \subset X$ is equal to a transversal intersection of $X_{-(i-1)}$ with a smooth closed oriented hypersurface $H_i \subset X$,

$$H_i \cap X_{-(i-1)} = X_{-i}.$$

If $n \leq 7$, then

there exists a closed oriented m-dimensional submanifold $Y \subset X$ homologous to X_{-N} and a warped product $\mathbb{T}^N$-extension $Y^{\rtimes}$ of $Y = (Y, h)$ for the Riemannian metric h on Y induced from g on X, such that the scalar curvature of $Y^{\rtimes}$, that is, being $\mathbb{T}^N$-invariant, is represented by a function on Y, is bounded from below by the Scalar curvature of X on $Y \subset X$,

$$Sc(Y^{\rtimes}, y) \geq Sc(X, y), \; y \in Y.$$

Proof. Proceed by induction on codimension $i = 1, 2, , ...N$ and construct submanifolds

$$X \supset Y_1 \supset ... \supset Y_i \supset ... \supset Y_N = Y \subset X$$

as follows.

At the first step, let $Y_1 \subset X$ be a volume minimizing, hence stable, hypersurface homologous to X_{-1} where, the positivity of the second variation implies the positivity of the

$$-\Delta + \frac{1}{2}(Sc(Y_1) - Sc(X)|_{Y_1},$$

for the Laplace $\Delta = \Delta_{h_1}$ on Y_1 with the metric h_1 induced from X and let $\psi_1 > 0$ be the first eigenfunction of this with the positive eigenvalue λ_1, thus

$$-\Delta\psi = \left(\lambda - \frac{1}{2}(Sc(Y, h_1) - Sc(X))\right) \cdot \psi_1.$$

Here, let $h_1^{\rtimes}(y) = h_1(y) + \psi^2 dt^2$ be the warped product metric on $Y_1 \times \mathbb{T}^1$ and observe

$$Sc(h_1^{\rtimes}, y) = Sc(h_1, y) - \frac{2}{\psi}\Delta\psi_1 = Sc(X, y) + 2\lambda_1.$$

Then, at the second step, let $Y_2 \subset Y_1$ be a hypersurface, such that $Y_2 \times \mathbb{T}^1 \subset Y_1 \times \mathbb{T}^1$ is volume minimizing for the metric $h_1^{\rtimes}$, which is equivalent for Y_2 to be volume minimizing in Y_1 with respect to the metric $\psi_1^{l_1} h_1$ for $l_1 = \frac{2}{n-1}$.

Thus we obtain Y_2', where the corresponding metric on $Y_2' \times \mathbb{T}^2$ is

$$h_2' + \psi_1^2 dt_1^2 + \psi_2^2 dt_2^2.$$

Repeating this $N-2$ more times, we arrive at Y_N' and an (iterated) warped product metric

$$h_N' + \sum_{i=1}^{N} \psi_i^2 dt_i^2 \text{ on } Y_N' \times \mathbb{T}^N,$$

which can be symmetrised further to the required $h^{\rtimes}$ by applying the above infinitely many times to hypersurfaces $Y_N' \times T^{N-1} \subset Y_N' \times T^N$ for all subtori $T^{N-1} \subset Y_N' \times T^N$.[50] (The luxury of the extra $O(N)$-symmetry is unneeded for most purposes.)

Exercise. Apply $[\rtimes_\varphi]^N$-symmetrization to n-manifolds with isometric $\mathbb{T}^{n-2}$-actions and prove the above equivariant area inequality by reducing it to the warped product case that was already settled in section 2.4.1.

Symmetrization by Reflections and Convergence Problem. Let Y be a closed minimal co-orientable (i.e. two sided) hypersurface in a Riemannian manifold. If Y is locally volume minimizing, then it admits arbitrarily small neighbourhoods $V_\varepsilon \supset Y$ in X with smooth *strictly mean convex* boundaries. Then by reflecting such a varepsilon in the two boundary components, one obtains manifolds $\hat{V}_\varepsilon$ with isometric actions of $\mathbb{Z} \rtimes \mathbb{Z}_2$.

If these Y are non-singular, e.g. if $dim(X) \leq 7$, then one can take solutions of the isoperimetric problem for these V_ε, where one minimize the volumes of both components of the boundaries of V_ε per given (small) volume contained between them and Y. In this case, $\hat{V}_\varepsilon$, $\varepsilon \to 0$, converge to smooth Riemannian manifolds

[50]See in, §12[GL(complete)1983], [G(inequalities) 2018] and also the sections 3.6, **??** for details of this argument and for generalizations.

$V^{\rtimes}$ with isometric actions of $\mathbb{R}$ and with their scalar curvatures bounded from below by $Sc(X)|_Y$.

If Y is singular, the boundaries of these V_ε, even if singular, [51] can be smoothed with positive mean curvatures, but it is unclear if they converge to a reasonable object for $\varepsilon \to 0$: what is *missing for convergence* is a *Harnack type inequality* for the boundary components of $\partial_1, \partial_2 \subset \partial V_\varepsilon$, that is a uniform bound for the ratios of the distances

$$\frac{dist(y, \partial_i)}{dist(y', \partial_i)}, y, y' \in Y,$$

$i = 1, 2$, and /or of distances $dist(x, x', Y)$, $x, x' \in \partial_i$.

Notice, that "symmetrization by reflections", albeit open to generalizations to singular Y, is not, apparently, applicable, to stable μ-bubbles Y, where the warped product construction does apply. [52]

Symmetrization versus Normalization. $\mathbb{T}^{\rtimes}$-Symmetrization of metrics g typically) makes their scalar curvatures *constant* by paying the price of modification of the topology of the underlying manifolds, $X \rightsquigarrow X \times \mathbb{T}^1$.

As far as sets of "interesting" maps between Riemannian manifolds are concerned a similar effect effect is achieved by keeping the same manifold X but modifying the metric by $g = g(x) \rightsquigarrow g^\circ = g^\circ(x) = Sc(X, x) g(x)$.

In fact, we shall see later in many examples, that

there is a close (but not fully understood) similarity between the sets of λ°-*Lipschitz* maps $(X, g^\circ) \to (Y, h^\circ)$ and of $\mathbb{T}^1$-*equivariant* $\lambda^{\rtimes}$-*Lipschitz* maps $(X \times \mathbb{T}^1, g^{\rtimes}) \to (Y \times \mathbb{T}^1, h^{\rtimes})$ for λ° and $\lambda^{\rtimes}$ related in a certain way.

2.9 Positive Eigenfunctions and the Maximum Principle

Let X be a *compact connected* Riemannian manifold and let

$$\Delta f = \sum_i \nabla_{ii} f = \mathsf{trace\,Hess} f = \mathsf{div\,grad} f$$

denote the Laplace (Beltrami) on X, which, recall, is a *negative* , since

$$\int_X \langle f, \Delta f \rangle dx = -\int_X ||\mathsf{grad} f||^2 dx \le 0$$

by Green's formula.

Non-Vanishing Theorem. Let $s(x)$ be a smooth function, such that the

$$L = L_s : f(x) \mapsto -\Delta f(x) + s(x) f(x)$$

is *non-negative*, that is $\int_X \langle f(x), Lf(x) \rangle dx \ge 0$ for all f or, equivalently, if L the lowest eigenvalue $\lambda = \lambda_{min}$ is ≥ 0.[53]

Then

the eigenfunction $f(x)$ associated with λ doesn't vanish anywhere on X.

[51] If $n = 8$, then, by adapting Nathan Smale's argument, one can show that these V_ε are non-singular for an open dense set of values of ε; but this is problematic for $n \ge 9$.

[52] See §8 in [G(billiards) 2014], §4.3 in [G(inequalities) 2019] and section **??** for more about all this.

[53] This is equivalent since our L has *discrete spectrum.*

Start with two lemmas.

1. C^1-*Lemma.* If the minimal eigenvalue of the $f(x) \mapsto Lf(x) = -\Delta f(x) + s(x)f(x)$ on a compact Riemannian manifold is *non-negative,* $\lambda = \lambda_{min} \geq 0$, then the *absolute value* $|f(x)|$ of the eigenfunction f associated with λ is C^1-*smooth.*

2. Δ-*Lemma.* Let $f(x)$ be a *non-negative* continuous function on a Riemannian manifold, such that

(i) $f(x)$ *vanishes* at some point in X,

$$f(x_0) = 0,\ x_0 \in X,$$

(ii) $f(x)$ is *not identically zero* in any neighbourhood of the point $x_0 \in X$,

(iii) $f(x)$ is everywhere C^1-*smooth* and it is C^2-*smooth* at the points x where it doesn't vanish.

Then there exists a sequence of points $x_1, x_2, ... \in X$ convergent to x_0, *where* $f(x_i) > 0$ *and such that*

$$\frac{\Delta f(x_i)}{f(x_i)} \to \infty,\ \textit{for } i \to \infty.$$

Derivation of Non-vanishing Theorem from the Lemmas. Since $|f|$ is C^1 by the first lemma, the Δ-lemma, applied to $|f(x)|$, shows that there exists a point x, where $f(x) \neq 0$ and

$$\frac{\Delta f(x)}{f(x)} = \frac{\Delta |f(x)|}{|f(x)|} > |s(x)|,$$

that is incompatible with $-\Delta f(x) + s(x)f(x) = \lambda f(x) \geq 0$ for $\lambda \geq 0$.

Proof of C^1-*Lemma.* Recall that the eigenvalues of the $L = L_s = -\Delta + s$ are equal to the critical values of *the energy functional*

$$E(f) = \int_X (||\mathrm{grad} f(x)||^2 + s(x)) f^2(x) dx$$

on the sphere

$$||f||^2 = \int_X f^2(x) dx = 1$$

in the Hilbert space $L_2(X)$ and the critical points of E are represented by eigenfunctions

Indeed,

$$E(f) = \langle f, Lf\rangle = \int_X \langle f(x), Lf(x)\rangle dx$$

by Green's formula and the differential of the quadratic function $f \mapsto \langle f, Lf\rangle$ on the sphere $||f||^2 = 1$ is

$$(dE)_f(\tau) = \langle \tau, Lf\rangle \text{ for all for all } \tau \textit{ normal to } f.$$

Thus, vanishing of dE at f on the unit sphere says, in effect, that Lf is a multiple of f, i.e. $Lf = \lambda f$.

All this makes sense in the present case, albeit the space $L_2(X)$ is *infinite dimensional* and L an *unbounded*, because L is an *elliptic* operator, which implies, for compact X, that *the spectrum of L is discrete, bounded from below and all eigenfunctions are smooth.*

In particular – this is all we need,

all minimizes of $E(f)$ on the unit sphere, that are, a priori, only Lipschitz continuous, are smooth.[54]

Now, observe that,

taking absolute values of smooth functions $f(x) \mapsto |f(x)|$ doesn't change their energies, as well as their L_2-norms,

$$\||f|\| = \|f\| = \sqrt{\int_X |f|^2(x)dx},$$

$$E(|f|) = E(f) = \int_X (||\mathsf{grad}|f|(x)||^2 + s(x))|f|^2(x)dx,$$

Indeed, absolute values $|f|(x)$ are Lipschitz for Lipschitz f, hence, they are almost everywhere differentiable functions, such that $\mathsf{grad}|f|(x) = \pm\mathsf{grad} f(x)$ at all differentiability points x of $|f|$.

It follows that the absolute value of the eigenfunction f with the smallest energy $E(f) = \lambda_{min}$ is also a minimizer; hence, *this $|f|$ is smooth.* QED.

Poof of Δ-Lemma. The common strategy for locating points $x \in X$ with "sufficiently positive" second differential of a function $f(x)$ is by using simple auxiliary functions $e(x)$ with this property and looking for minima points for $f(x) - e(x)$.

The basic example of such a function $e(x)$ in one variable is e^{-Cx}, $x > 0$, for large C, where $\frac{e''}{e} = C^2$, and where observe that the ratio $\frac{e''}{e'} = C$ also becomes large for large C.

It follows that that the Laplacians of the corresponding radial functions in small R-ball $B_y(R)$ in Riemannian manifolds X,

$$e(x) = e_C(x) = e_{y,C}(x) = e^{-C \cdot r_y(x)} \text{ for } r_y(x) = dist(y,x) \leq R$$

satisfy

$$\Delta e(x) \geq C^2 e(x) - C \cdot mean.curv(\partial B_y(r), x) \text{ for } r = r_y(x) = dist(y,x)$$

Now, in order to find a point x close to a given $x_0 \in X$ where $f(x) = 0$, take $y \in X$ very close to x_0, where $f(y) > 0$, let $B_y(R) \subset X$ be the *maximal* ball, such that $f(x) > 0$ in its interior, let

$$e(x) = e_C(x) = e^{-C \cdot r_y(x)} - e^{-C \cdot R}$$

and observe that $e(x)$ vanishes on the boundary of the ball $B_y(R)$ and is strictly positive in the interior. Moreover

$$e(x) \geq \varepsilon\rho,$$

for all x on the geodesic segment between y and x_0 within distance $\geq \rho$ from x_0 for all $\rho_0 \leq R$.

[54]Recall that our "smooth" means C^∞ and all our Riemannian manifolds are assumed smooth.

Notice that this $\varepsilon = \varepsilon_C$ albeit *strictly positive*, tends to zero for $C \to \infty$.

Assume without loss of generality that x_0 is the only point in $B_x(R)$ where $f(x)$ vanishes (if not, move y closer to x_0 along the geodesic segment between the two points), let C be *very very large* and see what happens to $f(x)$ and $e(x)$ in the vicinity of $x_0 \in \partial B_y(R)$, say in the intersection

$$U_0 = B_y(R) \cap B_{x_0}(R/3).$$

Observe the following.

• Since $f(x) > 0$ for $x \in B_y(R)$, $x \neq x_0$, and since $e_C(x) \to 0$ for $C \to \infty$ for $r_y(x) = dist(y,x) \geq r_0 > 0$, *the function* $e(x) = e_C(x)$, *for large* C, *is bounded by* $f(x)$ *on the boundary of* U_0,

$$e(x) \leq f(x), \; x \in \partial U_0,$$

where $e(x) < f(x)$ *unless* $x = x_0$.

• Since f is differentiable at x_0 and assumes minimum at this point, the differential *df* vanishes at x_0, which makes $f(x) = o(\rho)$ for $\rho = dist(x, x_0)$, *there is a part of (the interior of)* U_0, *where* $e(x) > f(x)$.

Hence, the difference $f(x) - e(x)$ assumes minimum at an interior point $x = x_{y,C} \in U_0$, such that $x = x_{y,C} \to x_0$ for $C \to \infty$ and

$$\frac{\Delta f(x)}{f(x)} \geq \frac{\Delta e(x)}{e(x)} \to \infty.$$

The proof of the Δ-lemma and of the non-vanishing theorem are thus concluded.

Discussion. The non-vanishing theorem, which, probably, goes back to Rayleigh, is often used without being even explicitly stated as, for instance, by Kazdan and Warner in their "conformal change" paper. But I couldn't find an explicit reference on the web, except for the paper by Doris Fischer-Colbrie and Rick Schoen, where they prove such a non-vanishing for non-compact manifolds needed for their

non-existence theorem for non-planar stable minimal surfaces in $\mathbb{R}^3$.

Their argument relies on the *"strong maximum principle"* for the L, for which they refer to pp. 33-34 of the canonical Gilbarg-Trudinger textbook, where the relevant case of this principle is stated (on p. 35 in the 1998 edition which is available on line) after the proof of theorem 3.5 as follows.

"Also, if $u = 0$ at an interior maximum (minimum), then it follows from the proof of the theorem that $u = 0$, irrespective of the sign of c."

(The assumptions of the theorem specifically rule out c with variable signs, where this $c = c(x)$ is the coefficient at the lowest term in the equation $Lu = a^{ij}(x)D_{ij}u + b^i D_i u + c(x)u = 0$ introduced on p. 30.)

What is actually proven in this book on about twenty lines on p. 34, is a version of "Δ-lemma" for L.

In our proof, we reproduce what is written on these lines, except for "*direct calculation gives*" that is replaced by an explicit evaluation of $\Delta e(x)$ [55]

[55] In truth, the only non-evident aspect of the argument resides with the identities $(e^{-Cx})' = -Ce^{-Cx}$ and $(e^{-Cx})'' = (-Ce^{-Cx})' = C^2 e^{-Cx}$ with the issuing inequalities

The following (obvious) corollary to the non-vanishing theorem will be used for construction of stable symmetric μ-bubbles in sections **??**, **??**.

Uniqueness/Symmetry Corollary. *If X is compact connected, then the lowest eigenfunction f of the L is unique up to scaling. Consequently, if L is invariant under an action of an isometry group on X, then, even if X is disconnected, there exists a positive f invariant under this action.*

Exercises. (a) **Multi-Dimensional Morse Lemma.** Show that two *non-coinciding* volume minimizing hypersurfaces in the same indivisible homology integer homology class of an orientable manifold X have *empty intersection and that, consequently, volume minimizing hypersurfaces must be invariant under symmetries of* X.[56]

(b) Generalize this to μ-bubbles, that are boundaries of domains V in a Riemannian manifold X that *minimize* the functional

$$V \to vol_{n-1}(\partial V) - \int_V \mu(x)dx$$

for a smooth function $\mu(x)$. (Unit spheres $S^{n-1}\mathbb{R}^n$ are *not minimizing* μ-bubbles for $\mu = (n-1)dx$.)

(b) **Courant's Nodal Theorem.** Show that the that is the number of connected components of the complement to the "k-th nodal set", i.e. the zero set of the k-th eigenfunction of $L = L_s = \Delta + s$ on a compact connected manifold, can't have more than k connected components.

Question. Is there a counterpart to this for non-quadratic functionals in spaces of functions, or, even better, spaces of hypersurfaces?

3 Dirac Operator and Scalar Curvarure

3.1 Spin Structure, Dirac Operator, Index Theorem, $\hat{A}$-Genus, $\hat{\alpha}$-Invariant and Simply Connected Manifolds with and without $Sc > 0$

Let $L \to X$ be a real orientable vector bundle of rank r and $F \to X$ be the oriented frame bundle of L. If $r \geq 2$ the fundamental group of the fiber $F_x =$

$(e^{-Cx})'' >> e^{-Cx}$ and $(e^{-Cx})'' >> |(e^{-Cx})'|$, which *can't be done by just staring* at the exponential function. (The appearance of e^x, that is an isomorphism between the *additive* $\mathbb{R}$ and *multiplicative* $\mathbb{R}_+^\times$ with all its counterintuitive properties, is amazing here – there is nothing visibly multiplicative in Δ; besides, the geometric proof of the existence of e^x via the *conformal* infinite cyclic covering map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ and analytic continuation is non-trivial.)

The rest of the proof is geometrically effortless: you *just look* at the graph Γ_e of the function $e(x) = \exp -C \cdot dist(y,x)$ in a small R-ball $B \subset X$ outside zero set of f with the center of your choice, such that B touches this set at x_0, and let $C = C_i \to \infty$. Then you *see* a tiny region in this ball close to x_0, where Γ_e mounts above Γ_f, and you take the point in X just under the *top* of this mountain, i.e. where the distance measured vertically between the two graphs is maximal, for you $x = x_i$.

[56]This was used by Marston Morse to show that

if the $(n-1)$-dimensional homology group of some covering of a compact Riemannian n-manifold, doesn't vanish then the universal covering $\tilde{X}$ of X contains an infinite minimal hypersurface the image of which under the covering map $\tilde{X} \to X$ is compact.

Morse was concerned in his paper "*Recurrent Geodesics on a Surface of Negative Curvature*" with the case of $n = 2$ but his argument, transplanted to the environment of the geometric measure theory, applies to manifolds of all dimensions n.

$SL(k)$ is infinite cyclic and if $k \geq 3$ this group is cyclic of order 2. In both cases, F comes with a canonical double cover $\tilde{F}_x \to F_x$.

The bundle L is called *spin*, if $\tilde{F}_x \to F_x$ extends to a double cover $\tilde{F} \to F$, and smooth orientable manifold X is *spin* if its tangent T(X) bundle is spin.

Extension of the covering $\tilde{F}_x \to F_x$, if it exists, is, in general, non-unique. In the case of of $L = T(X)$ such an extension is called a *spin structure* on X.

When you speak of spin, it is common in geometry and for a good reason, to reduce the structure group of L from $SL(r)$ to $SO(r) \subset SL(r)$ and to deal with the orthonormal frame bundle $OF \to X$ instead of F, where the double cover group $\tilde{SO}(r) = OF_x$ is called *spin group* $Spin(r)$.

Example. The tangent bundle of the 2-sphere is spin, but the Hopf bundle over S^2 is not, since OF, that is S^3 for the Hopf bundle, is simply connected.

Similarly – this an exercise in elementary topology,

an oriented bundle L of rank two over an oriented surface X is spin if and only if its *Euler class*, that is the *self-intersection number of* $X \subset L$ is *even*; if X is non-orientable, then L is spin if the *second Stiefel-Whitney class*, that is the self-intersection number mod 2 of $X \subset L$ *vanishes*. In either case L is spin if and only if the Whitney sum of L with the trivial line bundle $l \simeq X \times \mathbb{R}^1$ is trivial, $L \oplus l \simeq X \times \mathbb{R}^3$.

In general,

a bundle L over a manifold X of dimension $n \geq 3$ is spin, if and only if its restriction to all surfaces in X is spin, which is again equivalent to the vanishing of the second Stiefel-Whitney class $w_2(L)$.[57]

Half-spin Bundles. There exit two (remarkable) irreducible unitary representations of the group $Spin(r)$ for $r = 2k$ of complex dimensions 2^{k-1}, say $S^\pm(r)$. Accordingly, Riemannian spin manifolds, (i.e. with spin structures on them) X support two $Spin(n)$ bundles $\mathbb{S}^\pm$ with the fibers $S^\pm(r)$ that are associated with principal spin bundle $\tilde{SO} \to X$ for the double covering representing the spin structure on X. We let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ and call this $\mathbb{S}$ the *spin bundle.*[58]

The Dirac operator

$$\mathcal{D} : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$$

is a first order differential operator constructed in a canonical geometrically invariant way universally applicable to all X (see section).

This is an *elliptic selfadjoint* operator, which interchanges $C^\infty(\mathbb{S}^+)$ and $C^\infty(\mathbb{S}^-)$ where the operators

$$\mathcal{D}^+ : C^\infty(\mathbb{S}^+) \to C^-(\mathbb{S}^-) \text{ and } \mathcal{D}^- : C^\infty(\mathbb{S}^-) \to C^-(\mathbb{S}^+)$$

are mutually adjoint.

We explained already in section 1.1.1 how, following Lichnerowicz, that the ***Atiyah-Singer index theorem*** for the Dirac's $\mathcal{D}$ and the S-L-W-(B) identity

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4} Sc,$$

[57]The value of $w_2(L) \in H^2(X;\mathbb{Z}_2)$ on a homology class $h \in H^2(X;\mathbb{Z}_2)$ is, almost by definition, equal to zero if and only if the restriction of L to surfaces in X that represent h is trivial.

Geometrically, the double cover $\tilde{F}_x \to F_x$ extends to F over the complement to a subvariety $\Sigma \subset X$ of codimension two, the homology class of which is Poincare dual to $w_2(X)$. This $\Sigma \subset X$ is waht stands on the way of applying Dirac theoretic methods to non-spin manifolds.

[58]In reality, $\mathbb{S}$ comes first and then splitting $\mathbb{S}^- \oplus \mathbb{S}^+$ follows, see section 3.2.3.

imply that

there are smooth *closed simply connected* manifolds X of all dimensions $n = 4k$, $k = 1, 2, ...,$ that admit no metrics with $Sc > 0$.

The simplest example of these for $n = 4$ is *the Kummer surface* X_{Ku} given by the equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$$

in the complex projective space $\mathbb{C}P^3$.

In fact, all complex surfaces of *even degrees* $d \geq 4$ as well as their Cartesian products, e.g $X_{\mathsf{Ku}} \times ... \times X_{\mathsf{Ku}}$ *admit no metrics with* $Sc > 0$.

Also we know that the Atiyah-Singer $\mathbb{Z}_2$-index theorem of 1971 allowed an extension of Lichnerowicz' argument to manifolds of dimensions $8k+1$ and $8k+2$, e.g. to exotic spheres in

Hitchin's theorem: there exist manifolds Σ homeomorphic (but no diffeomorphic!) to the spheres S^n, for all $n = 8k+1, 8k+2$, $k = 1, 2, 3...$, which admit no metrics with $Sc > 0$.

(What makes the *differential* structures of Hitchin's *topological* spheres Σ incompatible with $Sc > 0$ is that to these Σ are *not boundaries of spin manifolds.*)

The actual Lichnerowicz-Hitchin theorem says that if a certain topological invariant $\hat{\alpha}(X)$ *doesn't vanish*, then X *admits no metric with* $Sc > 0$, since, by the Atiyah and Singer index formulae,[59]

$$\hat{\alpha}(X) \neq 0 \Rightarrow Ind(\mathcal{D}_{|X}) \neq 0 \Rightarrow \exists \text{ harmonic spinor} \neq 0 \text{ on } X,$$

which is incompatible with the identity $\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$ for $Sc(X) > 0$

Conversely,

☞ if X is a *simply connected* manifold of *dimension* $n \geq 5$, and if $\hat{\alpha}(X) = 0$, then, an application of "thin surgery" (see section **??**) to suitably chosen generators $O(n)$- and $Sp(n)$- cobordism groups in dimensions $n \geq 5$, where these generators carry metrics with $Sc > 0$, yields[60] that X admits a metric with positive scalar curvature.

Thus, for instance

all simply connected manifolds of dimension $n \neq 0, 1, 2, 4 \mod 8$ *admit metrics with* $Sc > 0$,[61] since $\hat{\alpha}(X) = 0$ is known to vanish for these n.[62]

Topology of Scalar Flat. By *Yau's solution of the Calabi conjecture,* the Kummer surface admits a metric with $Sc = 0$, even with $Ricci = 0$, but there is

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[59]The Dirac operator is defined only on *spin* manifolds; we postulate at the present moment that $\hat{\alpha}(X) = 0$ for non-spin manifolds X.

(In fact, if $n = dim(X) = 4k$, this $\hat{\alpha}(X)$ is a certain linear combination of the *Pontryagin numbers* of X, called $\hat{A}$-*genus* and denoted $\hat{A}[X]$.

Accidentally, since all *compact homogeneous* spaces $X = G/H$, except for tori, support metrics with $Sc > 0$, Lichnerowicz' theorem says that they *either non-spin or* $\hat{A}[X] = 0$.)

[60][GL(classification) 1980], [Stolz 1992].

[61]If $dim(X) = 3$, this follows from Perelman's solution of the Poincaré' conjecture.

[62]As far as the exotic spheres Σ are concerned, these Σ admit metrics with $Sc > 0$ if and only if $\hat{\alpha}(\Sigma) = 0$, i.e. if Σ bound spin manifolds, which directly follows by the codimension 3 surgery of manifolds with $Sc > 0$ described in [SY(structure) 1979] and in [GL(classification) 1980]. Moreover, many of these Σ, e.g. all 7-dimensional ones, admits metrics with *non-negative sectional curvatures* but the full extent of "curvature positivity" for exotic spheres remains problematic (see [JW(exotic) 2008] and references therein.

no metrics with $Sc = 0$ on Hitchin's exotic spheres Σ.

In fact,

if a compact simply-connected scalar-flat manifold X of dimension ≥ 5 admits no metric with $Sc > 0$,[63] *then there are cohomology classes $\alpha \in H^2(X)$ and $\beta \in H^4(X)$, such that*

$$\langle \exp\alpha \smile \exp\beta \smile p_1(X) \rangle \neq 0,$$

where $p_1(X)$ is the first Pontryagin number, [Futaki(scalar-flat)1993], [Dessai(scalar flat) 2000].

And if X is non-simply connected then

a finite covering of X isometrically splits into the product of a flat torus and the above kind simply connected manifold,

as it follows from Cheeger-Grommol splitting theorem + Bourguignon-Kazdan-Warner perturbation theorem.

A Few Words on $n = 4$ and on $\pi_1 \neq 0$. If $n = 4$ then, besides vanishing of the $\hat{\alpha}$-invariant (which is equal to a non-zero multiple of the first Pontryagin number for $n = 4$), positivity of the scalar curvature also implies the vanishing of the *Seiberg-Witten invariants* (See lecture notes by Dietmar Salamon, [Salamon(lectures) 1999]; also we say more about it in section **??**).

If X is a closed spin manifold of dimension $n \geq 5$ with the fundamental group $\pi_1(X) = \Pi$, then, again by an application of the thin surgery,

the existence/non-existence of a metric g on X with $Sc(g) > 0$ is an invariant of the spin bordism class $[X]_{sp} \in bord_{sp}(\mathsf{B}\Pi)$ in the classifying space $\mathsf{B}\Pi$,

where, recall, that (by definition of "classifying") the universal covering of $\mathsf{B}\Pi$ is contractible and $\pi_1(\mathsf{B}\Pi) = \Pi$. [64]

There is an avalanche of papers, most of them coming under the heading of "Novikov Conjecture", with various criteria for the class $[X]_{sp}$, and/or for the corresponding homology class $[X] \in H_n$ ($\mathsf{B}\Pi$) (not) to admit g with $Sc(g) > 0$ on manifolds in this class, where these criteria usually (always?) linked to generalized index theorems for twisted Dirac operators on X with several levels of sophistication in arranging this "twisting".

Yet, despite the recent progress in this direction for dimensions 4 and 5[65] proving/disproving the following for $n \geq 4$ remains beyond the present day means. [66]

(Naive?) **Conjecture.**[67] If a closed oriented n-manifold X admits a continuous map to an *aspherical space* B,[68] such that the image of the rational fundamental homology class of $[X]_{\mathbb{Q}}$ in the rational homology[69] $H_n(\mathsf{B};\mathbb{Q})$ doesn't vanish, then

[63] These X are Ricci flat, [Bourguignon (these) 1974], [Kazdan[complete 1982].

[64] See lecture notes [Stolz(survey) 2001].

[65] See [Chodosh-Li(bubbles) 2020], [G(aspherical) 2020] and section **??**

[66] The case $n = 3$ follows from the topological classification of compact 3-manifolds X with positive scalar curvature *these are connected sums of quotients of spheres S^3 and products $S^2 \times S^1$ by finite isometry groups* [GL(complete) 1983],[Ginoux(3d classification) 2013].)

[67] This, as many other our conjectures, is based on a limited class of examples with no idea of where to look for counter examples.

[68] That is the universal covering of B is contractible, hence, B is $\mathsf{B}(\Pi)$ for $\Pi = \pi_1(\mathsf{B})$.

[69] Bernhard Hanke pointed out to me that non-vanishing of this image in homology with *finite coefficients*, e.g. for finite groups Π, may also prohibit $Sc > 0$, but this remains obscure even on the level of conjectures.

X admits no metic g with $Sc(g) > 0$.

(We shall describe the status of this problem together with the *Novikov conjecture* in section **??**.)

3.2 Unitary Connections, Twisted Dirac Operators and Almost Flat Bundles Induced by ε-Lipschitz Maps

We turn now to twisted Dirac operators $\mathcal{D}_{\otimes L}$ that act on tensor products $\mathbb{S} \otimes L$ for vector bundles $L \to X$ with linear (most of the time, unitary) connections ∇.

One can think of such a $\mathcal{D}_{\otimes L}$ as an *infinitesimal family* of $\mathcal{D}$-s parametrized by L, where the action takes place along $\mathbb{S}$ with no differentiation in the L-directions.

For instance if $L = (L, \nabla)$ is a trivial flat bundle, $L = X \times L_0$, where L_0 is a vector space (fiber), then $C^\infty(\mathbb{S} \otimes L) = C^\infty(\mathbb{S}) \otimes L_0$ and the $\mathcal{D}_{\otimes L}$ doesn't act on L at all:

$$\mathcal{D}_{\otimes L}(f \otimes l) = \mathcal{D}(f) \otimes l \text{ for all vectors } l \in L_0.$$

In general, the $\mathcal{D}_{\otimes L}$ differs from that in the flat case by a zero order term, which is, bounded by the curvature of L and, strictly speaking, is defined only locally, where the bundle L is topologically trivial. But exactly this impossibility of global comparison of $\mathcal{D}_{\otimes L}$ on $C^\infty(\mathbb{S} \otimes L)$ with $\mathcal{D}$ on $C^\infty(\mathbb{S}) \otimes L_0$ creates a correction term in the index formula.

This correction, unlike the background operator $\mathcal{D}$, carries no subtle topological information about X, such as $\hat{A}(X)$ for $n = 4k$, which is not a homotopy invariant for $n > 4$ and even less so about $\hat{\alpha}(X)$ for $n = 8k + 1, 8k + 2$, which is not even invariant under p.l. homeomorphisms and which is far removed from anything even remotely, geometric about X, while the topology (Chern classes) of L reflects the area-wise size of the metric g on X, which, in turn, influences *homotopy theoretic properties* of X linked to the fundamental group.

The following definition gives you a fair idea of what kind of properties these are.

Profinite Hypersphericity. A Riemannian n-manifold X is *profinitely hyperspherical* if

given an $\varepsilon > 0$, there exists an orientable finite covering $\tilde{X} = \tilde{X}_\varepsilon$, which admits an *$\varepsilon$-Lipschitz* map between [70] $\tilde{X} \to S^n$ of *non-zero* degree.

This property of *compact* manifolds (the definition of this hypersphericity extends too open manifolds) doesn't depend on the Riemannian metric on X. Moreover

If X_1 is profinitely hyperspherical and X_2 admits a map of non-zero degree to X_1 then, obviously, X_2 is also profinitely hyperspherical; in particular, this property is a *homotopy invariant* of X.

Example. Manifolds X, which admit *locally expanding self-maps* $E : X \to X$, e.g. the n-torus $\mathbb{T}^n$, where the endomorphism $t \mapsto Nt$ locally expands the metric by N, are profinitely hyperspherical.

[70] A map between metric spaces, $f : X \to Y$, is ε-Lipschitz if $dist_Y(f(x_1), f(x_2)) \le \varepsilon dist_X(x_1, x_2)$ for all $x_1, x_2 \in X$. For instance, "1-Lipschitz" means "distance non-increasing". ε-Lipschitz for smooth maps f between Riemannian manifolds is equivalent to $\|d(f(x)\|\varepsilon$, $x \in X$.

Indeed, such an E defines a *globally* expanding homeomorphism, call it $\hat{E}$, from X onto a finite covering $\tilde{X} = \tilde{X}(E)$, where the inverse map $\hat{E}^{-1} : \tilde{X} \to X$ contracts as much as E expands.

Therefore, the covering corresponding to the i-th iterate of E comes with an ε_i-Lipschitz map to X, where $\varepsilon_i \to 0$ for $i \to \infty$ and compositions of these with a map $X \to S^n$ of non-zero degree also have $deg \neq 0$, while their Lipschitz constants go to zero.[71]

Now, if you recall Atiyah-Singer index theorem for the twisted Dirac operator and $\mathcal{D}_{\otimes L}$ and the (untwisted) S-L-W-(B) formula $\mathcal{D}^2 = \nabla\nabla^* + \frac{1}{4}Sc$[72] you arrive at the following.

[Sc $\not>$ 0]: **Provisional Proposition.**[73] *Compact orientable*[74] *profinitely hyperspherical spin manifolds X of all dimensions n support no metrics with $Sc > 0$.*

Proof. This is obvious once said. Indeed, a *simple special case of the Atiyah-Singer index theorem* says that,

if a complex vector bundle L of rank k over a compact orientable *spin* Riemannian manifold X of dimension $n = 2k$, has *non-zero Euler* (*Chern*) *number*, that is the self-intersection index of the zero section $X \to\hookrightarrow \underline{L}$, then

the twisted Dirac $D_{\otimes L} : C^\infty(\mathbb{S} \otimes L) \to C^\infty(\mathbb{S} \otimes L)$ has non-zero kernel, for all linear connections in L, provided,

the number k is *odd*, and the restriction of L to the complement to a point in X is a *trivial bundle.*[75]

Then, by elementary algebraic topology,

the $2k$-sphere supports a complex vector bundle of rank k, say $\underline{L} \to S^{2k}$, which has *non-zero* Euler (Chern) number,

and

bundles $L = f^*(\underline{L}) \to X$ induced from $\underline{L}$ by continuous maps $f : X \to S^{2k}$ have their Euler numbers $e(L) = deg(f)e(\underline{L})$.

It follows that finite coverings $\tilde{X}_\varepsilon$ of X admit smooth ε-Lipschitz-maps $f_\varepsilon : \tilde{X}_\varepsilon \to S^n$ with arbitrary small ε and such that the twisted Dirac operators $\mathcal{D}_{\otimes L_\varepsilon}$ on $\tilde{X}_\varepsilon$ for $L_\varepsilon = f_\varepsilon^*(\underline{L})$, have non-zero kernels for all connections in L_ε.

Apply this to connections ∇_ε in L_ε induced by f_ε from a fixed smooth linear (unitary if you wish) connection $\underline{\nabla}$ in $\underline{L} \to S^{2k}$, let $\varepsilon \to 0$ and observe that, since the maps f_ε converge to constant ones on all unit balls in $\tilde{X}_\varepsilon$, the bundles $(L_\varepsilon, \nabla_\varepsilon)$ converge to trivial ones with trivial flat connections on all balls. Therefore the difference between the Dirac operator $\mathcal{D}_{\otimes L_\varepsilon}$ and $\mathcal{D}$ twisted with the trivial flat bundle L_{flat} of rank k becomes arbitrary small for $\varepsilon \to 0$, and the S-L-W-(B) formula applied to $\mathcal{D}_{L_{flat}}$ shows that $\inf_X Sc(X) = \inf_{\tilde{X}} Sc(\tilde{X}_\varepsilon) \leq 0$.

[71] Further examples of this phenomenon and issuing topological obstruction to $Sc > 0$ for manifolds with *residually finite* fundamental groups are given in [GL(spin) 1980] under the heading of "enlargeability". Since the residual finiteness condition was eventually lifted, this terminology now applies to a broader class of manifolds, including spaces X the *universal covers* of which admit contracting self-maps of positive degrees, see section**??**

[72] This ∇ stands for the Levi-Civita connection in the spin bundle.

[73] This will be significantly generalized later on.

[74] If X is non-orientable, take an oriented double cover of it.

[75] These are minor technical conditions, the role of which is to avoid undesirable consequences of possible cancellation in the index formula (see section **??**). For instance if X can be embedded or immersed into $\mathbb{R}^{2k+1}$, or if it admits a metric with positive scalar curvature then even k is allowed. (Observe in passing that these X are spin.)

This completes the proof for $n = 4l + 2$ and the general case follows by (shamelessly) taking the product $X \times \mathbb{T}^{3n+2}$.

Well..., this is convincing but it is not quite a proof. We still have to define $\mathcal{D}_{\otimes L}$ and to make sense of the "difference" between the operators $\mathcal{D}_{\otimes L_\varepsilon}$ and $\mathcal{D}_{\otimes L_{flat}}$ that are defined in *different spaces*. We do all this below closer to the end of this section.

Why Spin? The essential new information delivered by $\mathcal{D}_{\otimes L}$ does not visibly depend on the spin structure (unlike to how it is with the Dirac operator $\mathcal{D}$ itself).[76]

However, one doesn't know how to get rid of the spin condition, in the cases where it appears irrelevant. For instance, there is no single known area-wise bound on the size of a non-spin manifold with a large scalar curvature.[77]

All in all, although "twisted Dirac" proofs are short and simple, their nature remains obscure.

Partly, this is why we explain below with such a care standard "trivial" properties of the "twist" $\mathcal{D} \rightsquigarrow \mathcal{D}_{\otimes L}$, hoping this may help us to visualize *something* behind this "trivial" that makes the Dirac's $\mathcal{D}$ work in geometry, "something", which is only tangentially related to the Dirac operator itself and, if untangled from $\mathcal{D}$ with its bondage to spin, would open up new possibilities.

3.2.1 Recollection on Linear Connections and Twisted Differential Operators

A connection in a smooth fibration $L \to X$ is a *retractive homomorphism* from the tangent bundle $T(L)$ to the subbundle $T_{vert} = T_{ver}(L) \subset T(L)$ of the vectors tangent to the fibers of L.[78]

Denote this by

$$\hat{\nabla} : T(L) \to T_{vert} \subset T(L),$$

and observe that $\hat{\nabla}$ is uniquely defined by its kernel, that is what is called a horizontal subbundle, $T_{hor} = T_{hor}(L) \subset T(L)$ that is complementary to T_{vert} such that $T(L) = T_{vert} \oplus T_{hor}$.

If L is a trivia?l (split) fibration $L = X \times L_0$, then it comes with the trivial or split flat connection, where T_{hor} is the bundle of vectors tangent to the graphs of constant maps $X \to L_0$, $l \in L$.

A connection is called *flat* at $x_0 \in X$ if, over a neighbourhood $U \subset X$ of x, it is *isomorphic* to the trivial flat connection on $X \times L_{x_0}$, for the fiber L_{x_0} of L over x_0.

---

[76]Sometimes, e.g. for lower bounds on the (area) norms of differentials of maps $X \times X_{\mathsf{kum}} \to S^n$, $n = dim(X)$, for metrics g on $X \times X_{\mathsf{kum}}$ with large scalar curvatures, the spin is irreplaceable.

[77]In truth, this applies only to *non-spin*$^{\mathbb{C}}$ manifolds, where spin$^{\mathbb{C}}$ means that the second Stiefel-Whitney class is equal to the mod 2 reduction of the Chern class of a complex line bundle $L \to X$.

Such bounds are available for spin$^{\mathbb{C}}$ manifolds. For instance (a special case of) *Min-Oo extremality/rigidity theorem* says that

if the scalar curvature a Riemannian metric g on on $\mathbb{C}P^m$ is (non-strictly) greater than that of the Fubini-Study metric, $Sc(g) \geq Sc(g_{FuSt})$, and $area_g(S) \geq area_{g_{FuSt}}(S)$ for all smooth surfaces $S \subset \mathbb{C}P^m$, than $g = g_{FuSt}$.

(The complex projective spaces $\mathbb{C}P^m$ are non-spin for even m, yet they are all spin$^{\mathbb{C}}$).

[78]Here, "retractive" means being the identity on T_{vert}.

If the fibration L carries a fiber-wise geometric structure $\mathscr{S}$, say, linear, affine, unitary, etc, then "flat" signifies that the implied isomorphism, that is a fiber preserving diffeomorphism $L_{|U} \to U \times L_x$, preserves $\mathscr{S}$, i.e. it is fiber-wise linear, affine, unitary, etc.

A connection $\hat{\nabla}$ in L is called $\mathscr{S}$: linear, affine, unitary, etc if, for each $x \in X$, there exist a flat $\mathscr{S}$-connection $\hat{\nabla}_{x,flat}$ *adapted to* $\hat{\nabla}$ at x, i.e.such that the restriction of $\hat{\nabla}_{x,flat}$ to the fiber $L_x \subset L$, denoted $(\hat{\nabla}_{x,flat})_{|L_x}$ is equal to $\hat{\nabla}_{|L_x}$.

Twisting Differential s. A first order differential between (sections of) vector bundles (linear fibrations) K_1 and K_2 over a manifold X, is a linear map

$$D : C^\infty(K_1) \to C^\infty(K_2),$$

such that the value $Df(x) \in K_2$ depends only on the differential $df(x) : T_x(X) \to T_{f_x}(K_1)$ for all $x \in X$.

For instance, a linear connection in L defines a differential , denoted just ∇, from L to the bundle $Hom(T(X), L) = T^*(X) \otimes L$, that is the composition of the differential $df : T(X) \to T(L)$ with $\hat{\nabla} : T(L) \to T_{vert}$ combined with the canonical identifications of all (vertical) tangent spaces of the fiber L_x with L_x itself.

Such a ∇ uniquely determines (linear) $\hat{\nabla}$, it is also called "connection". where the values $\nabla f(\tau)$ at tangent vectors τ are written as (covariant) derivatives $\nabla_\tau f$.

Basic Example. If ∇ is the flat split connection in $X \times L_0$, then this is applies to sections $X \to X \times L_0$, that are the graph of maps $f : X \to L_0$, as the ordinary differential $df : T(X) \to L_0$.

If a section $f : X \to L$ vanishes at a point $x \in X$, then, clearly, $\nabla f(x) = \nabla_{flat} f(x)$ for all *nabla.*

It follows that the difference between two connections in L, $\nabla_1 - \nabla_2$, is a it zero order defined by a homomorphism $\Delta = \Delta_{1,2} : L \to Hom(T(X), L)$, that can be thought of as a $Hom(L, L)$-valued 1-form on X.

Thus any ∇ in a flat, e.g. split, bundle is $df + \Delta$.

If ∇ is a *flat split connection*, in $L = X \times L_0$, then the twisted $D_{\otimes L} : C^\infty(K_1 \otimes L) \to C^\infty(K_1 \otimes L)$ is defined via the identity $C^\infty(K \otimes L_{split}) = C^\infty(K) \otimes L_0$, as it was explained above for the Dirac operator.

If ∇ is *flat*, then $D_{\otimes\nabla} = D_{\otimes(L,\nabla)}$ is defined on all neighbourhoods where this connection splits and local $\Rightarrow$ global by locality of differential s.

Finally, for *general* (L, ∇), the twisted $D_{\otimes\nabla}(\psi)$ for sections $\psi : X \to K_1 \otimes L$ is determined by its values at all points $x \in X$ that are defined as follows

$$D_{\otimes\nabla}(\psi)(x) = D_{\otimes\nabla_{x,flat}}(\psi)(x)$$

for flat connections $\nabla_{x,flat}$ adapted to ∇ at x.

Since the difference $\nabla - \nabla_{flat}$ is a *zero order* for all connections ∇ in flat split bundles $L = (X \times L_0 \nabla_{flat})$, the same is true for D twisted with ∇: the difference

$$\Delta_\otimes = D_{\otimes\nabla} - D_{\otimes\nabla_{flat}}$$

is a zero order – "vector potential" in the physicists' parlance.

A similar representation $D_{\otimes\nabla} = D_{\otimes\nabla_{flat}} + \Delta_\otimes$ for topologically non-trivial bundles L is achieved as follows.

Let $L^\perp \to X$ be a bundle complementary to L such that the Whitney sum of the two bundles topologically splits,

$$L \oplus L^\perp = L^\oplus \simeq X \times (L_0 \oplus L_0^\perp)$$

and let $\nabla^\perp$ be an arbitrary connection in $L \oplus L^\perp$ and

Define the connection $\nabla^\oplus = \nabla + \nabla^\perp$ in $L^\oplus$ by the rule

$$\nabla^\oplus_\tau(l + l^\perp) = \nabla_\tau l + \nabla^\perp_\tau l^\perp$$

and observe that the $\nabla^\oplus$-twisted operator $D_{\otimes\nabla^\oplus}$, that maps the space

$$C^\infty(K_1 \otimes L^\oplus) = C^\infty(K_1 \otimes L) \oplus C^\infty(K_1 \otimes L^\perp)$$

to

$$C^\infty(K_2 \otimes L^\oplus) = C^\infty(K_2 \otimes L) \oplus C^\infty(K_2 \otimes L^\perp)$$

respects this splitting:

$$D_{\otimes\nabla^\oplus} = D_{\otimes\nabla} \oplus D_{\otimes\nabla^\perp}.$$

Thus, if not the $D_{\otimes\nabla}$ itself, then its $\oplus$-sum with another is

$$D_{\otimes\nabla_{flat}} + \text{zero order}\ .$$

3.2.2 [Sc $\not> 0$] for Profinitely Hyperspherical Manifolds, Area Decreasing Maps and Upper Spectral Bounds for Dirac Operators

Conclusion of the proof of provisional proposition [**Sc $\not> 0$**] *from 3.2.* Return to the bundles $L_\varepsilon = f^*(\underline{L}) \to X$ induced by smooth ε-Lipschitz maps $f : X \to S^n$, $n = dim(X) = 4l + 2$, with non-zero degrees and $\varepsilon \to 0$ from a complex vector bundle $\underline{L} \to S^n$, with the Euler number $e(\underline{L}) \neq 0$.

Let $\underline{L}^\perp \to S^{2k}$ be a bundle *complementary* to $\underline{L} \to S^{2k}$, i.e. the sum $\underline{L} \oplus \underline{L}^\perp$ is a trivial bundle, endow $\underline{L}$ and $\underline{L}^\perp$ with a connections $\underline{\nabla}$ and $\underline{\nabla}^\perp$ and let $\nabla^\oplus_\varepsilon$ be the connection on the (topologically trivial!) bundle

$$L^\oplus_\varepsilon = f^*(\underline{L} \oplus \underline{L}^\perp)$$

induced from $\underline{\nabla}^\oplus = \underline{\nabla} \oplus \underline{\nabla}^\perp$, where the latter is defined by the component-wise differentiation rule:

$$\underline{\nabla}^\oplus(\phi, \psi) = (\underline{\nabla}\phi, \underline{\nabla}^\perp\psi) \text{ for sections } (\phi, \psi) = \phi + \psi : S^n \to \underline{L} \oplus \underline{L}^\perp.$$

Then (see the proof of [**Sc $\not> 0$**]) the twisted Dirac operator decomposes into the sum of (essentially) untwisted $\mathcal{D}$ and a zero order (vector potential)

$$\mathcal{D}_{\otimes\nabla^\oplus_\varepsilon} = \mathcal{D}_{\otimes\nabla_{flat}} + \Delta_\varepsilon$$

where ∇_{flat} is the flat split connection in the bundle $L^\otimes_\varepsilon$ with the splitting induced by f_ε from a splitting of $\underline{L} \oplus \underline{L}^\perp$, obviously (but most significantly), $\Delta_\varepsilon \to 0$ for $\varepsilon \to 0$.

Now, the (untwisted) S-L-W-(B) formula, applied to $\mathcal{D}_{\otimes\nabla_{flat}}$ says that

$$\mathcal{D}^2_{\otimes\nabla^\oplus_\varepsilon} = \nabla_{flat,\mathbb{S}}\nabla^*_{flat,\mathbb{S}} + \frac{1}{4}Sc + \Delta^\square_\varepsilon,$$

where $\nabla_{flat,\mathbb{S}}$ denotes the flat connection $\nabla_{flat,\mathbb{S}}$ in the twisted spin bundle associated with ∇_{flat}.

The correction term $\Delta_\varepsilon^\square$ in this formula is a first order differential (it depends on how you trivialise the bundle $\underline{L}\oplus\underline{L}^\perp$) which tends to 0 for $\varepsilon\to 0$,

$$\Delta_\varepsilon^\square \to 0 \text{ for } \varepsilon\to 0.$$

A priori, the ε-bound on the differential of f_ε doesn't make the coefficients of the $\Delta_\varepsilon^\square$ small, but an obvious smoothing allows an approximation of f_ε by maps their derivatives of which of *all orders converging to* 0.

Because of this, we may assume $\Delta_\varepsilon^\square\to 0$ in the *strongest conceivable sense*, while al is needed is that $\Delta_\varepsilon^\square\to 0$ becomes negligibly small compare to $\nabla_{flat,\mathbb{S}}\nabla^*_{flat,\mathbb{S}}+\frac{1}{4}Sc$, which implies *strict positivity* of the $\mathcal{D}^2_{\otimes\nabla^\oplus_\varepsilon}=\nabla_{flat,\mathbb{S}}\nabla^*_{flat,\mathbb{S}}+\frac{1}{4}Sc+\Delta_\varepsilon^\square$, for ε much smaller than the lower bound $\sigma=\inf_{x\in X}Sc(X,x)>0$.

Thus, the condition $Sc(X)>0$ leads to a contradiction with the index formula, which in this case, as we already know from the proof of **[Sc $\not>$ 0]** yields non-zero harmonic ∇_ε-twisted, hence $\nabla^\oplus_\varepsilon$ twisted, spinors, because the subbundle $\underline{L}\subset\underline{L}\subset$ is invariant under the parallel transport by the connection $\underline{\nabla}^\oplus=\underline{\nabla}\oplus\underline{\nabla}^\perp$, by the very definition of the sum of connections and this property is inherited by the induced connection $\nabla^\oplus_\varepsilon$.

This concludes the proof of **[Sc $\not>$ 0]** for $n=4l+2$ and, as we have already explained, the general case follows by stabilization $X\rightsquigarrow X\times\mathbb{T}^{3n+2}$.

Area Contraction instead of Length Contraction. Say that X is $\wedge^2$-*profinitely hyperspherical* if, instead of ε-Lipschitz property of maps $f_\varepsilon\tilde{X}_\varepsilon\to S^n$ of no-zero degree, we require that the second exterior power of the differential of f_ε is bounded by ε^2,

$$\|\wedge^2 df_\varepsilon(x)\|\le\varepsilon^2.$$

Geometrically, this means that f_ε decreases the areas of the smooth surfaces in X by factor ε^2, (This, obviously, is satisfied by ε-Lipschitz maps.)

It is clear, heuristically, that the Dirac operator *twisted with* ∇_ε in this case, similarly how it is for ε-Lipschitz maps, is *close to the untwisted* $\mathcal{D}$; this

rules out positive scalar curvature for $\wedge^2$-profinitely hyperspherical manifolds.

However, the above proof with the complementary bundle $L^\perp$ doesn't apply here; to justify heuristics, one has to pursue algebraic similarity between ∇ and the ordinary differential d a step further.

This can be done by pure thought, on the basis of general principles only, (no tricks like $L^\perp$) but writing down this "thought" turned out more space and time consuming than what is needed for (a few lines of) the twisted version of the S-L-W-(B) formula, as we shall see in section 3.2.4.

So, we conclude here with two remarks.

(i) It is unknown if "length contractive" is *topologically* more restrictive than "area contractive".

For instance one has no idea if there exist $\wedge^2$-profinitely hyperspherical manifolds which are *not* profinitely hyperspherical.

(ii) Representation of ∇-twisted differential operators by vector-potentials Δ in larger bundles has further uses, such as *Vafa-Witten's lower bounds on the spectra of Dirac operators.* For instance,

if a compact Riemannian spin n-manifold X admits a distance decreasing map to S^n of degree d, then the number N of eigenvalues λ of the Dirac on X in the interval $-C_n\le\lambda\le C_n$ satisfies $N\ge d$, where $C_n>0$ is a universal constant.[79]

[79]See §6 in [G(positive) 1996] for related spectral geometric inequalities.

3.2.3 Clifford Algebras, Spinors, Atiyah-Singer Dirac Operator and Lichnerowicz Identity

The Dirac on $\mathbb{R}^n$ is a particular first order differential , which acts on the space of smooth $\mathbb{C}^N$-valued functions,

$$\mathcal{D} : C^\infty(\mathbb{R}^n, \mathbb{C}^N) \to C^\infty(\mathbb{R}^n, \mathbb{C}^N),$$

where $N = 2^{\frac{1}{2}n}$ for even n and $N = 2^{\frac{1}{2}(n-1)}$ for odd n and where essential properties of this $\mathcal{D}$ are as follows.

I. *Ellipticity.* The $\mathcal{D}$ is an *elliptic*, which means that the initial value (Cauchy) problem for the equation $Df = 0$ is *formally uniquely solvable* for *all* initial data on *all* smooth hypersurfaces in $\mathbb{R}^n$, where "formally" can be replaced by "locally" in the real analytic case.

Basic Example. The Cauchy-Riemann (system of two) equation(s) $D_{CR}f = 0$ for maps $f : \mathbb{R}^2 \to \mathbb{C}^1$, defines conformal orientation preserving maps $\mathbb{R}^2 \to \mathbb{C}$. These are called *holomorphic* if $\mathbb{R}^2$ is "identified" with $\mathbb{C}$, where the ambiguity inherent in this identification is responsible for spin.

D_{CR} *is elliptic*: real analytic functions locally uniquely extend from real analytic curves in $\mathbb{C}^1$ to holomorphic functions.

Let us describe ellipticity in linear algebraic terms applicable to all (systems of) partial differential equations of first order for maps between smooth manifolds, $f : X \to Y$. Such a system, call it S, is characterised by subsets in the spaces of linear maps between the tangent spaces of X and Y at all $x \in X$ and $y \in Y$, denoted $\Sigma_{x,y} \subset Hom(T_x \to T_y)$, where $T_x = \mathbb{R}^n$, $n = dim(X)$, where $T_y = \mathbb{R}^m$, $m = dim(Y)$ and where f *satisfies* S if $df(x) \in \Sigma_{x,f(x)}$ for all $x \in X$.

Let $R_L : Hom(\mathbb{R}^n, \mathbb{R}^m) \to Hom(L, \mathbb{R}^m)$ denote the restriction of linear maps to $\mathbb{R}^m$ from $\mathbb{R}^n$ to linear subspaces $L \subset \mathbb{R}^n$, that is $R_L : h \mapsto h_{|L} : L \to \mathbb{R}^m$.

Call a smooth submanifold $\Sigma \subset Hom(\mathbb{R}^n, \mathbb{R}^m)$ *elliptic* if the map R_L *diffeomorphically* sends Σ *onto* $Hom(L, \mathbb{R}^m)$ for all *hyperplanes* $L \subset \mathbb{R}^n$.

Now, a PDE system S is called *elliptic* if the subsets

$$\Sigma_{x,y} \subset Hom(T_x \to T_y) = H_{n,m} = Hom(\mathbb{R}^n, \mathbb{R}^m)$$

are elliptic for $x \in X$ and $y \in Y$.

Put it another way, let $K_p \in H_{n,m}$, $p \in \mathbb{R}P^{n-1}$, be the family of m-dimension linear subspaces that are the kernels of the linear maps $R_{L_p} : H_{n,m} \to H_{n-1,m,p} = Hom(L_p, \mathbb{R}^m)$ parametrized by the projective space $\mathbb{R}P^{n-1}$ of hyperplanes $L = L_p \subset \mathbb{R}^n$. Then ellipticity says that

Σ *transversally* intersect K_p at a single point for all $p \in \mathbb{R}P^n$.

Finally, back to the linear case, observe that systems $Df = 0$ for maps

$$f : \mathbb{R}^n \to \mathbb{R}^m, \ x \in \mathbb{R}^n$$

are depicted by *linear* subspaces

$$\Sigma = \Sigma_x \subset Hom(T_x(\mathbb{R}^n), \mathbb{R}^m = T_0(\mathbb{R}^m)), x \in \mathbb{R}^n$$

and ellipticity says in these terms that

firstly, $dim(\Sigma) = n$

secondly

$\bigcirc$ *the linear maps* $h : T_x(\mathbb{R}^n) \to \mathbb{R}^m$ *have* $rank(h) = n$ *for all non-zero* $h \in \Sigma$.
and finally
differential operators between sections of vector bundles over a smooth manifold X *are elliptic if these properties are verified locally over small neighbourhoods of all points in* X.

Exercises. (a) Twisting with ∇. Show that

D is elliptic $\Rightarrow$ $D_{\otimes\nabla}$ is elliptic:

twisting with connections doesn't hurt ellipticity.

(b) Symmetric but non-Elliptic. Figure out what makes the exterior differential

$$d : C^\infty(\bigwedge^k(T(X)) \to C^\infty(\bigwedge^{k+1}(T(X))$$

on $(2k+1)$-dimensional manifolds *non-elliptic.*

II. *Symmetry and Spinors.* The Dirac operator $\mathcal{D}$ on $\mathbb{C}^\infty(\mathbb{R}^n, \mathbb{C}^N)$, is $Spin(n)$-*invariant*, where

$\bullet_1$ $Spin(n)$ denotes the double cover of the special orthogonal group $SO(n)$,

$\bullet_2$ the group $Spin(n)$ acts on $\mathbb{R}^n$ via the (2-sheeted covering map) homomorphism $Spin(n) \to SO(n)$,

$\bullet_3$ the action of $Spin(n)$ on $\mathbb{C}^k$, called *spin representation*, is *faithful*: it *doesn't factor* through an action of $SO(n)$,[80]

$\bullet_4$ "invariant" here means *equivariant* under the (diagonal) action of $Spin(n)$ on the space of maps $\psi : \mathbb{R}^n \to \mathbb{C}^N$, that is

$$g(\psi)(x) = g(\psi(g(x)), \ g \in Spin(n),$$ [81]

and "*equivariant*" says that

$$\mathcal{D}(g(\psi)) = g(\mathcal{D}(\psi)).$$ [82]

Cauchy-Riemann Example. The group $Spin(2)$ diagonally acts on on maps $f : \mathbb{R}^2 \to \mathbb{C}^1$, where all actions (representations) of $Spin(2) = \mathbb{T} = U(1) \subset \mathbb{C}^\times$ on $\mathbb{C}^1$ are possible: these are $t(z) = t^m z$, $m = ... - 1, 0, 1, 2, ...$ (There are no such possibilities for for $n > 1$.)

The corresponding operators $\bar\partial = \bar\partial_m$ are all *locally non-canonically* isomorphic (this makes them often confused in the literature), but this m (spin quantum number), becomes the major feature of the $\bar\partial_i$ globally, where it controls its very existence and its index.

III. *Spin Representations and Clifford Algebras* $Cl_n = Cl(V)$.[83]The lowest dimension complex vector space, where $Spin(n)$, can linearly *faithfully* act is

[80]The spin representation, as we shall explain below, is *irreducible* for odd n and it splits into *two irreducible half-spin representations* for even n. There are no *faithful* representations of $Spin(n)$ in lower dimensions (except for $n = 1, 2$), where, apparently, this faithfulness is necessitated by ellipticity of $\mathcal{D}$.

[81]To visualize this, think of the graphs $\Gamma_\psi \subset \mathbb{R}^n \times \mathbb{C}^N$ moved by the diagonal actions of $g \in Spin(n)$ on this product.

[82]This Dirac operator has "constant coefficients", which means is *invariant under parallel translations* t_y of $\mathbb{R}^n$ that act on our maps: $D(t_y(\psi)) = t_y(D(\psi))$ for $(t_y(\psi))(x) = \psi(x + y)$, $x, y \in \mathbb{R}^n$.

[83]The basic reading on this subject matter is the book [Lawson&Michelsohn(spin geometry) 1989] and a (very) brief outline of the main points is contained in [Min-Oo(K-Area) 2002], [Min-Oo(scalar) 2020].

$\mathbb{C}^{2^k}$ for $k = \frac{1}{2}n$ for n even and $k = \frac{1}{2}(n-1)$ for odd n, where such an action (representation) is obtained by realizing $Spin(n)$ as a subgroups in the multiplicative semigroups of the *Clifford algebra*, denoted $Cl_n = Cl(\mathbb{R}^n) = Cl(\mathbb{R}^n, -\sum_1^n x_i^2)$.

Recall that Cl_n is an *unital*[84] *associative algebra* A over the field of real numbers with a distinguished *Clifford basis* that is linear subspace $V = V_{Cl} \subset Cl_n$ *endowed with a Euclidean structure*, that is represented by a *negative definite* quadratic form.[85]

We denote the *Clifford product* by $a_1 \cdot a_2$ an let "1" stand for the unit in A.

(There is nothing especially exciting about Cl_n understood as "just an algebra", especially if you tensor it with $\mathbb{C}$, which we do at the end of the day anyway. For instance, we shall see it presently, $Cl \otimes \mathbb{C}$ is isomorphic to a matrix algebra for even n and to the sum of two matrix algebras for odd n.

What gives to a particular favour to Cl_n is the distinguished linear subspace $V \subset Cl_n$, which, on the one hand, *generates all* of Cl_n, on the other hand, the matrices corresponding to all $v \neq 0$ in V, have maximal possible ranks, since all non-zero $v \in V$ are *invertible* in the multiplicative semigroup $CL_n^\times$. This "maximal rank property" is exactly what makes the Dirac operator *elliptic* and, because of this, so powerful in the Riemannian geometry.)

The fundamental feature of the pair (A, V) is that $A = Cl(V)$ is *functorially* determined by V:

isometric embeddings $V_1 \to V_2$ canonically extend to monomorphisms $A_1 \to A_2$.

where this Clifford functor is uniquely characterised by the following two properties.

A. *$V = V_{Cl}$ is a Basis in A.* The subspace V generates A as an $\mathbb{R}$-algebra.

B. *Specification.* The algebra $Cl_1 = Cl(\mathbb{R}^1)$ is isomorphic to $(\mathbb{C}, i\mathbb{R})$, for $i = \pm\sqrt{-1}$.

(It is impossible to *mathematically*, distinguish i and $-i$; this unresolvable ±i-ambiguity is grossly amplified, at least psychologically,, when it comes to spinors. [86])

In simple words, the Clifford squares of all unit vectors $v \in V$ are equal to -1, or, equivalently,

$$v \cdot v = -||v||^2 = \langle v, v \rangle \text{ for all } v \in V.$$

A&B. *Anti-commutativity.* The Clifford product is *anti-commutative on orthogonal vectors.*

$$v_1 \cdot v_2 = -v_2, v_1, \text{ for } \langle v_1, v_2 \rangle = 0.$$

[84]This means possessing a unit in it

[85]It is negative to agree with the Laplacian $\sum_i \partial_i^2$, which is a negative operator.

[86]To be blameless, write $\pm i$ (even better, $\{\pm i, \mp i\}$) and never dare utter "left ring ideal" and "right group action", even in absence of left-handed (left-minded?) persons. (Defending such an action by *biological molecular homochirality* and *parity violation by weak interactions* is not recommended for being politically incorrect.)

Jokes apart, arbitrary terminological conventions presented as mathematical definitions sow confusion and undermine "rigor" in mathematics.

Who are the lucky ones who are able to tell if $f \circ g$ means $f(g(x))$ rather than $g(f(x))$ or vice versa?

Can *encoding formulas by Peano's integers*, e.g. in the proof of *Gödel's incompleteness theorem*, be accepted as "logically rigorous", unless you face the issue of "directionality" inherent in the decimal representation of integers?

Indeed, since $||v_1 - v_2||^2 = ||v_1 + v_2||^2$ for orthogonal vectors, bilinearity of the the Clifford product implies that

$$0 = (v_1 - v_2)^2 - (v_1 + v_2)^2 = -v_1 \cdot v_2 - v_2 \cdot v_1 + v_1 \cdot v_2 + v_2 \cdot v_1 = 2(v_1 \cdot v_2 + v_2 \cdot v_1).$$

Exercise. Show that

$$v_1 \cdot v_2 + v_2 \cdot v_1 = -2\langle v_1, v_2\rangle \text{ for all } v_1, v_v \in V.$$

IV. *Groups $Pin(n)$ and $Spin(n)$ and G_n.* The group $Pin(n)$ is defined in Cl_n-terms as the subgroup of the multiplicative semigroup of $Cl_n^\times \subset Cl_n$ multiplicatively generated by the unit vectors $v \in V \subset Cl_n$.

The subgroup $Spin(n) \subset Pin(n)$ consists of the products of *even numbers* of unit vectors from V. [87]

Existence & Uniqueness. Let us explain why the algebra $Cl(\mathbb{R}^n)$, if exists at all, is large enough to (multiplicatively) contain the group $Spin(n)$ that double covers the special orthogonal group $SO(n)$. [88]

Observe that the *Clifford relations* [89]

$$[\mathbf{Cl}] \qquad e_i \cdot e_j = -e_j \cdot e_i \text{ and } e_i^2 = -1$$

for an orthonormal frame $\{e_i\} \subset V$, $i = 1, ..., n$,

on the one hand, imply $v_1 \cdot v_2 + v_2 \cdot v_1 = -2\langle v_1, v_2\rangle$ for all $v_1, v_v \in V$, hence,

fully characterize Clifford's algebras,

on the other hand, define

a finite group G_n of order 2^{n+1} that is a central extension of $\mathbb{Z}_2^n$,

with an additional generator (central element) c of order 2 and the following relations,

$$[\mathbf{Cl_c}] \qquad ce_i = e_i c,\ c^2 = 1,\ e_i e_j = ce_j e_i \text{ and } e_i^2 = c.$$

where the central element c in G_n corresponds to $-1 \in Cl_n$.

Non-triviality of this G_n is apparent, since letting $c = 1$ defines a *surjective* homomorphism $G_n \to \mathbb{Z}_2^n$ with kernel $\mathbb{Z}_2$.

(What is not immediately apparent, is a pretty combinatorics of shuffling indices in $e_{i_1} e_{i_2} ... e_{i_m} \in G_n$, $i_1 < i_2 < ... < i_m$, under multiplication by e_k, which is rightly appreciated by people working on quantum computers.)

One look at G_n is sufficient to make it obvious that there is a *homomorphism from G_n to the multiplicative (semi) group $Cl_n^\times$* of the Clifford algebra (with the image in $Pin(n) \subset Cl^\times$), such that

[87] This parallels the definition of $SO(n) \subset O(n)$ as the subgroup consisting of products of *even numbers* of reflections of $\mathbb{R}^n$. In fact, $Spin(n)$ equals the connected component of the identity in $Pin(n)$ and $Pin(n)/Spin(n) = O(n)/SO(n) = \mathbb{Z}_2 = \{-1, 1\}$.

[88] To appreciate non-triviality of the problem, try to construct geometrically more than two, say three, anti-commuting linear isometric involutions represented by reflections around linear subspaces in some Euclidean space.

[89] This must be written in Clifford's unpublished note *On The Classification of Geometric Algebras* see [Diek-Kantowski (Clifford History)1995] for further references.

the algebra homomorphism from the *real group algebra* $\mathbb{R}(G_n)$[90] to Cl_n associated with this group homomorphism $G_n \to Cl_n^\times$ is *surjective* and the kernel of this homomorphism is defined by the relation $c = -1$, that is

$$Cl_n = \mathbb{R}(G_n)/(c+1),^{91}$$

Amazingly, nowhere, except for a few papers on quantum computers, G_n is called "finite Clifford group",[92] while the authors of the only mathematical papers found on the web (unless I missed some) call G_n a "*Salingaros vee group.*"[93]

The structure this "vee group" $G_,$, which tells you everything about Cl_n, is transparently seen in the combinatorics of its multiplication table, where $g \in G$ are written as *lexicographically ordered products* of e_i and (if it is there) c. Here are a few relevant properties of G_n.

All elements in G_n have orders 2 and/or 4.

The commutator subgroup $[G_n, G_n] = \{g_1 g_2 g_1^{-1} g_2^{-1}\}$ equals to the 2-element (central) subgroup $\{1, c\}$.

If n is even, it coincides with the center of G_n;

$$center(G_n) = [G_n, G_n] = \{1, c\}$$

If n is odd, the center has order 4. For instance $G_1 = \mathbb{Z}_4$; in general, the extra central element for $n = 2k+1$ is the product $e_1 e_2, ..., e_n$.

If n is even, then the number $N_{conj}(G_n)$ of the conjugacy classes of G_n is $2^n + 1$ where 2^n of them comes from $\mathbb{Z}_2^n$ and the extra one is that of c. If n is odd, there are $2^n + 2$ classes, where centrality of $e_1 e_2, ..., e_n$ is responsible for the additional one.

V. *Representations of the Group G_n.* The space $\Psi_n = \mathbb{C}(G_n) = \mathbb{C}^{G_n}$ of *complex* functions on G splits into the sum $\Psi_n = \Psi_n^+ \oplus \Psi_n^-$, where Ψ_n^+ consists of *c-symmetric functions* $\psi(g)$ that are *invariant* under the action of the central

---

[90] $\mathbb{R}(G)$ is the space of formal linear combinations $\sum_{g \in G} c_g g$ with the obvious product rule, where the identity element $id \in G$ serves as the unit of this algebra.

Alternatively, $\mathbb{R}(G)$ is defined as the algebra of linear operators $\mathbb{R}(G)$ on functions $\psi(g)$ that is generated by translations on the space of functions on G, for $\psi(g) \mapsto \psi(g'g)$, $g' \in G$.

The same space $\mathbb{R}(G) = G^{\mathbb{R}}$ of functions on G with the action of G by $\psi(g) \mapsto \psi(g'g)$ is called (not very inventively) the *regular $\mathbb{R}$-representation* of G, where just "regular representation" stands for regular $\mathbb{C}$-representation.

[91] Recall that $c \in G_n \subset \mathbb{R}(G_n)$ is the central involution in G_n and "1" is the unit in the algebra $\mathbb{R}(G_n)$ that is represented by the unit function, where $(c+1) \subset \mathbb{R}(G_n)$ denotes the ideal generated by $c+1 \in \mathbb{R}(G_n)$. (The quotient algebra $\mathbb{R}(G_n)/(c+1)$ has the same underlying linear space as the group algebra $\mathbb{R}(G_n/(c))$, for the normal subgroup $(c) \subset G_n$ generated by c, but multiplicatively $\mathbb{R}(G_n)/(c+1)$ is much different from the (commutative) group algebra of $G_n/(c) = \mathbb{Z}_2^n$.)

[92] The terms "Clifford group", sometimes "naive Clifford group", are reserved for the subgroup G of the multiplicative semigroup of Cl, the action of which on Cl by conjugation for $a \mapsto g \cdot a \cdot g^{-1}$ preserves V.

[93] See [AbVaWa(Clifford Salingaros Vee)2018] for more general definitions and references to the the original 1981-82 papers by Nikos Salingaros. (I don't know what is written in these papers, since these are not openly accessible on the web.)

Also, amazingly, no survey or tutorial on Clifford algebras I located on the web makes any use or even mentions G_n. Possibly, there is something about it in textbooks, but none seems to be openly accessible.

$c \in G_n$, i.e. $\psi(g) = \psi(cg)$ and where the functions $\psi \in \Psi_n^-$ are *antisymmetric*, $\psi(cg) = -\psi(g)$.

The space Ψ_n^+ obviously identifies with the space $\mathbb{C}(\mathbb{Z}_2^n)$ of functions on the Abelian group $\mathbb{Z}_2^n$, where action of G_n factors through the homomorphism $G_n \to \mathbb{Z}_2^n$.

Since the commutator subgroup of G_n is equal to $\{1, c\}$, all 1-dimensional representations of G_n are contained in Ψ_n^+.

Frobenius

Now, the *number one theorem* in the representation theory of finite groups reads:[94]

the regular representation of G uniquely decomposes into the sum of sub-representations $G^{\mathbb{C}} \bigoplus_i R_i^2$, $i = 1, 2, ..., N = N_{irrd}(G)$, where each R_i^2 is (non-canonically) isomorphic to the sum of k_i-copies of an irreducible representation R_i of dimension k_i and where every irreducible representations of G is isomorphic to one and only one of R_i.

Accordingly, the group algebra of G (the same linear space $G^{\mathbb{C}}$, but now with the group algebra structure) *decomposes into the sum of matrix algebras*

$$\mathbb{C}(G) = \bigoplus_i End(\mathbb{C}^{k_i}).$$

This is an exercise in linear algebra. What is less obvious is that

The number $N_{irrd}(G)$ of mutually non-isomorphic irreducible complex representations of G is equal to the number of the conjugacy classes in G.

$$N_{irrd}(G) = N_{conj}(G) \text{ for all finite group } G.$$

Consequently,

the sum of the squares of the dimensions of the irreducible representations of G is equal to the order of the group G,

$$\sum_i k_i^2 = card(G).^{95}$$

If we apply this to G_n for $n = 2k$, we shall see that, besides the one dimensional representations, this group has a *single irreducible* one of dimension 2^k, call it S_n, which enters the regular representation with multiplicity 2^k.

Now, clearly,

the 2^k-multiple S_n-summand of the regular representation is exactly the space Ψ_n^- of antisymmetric functions ψ on G_n.

Equally clearly,

the space of antisymmetric functions $\psi(g) = -\psi(cg)$ on G_n (here we speak of $\mathbb{R}$-valued functions ψ) is G_n-equivariantly isomorphic to Cl_n.

---

[94]This must be attributed to Frobenius (1896), since it follows by his character theory, see `file:///Users/misha/Downloads/Curtis2001_Chapter_RepresentationTheoryOfFiniteGr.pdf`

Unfortunately, this theorem has no name an can't be *instantaneously* found on Google.

[95]See `https://projecteuclid.org/download/pdf_1/euclid.lnms/1215467411` and the character sections in `https://web.stanford.edu/~aaronlan/assets/representation-theory.pdf` and `https://arxiv.org/pdf/1001.0462.pdf`.

VI. *Clifford Conclusion.* Since the Clifford algebra Cl_n is, as an algebra, generated by $G_n \subset Cl_n$, the representation S_n of G_n in $\mathbb{C}^{2^k}$, that is a multiplicative homomorphism $G_n \to End(C^{2^k})$, extends to an algebra homomorphism $Cl_n \to End(C^{2^k})$; hence, to

an irreducible representation of $Pin(N)$ in $\mathbb{C}^{2^k}$,

which extends (irreducible!) representation S_n of $G_n \subset Pin(n)$.

This is called *the spin representation* and still denoted S_n.

Why Clifford Algebra? Why algebras are needed here at all?

What we used for the construction of the spin representation S_n of $Pin(n)$ in $\mathbb{C}^{2^k}$ for even $n = 2k$ are the two following simple, not to say "trivial", but indispensable (are they?) algebra theoretic facts.

(i) The linear actions of $Pin(n)$ and G_n on the space Ψ_n^- (and also on Cl_n) generate the same subalgebras of operators on this space.

(ii) If an algebra A of operators on a linear space M, e.g. $M = \mathbb{C}^{N^2}$, is isomorphic to the (full matrix) algebra of endomorphisms of another space,

$$A \simeq End(L),$$

then M is A-equivariantly isomorphic to $End(L)$ for, say "left", action of the algebra $End(L)$ on itself.

(Also we were jumping back and forth between $\mathbb{R}$-linear and $\mathbb{C}$-linear spaces and actions, but with nothing non-trivial happening on the way.)

The correspondence $\Phi : L \rightsquigarrow A = End(L)$ *is a functor* from the category of vector spaces over $\mathbb{R}$ to the category of unital $\mathbb{R}$-algebras, but L can be reconstructed from $End(L)$ only *up to a homothety $l \mapsto rl$*, $r \in \mathbb{R}^\times$,where the projective space $P = L/\mathbb{R}^\times$ can be identified with the space of *maximal left ideals* in $End(L)$[96]

(Because of this ambiguity, one can't globally define the Dirac operator on a non-spin manifold X, because there is no vector bundle that would support $\mathcal{D}$.

And although the the projectivized spin bundle $\mathcal{PS} \to X$ with a real projective space as the fiber is still there, this fibration *admits no continuous section $X \to \mathcal{PS}$* – non-zero second Stiefel-Whitney class is an obstruction to that.)

VII. *Subgroup $G_n^0 \subset G$ and half-Spin Representations.* Let $\mathbb{Z}^n \to \mathbb{Z}_2$ be the (only) non-zero homomorphism, which is invariant under permutations of e_i, denote by $deg : G_n \to \mathbb{Z}_2 = \{-1, 1\}$ be the composition of this with the homomorphism $G_n \to \mathbb{Z}_2^n$ which sends $c \to 1$ and let G_n^0 be the *kernel of this "degree" homomorphism.*

In terms of Cl_n, this is the intersection of the subgroups G_n and $Spin(n)$ in $Pin(n)$,

$$G_n^0 = G_n \cap Spin(n) \subset Pin(N) \subset Cl_n.$$

Exercise. Show that G_{n+1}^0 is isomorphic to G_n.

Hint. Send $e_i \in G_n$, $i = 1, ..., n$, to the products $e'_{n+1}e'_i$ for $e'_1, ..., e'_{n+1} \in G_{n+1}$.

Let $\hat{e} = e_1 e_2 ... e_n$ and let us split the representation space $L = \mathbb{C}^{2^k}$ of S_n for even $n = 2k$ into ± 1-eigenspaces of $\hat{e}$, $L = L^+ \oplus L^-$

If n is even then this $\hat{e}$ anti-commute with all e_i, that is $\hat{e}e_i = ce_i\hat{e}$.

It follows that, for $n = 2k$,

[96]Left ideals $I \subset End(L)$ corresponds to linear subspaces $L_I \subset L$, such that $a \in I \Leftrightarrow a_{|L_I} = 0$.

all e_i that act via S_n on L send $\mathbb{L}^+ \leftrightarrow L^-$

and

the restriction of the representation S_n on $L = \mathbb{C}^{2^k}$ from the group G_n to the subgroup $G_n^0 \subset G_n$ sends $L^+ \to L^+$ and $L^- \to L^-$.

Furthermore, since the representation S_n is irreducible for G_n,

the representations $S_n^\pm$ on $L^\pm$ are irreducible for G_n^0.

Extend these representations by linearity to the subalgebra $Cl_n^0 \subset Cl_n$ generated by $G_n^0 \subset Cl_n$, observe that Cl_n^0 contains the group $Spin(n)$ and restrict from Cl_n^0 to $Spin(n)$. Thus, for $n = 2k$, we obtain

two faithful irreducible representations, called half-spin representations $S^\pm$ of the group $Spin(n)$ of dimensions 2^{k-1}.

Remark/Question. The above shows that a linear space of dimension $< 2^k$ can't have $2k$ anti-commuting anti-involutions. *Is there a direct geometric proof of this?*

(The answer must be known to some people.)

VIII. *Clifford's $Spin(n)$ Covers $SO(n)$,* What remains (for $n = 2k$) to show is that this $Spin(n)$, which is defined as the subgroup of the multiplicative group of the Clifford algebra Cl_n generated by products of even numbers of unit vectors $V \in V \subset Cl_n$, double covers the special orthogonal group $SO(n)$.

To do this we define an orthogonal (i.e. linear isometric) action of all of $Pin(n) \supset Spin(n)$ on the (n-dimensional Euclidean) subspace $V = V_{Cl} \subset Cl_n$ as follows.

Let $\alpha : Cl_n \to Cl_n$ be the automorphism that linearly extends $v \mapsto -v$ on $V \subset Cl_n$ and let

$$p(v) = \alpha(p) \cdot v \cdot p^{-1} \text{ for } v \in V \text{ and } p \in Pin(n).$$

It is clear that if p is a unit vector in V, then the transformation $v \mapsto p(v)$ sends V to itself by reflection in the hyperplane $p^\perp \subset V$ normal to p. (You can think of this $p \in Pin(n)$ as the square root of such a reflection.[97])

Since α is an *automorphism* of the Clifford algebra, the map from $Pin(n)$ to the group $O(n)$, regarded as the group of linear Euclidean isometries of $V = (V, \sum_i x_i^2)$, is a *homomorphism* of groups, which sends $Spin(n)$ onto this $SO(n)$.

To conclude, we need to show that the kernel of the homomorphism $Pin(n) \to O(n) \subset End(V)$ is equal to $\{1, -1\} \subset Cl(n)$, which is done by induction on n starting from $Pin(1) = \mathbb{Z}_4 = \{1, i, -1, i\}$ and $\alpha(i) = -1$, and using the following.
.

Lemma. If $\alpha(p) \cdot v \cdot p^{-1} = v$ for a unit vector $v \in V$, then p is contained in the subalgebra $Cl(v^\perp) \simeq Cl_{n-1}$ generated by the hyperplane $v^\perp \subset V$.

Proof. Decompose the Clifford algebra into sum of four linear subspaces,

$$Cl_n = A_0 \oplus v \cdot A_1 \oplus A_1 \oplus v \cdot A_0,$$

where $A_0 \subset Cl(v^\perp)$ is equal to the +1-eigenspace of α, i.e. where $\alpha(a) = a$, and $A_1 \subset Cl(v^\perp)$ is the −1-eigenspace.

[97] If you omit α, the resulting transformation square$v \mapsto pvp^{-1}$ becomes *minus reflection* in $p^\perp$. Thus, if n is odd, all of $P(n)$ ends up in $SO(n)$.
Since one wants $Pin(n)$ to cover the full orthogonal group $O(n)$ one brings in this α.

Observe that all a_0 in A_0 are linear combinations of products of of *even numbers* of vectors from V, while all $a_1 \in A_1$ are combinations of *odd* products.

Now, by keeping track of parity of products we see that the relation $\alpha(p) \cdot v \cdot p^{-1} = v$ divides into two equalities,

$$(a_0 + v \cdot a_1') \cdot v = v \cdot (a_0 + v \cdot a_1') \text{ and } (a_1 + v \cdot a_0') \cdot v = -v \cdot (a_1 + v \cdot a_0'),$$

which imply that $a_1' = 0$ and $a_0' = 0$.

Indeed, since v *commutes* with a_0 and anti-commute with a_1,

$$(a_0 + v \cdot a_1') \cdot v = v \cdot (a_0 + v \cdot a_1') \Rightarrow v \cdot a_1' \cdot v = v \cdot v \cdot a_1' \Rightarrow -v \cdot v \cdot a_1' = v \cdot v \cdot a_1',$$

and $v \cdot v = -1 \Rightarrow a_1' = 0$.

Similarly, one shows that also $a_0' = 0$ and lemma follows.

Finally, we are through with *even n*:

the double cover group $Pin(n) \to O(n)$ for $n = 2k$ comes with a faithful irreducible complex representation $S_n = S_{2k}$ in $\mathbb{C}^{2^{2k}}$, called *spin representation*.[98]

The restriction of S_n to $Spin(n) \subset Pin(n)$, that is the double cover of $SO(n) \subset O(n)$, splits into the sum $S_n = S^+_{\frac{1}{2}n} \oplus S^-_{\frac{1}{2}n}$ of two complex conjugate[99] representations, called *half spin representations*.[100]

IX. *About Odd n.* A quick way to arrive at the spin representation S_{2k} of the group $Spin(n)$ in $\mathbb{C}^{2^k}$ for $n = 2k + 1$ is by imbedding $Spin(n) \hookrightarrow Cl^\times_{n-1}$ and then restricting the spin representation $S_{n-1=2k}$ the Clifford algebra Cl_{n-1} to the so embedded $Spin(n) \subset Cl^\times_{n-1}$.

To achieve this, we start, somewhat paradoxically, with a (somewhat artificial) embedding $Cl_{n-1} \to Cl_n$ that sends Cl_{n-1} onto the *even part* $Cl^0_n \subset Cl_n$, that is the +1-eigen space of the automorphism $\alpha : Cl_n \to Cl_n$ of the Clifford algebra induced by the central symmetry $v \mapsto -v$ of the Clifford base subspace $V = V_{Cl} \subset Cl_n$.

It is (nearly) obvious that Cl^0_n is a *subalgebra* in Cl_n and that (this is slightly less obvious) this subalgebra is isomorphic to Cl^0_{n-1}.

To prove the latter, imbed Cl_{n-1} to Cl_n with the image Cl^0_n as follows.

Map the orthogonal complement $v^\perp \subset V \subset Cl_n$ of a unit vector $v \in V$ back to Cl^0_n by $e \mapsto v \cdot e$ for all $e \in v^\perp$ and show that this map extends to an *injective algebra homomorphism* $Cl_{n-1} = CL(v^\perp) \to Cl^0_n$.

All you need for this is an (easy) check up of the identities

$$(v \cdot e)^2 = -1 \text{ and } v \cdot e \cdot v \cdot e' = -v \cdot e' \cdot v \cdot e$$

for all $v, v' \in v^\perp$ (implicit in the above exercise about the homomorphism $G_n \to G^0_{n+1}$).

Finally, since that the group $Spin(n)$, by its very definition, is contained in $(Cl^0_n)^\times$ it goes to $Cl^\times_{n-1}$ by inverting the isomorphism $Cl_{n-1} \to Cl^0_n$. QED.

IX. *Spin Representation of Pin(n) fo Odd n.* Just for completeness sake, let us explain why

[98]There in no faithful representation of $Pin(n)$ in a lower dimensional space, since even the subgroup $G_n \subset Pin(n)$ admits no such representation.

[99]We didn't prove these are complex conjugate but this follows from their construction

[100]Arriving at this point took unexpectedly long – not a page or two as I had expected.

the complexified Clifford algebra Cl_{2k+1}, which has dimension 2^{2k+1}, is isomorphic to the sum of of two matrix algebras $End(\mathbb{C}^{2^{k-1}})$.

Recall that the group G_{2k+1} has exactly two irreducible non-one-dimensional representations, where the sum of their dimensions is 2^k.

In fact both representation must have the same dimensions, because of another fundamental (also nameless?) theorem:

the dimensions of all irreducible representations of a finite group G divide the order order of G.[101])

Therefore the non-Abelian part of the group algebra of G_{2k+1}, hence the Clifford algebra Cl_{2k+1} is the sum of two matrix algebras of the same dimension. QED.

As a consequence, we get

two irreducible representations of the group $Pin(2k+1)$ of dimensions 2^{k-1}.

X. *Example: Pauli "Matrices".* The first interesting case of S_n is an irreducible 2-dimensional complex representation S_2 of the group G_2, hence of $Pin(2)$, where he latter is the non-trivial central $\mathbb{Z}_2$-extension of the circle group $\mathbb{T}^1 = U(1)$.

To obtain such a representation all you need is to find two *anti-commuting anti-involutions* σ_1, σ_2 of $\mathbb{C}^2$ corresponding to the generators of e_1, e_2 of the (sub)group $G_2 \subset Cl_2 \supset Pin(2)$.

This is kindergarten math: let $\underline{\sigma}_1, \underline{\sigma}_2$ be anti-commuting *involutions* of the real plane $\mathbb{R}^2$, namely reflections in two lines with the 45° between them. Their compositions, $\underline{\sigma}_1\underline{\sigma}_2$ and $\underline{\sigma}_2\underline{\sigma}_1$ are rotations by 90° in the opposite directions, thus $\underline{\sigma}_1$ and $\underline{\sigma}_2$ anti-commute:

$$\underline{\sigma}_1\underline{\sigma}_2 = -\underline{\sigma}_2\underline{\sigma}_1.$$

The *anti-involutions* $\sigma_1 = i\underline{\sigma}_1$ and $\sigma_2 = i\underline{\sigma}_2$, $i = \sqrt{-1}$, of $\mathbb{C}^2$ with $\sigma_3 = \sigma_1\sigma^2$ coming along are your Pauli guys.

$\hat{\otimes}$-*Remark.* This example can be amplified by taking tensor products, for

$$Cl_{m+n} = Cl_m \hat{\otimes} Cl_n,$$

where $\hat{\otimes}$ stands for $\mathbb{Z}_2$-*graded* tensor product, for which

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{deg(a')\deg(b)}(a \cdot a') \otimes (b \cdot b').$$

This allows a sleek construction of the spin representations but it doesn't make it more geometrical than the one via G_n.

X. *Clifford Moduli and Dirac operators.* It is convenient at this point to call a linear space L with an action S of the Clifford algebra $Cl(V)$ *"Clifford V-module"* and to write just S instead of $L = (L, S)$.

Also observe at this point that the actual action of $V \subset Cl(V)$ on such an S reduces to a single linear map $cl : V \otimes S \to S$, where the Clifford action is denoted by "$\cdot$",

$$cl(v \otimes s) = v \cdot s.$$

[101]See https://math.stackexchange.com/questions/243221/proofs-that-the-degree-of-an-irrep-divides-the-order-of-a-group for several proofs.

Now, recall, that such a map defines (and is defined by) a first order differential on the space of smooth maps $\psi : V \to S$, denoted $D : C^\infty(V,S) \to C^\infty(V,S)$, that is the composition of this cl with the differential $d : C^\infty(V) \to C^\infty(H)$ for $H = Hom(V,S)$ as we explained in the previous section.

Since $v^2 = -||v||^2$, $v \in V$, all $v \neq 0$ are *invertible* in the multiplicative semigroup $End^\times(S)$; thus,

the linear operators D are elliptic for all Cl_n-moduli S.

These D can be also defined with orthonormal frames $\{e_i\} \subset V$ by

$$D(\psi) = \sum_{i=1}^{n} e_i \cdot \partial_i \psi,$$

which shows that $D^2 = -\Delta^2 = -\sum_i \partial_i \partial_i$, since

$$D^2 = \sum_{i,j} e_i \partial_i e_j \partial_j = \sum_{i,j} e_i \cdot e_j \partial_i \partial_j = \sum_i e_i \cdot e_i \partial_i \partial_j + \sum_{i \neq j} (e_i \cdot e_j \partial_i \partial_j + e_j \cdot e_i \partial_j \partial_i) = -\sum_i \partial_i \partial_i.$$

or, where the symmetry is apparent, by integration over the unit sphere $\{||v|| = 1\} \subset V$,

$$D(\psi(v)) = const_n \int_{||v||=1} v \cdot \partial_v \psi(v) dv,$$

and if $V = \mathbb{R}^n$.

It follows by a simple symmetry consideration or by a one line computation that

$$D^2 = -\Delta = -\sum \partial_{e_i}^2.$$

Exercise. Prove, directly that

$$\int_{||w||=1} w \cdot \partial_w dw \int_{||v||=1} v \cdot \partial_v dv = const_n \int_{||v||=1} -\sum \partial_v^2 dv.$$[102]

Dirac operator $\mathcal{D}$ on Spinors. This $\mathcal{D}$ is defined with the spinor representation S_{2k} in $\mathbb{C}^{2^k}$,

$$\mathcal{D} : \mathbb{S}_{2k} \to \mathbb{S}_{2k},$$

where the "spinors" are understood here as smooth maps $\psi : \mathbb{R}^n \to S_{2k}$ for $n = 2k$ or $n = 2k+1$.

If n is even, the spin representation splits into two adjoint representation, accordingly $\mathbb{S}_{2k} = \mathbb{S}^+_{2k} \otimes \mathbb{S}^-_{2k}$, where the action of the Clifford algebra interchanges $\mathbb{S}^+_{2k} \leftrightarrow \mathbb{S}^-_{2k}$. It follows that $\mathcal{D} = \mathcal{D}^+ \otimes \mathcal{D}^-$ for

$$\mathcal{D}^+ : \mathbb{S}^+_k \to \mathbb{S}^-_k \text{ and } \mathcal{D}^- : \mathbb{S}^-_k \to \mathbb{S}^+_k,$$

the operators $\mathcal{D}^+$ and $\mathcal{D}^-$ are *formally adjoint.*

XI. *$\mathcal{D}$ on Manifolds and Schrödinger-Lichnerowicz-Weitzenböck-Bochner Formula.* Let X be a Riemannian spin manifold of dimension n and let $\mathcal{S}_{2k}$ be the spin bundle associated with the principal spin bundle over X that is the double cover of the orthonormal frame bundle, where this cover is what defines the spin structure on X.

[102] I myself got lost in this calculation.

Let ∇ be the Riemannian Levi-Civita connection, which is, observe, simultaneously and coherently defined on all bundles associated with the tangent bundle. (It is irrelevant whether his is done via the principal $O(n)$-bundle or $Spin(n)$-bundle.)

We know (this applies to all bundles with connections, see section3.2.1) that this ∇ decomposes at each point $x \in X$ into the sum $\nabla = \nabla_{flat} + E_\nabla$, where $E_\nabla = E_{\nabla,x}$ a smooth endomorphism of $\mathcal{S}_{2k}$ over a (small) neighbourhood of $x \in X$, which *vanishes* at x.

This allows a "functorial transplantation" of the above $\mathcal{D} = \mathcal{D}_{flat}$ to an $\mathcal{D}_\nabla$ on the space $\mathbb{S}$ of sections of the bundle $\mathcal{S}_{2k}$, where $\mathcal{D}_\nabla$ infinitesimally agree with $\mathcal{D}$ at each point $x \in X$,

$$\mathcal{D}_\nabla = \mathcal{D}_{flat} = E_{\mathcal{D}},$$

for a smooth (locally defined) endomorphism $E_{\mathcal{D}} = E_{\mathcal{D},x}$ of S_{2k}, which *vanishes at x.*

If n is even, then, clearly, $\mathcal{S}_{2k} = \mathcal{S}^+_{2k} \oplus \mathcal{S}^-_{2k}$ and the operator $\mathcal{D}_\nabla$, denoted just $\mathcal{D}$ from now on, splits accordingly: $\mathcal{D} = \mathcal{D}^+ \otimes \mathcal{D}^-$ for

$$\mathcal{D}^+ : \mathbb{S}^+_k \to \mathbb{S}^-_k \text{ and } \mathcal{D}^- : \mathbb{S}^-_k \to \mathbb{S}^+_k,$$

where the operators $\mathcal{D}^+$ and $\mathcal{D}^-$ are *formally adjoint.*

Since the $\mathcal{D}^2_{flat} = \mathcal{D}^2_{flat,x}$, which is defined locally, is equal to $-\Delta = \nabla_{flat}\nabla^*_{flat}$ at each x, the square of $\mathcal{D} = \mathcal{D}_\nabla$, now *globally*, satisfies what is called "*Weitzenboeck identity*" (this applies to all "geometric operators")

$$\mathcal{D}^2 = \nabla\nabla^* + R_{\mathcal{D}},$$

where ∇^* stands for the differential *formally adjoint* to ∇ (this spinor ∇ acts from (sections of) $\mathcal{S}_{2k}$ to (sections of) the bundle $Hom(T(X), \mathcal{S}_{2k})$, where $R_{\mathcal{D}} = R_{\nabla,\mathcal{S},\mathcal{D}}$ is a selfadjoint endomorphism of the bundle $\mathcal{S}_2k$.

It would be nice to continue this line of this reasoning and see without calculation that, why this $R_{\mathcal{D}}$, is a multiplication by a scalar. Regretfully, I couldn't do this and have resort to the (standard) symbolic manipulations. [103]

To perform these we, observe that the bundle of the Clifford algebras $Cl(T_x(X))$ acts on $\mathcal{S}_{2k}$, where this action agrees with the covariant differentiation ∇ in $\mathcal{S}_{2k}$. Then we see that, for all orthonormal framed of tangent vectors e_i, $i = 1, ..., n$, the Dirac operator is

$$\mathcal{D} = \sum_i e_i \cdot \nabla_i \text{ for } \nabla_i = \nabla_{e_i}$$

and

$$\mathcal{D}^2 = \sum_{i,j} e_i \cdot \nabla_i e_j \cdot \nabla_j = \sum_{i,j} e_i \cdot e_j \cdot \nabla_i \nabla_j = \sum_{i=j} e_i \cdot e_j \nabla_i \nabla_j + \sum_{i \neq j} e_i \cdot e_j \cdot \nabla_i \nabla_j =$$

$$= -\sum_i \nabla_i \nabla_i + \sum_{i<j} e_i \cdot e_j \cdot (\nabla_i \nabla_j - \nabla_j \nabla_i) = \nabla\nabla^* + \sum_{i<j} e_i \cdot e_j \cdot R_{\mathcal{S}}(e_i \wedge e_j),$$

[103]It doesn't feel right when you can't do mathematics solely in your mind: a piece of paper for this purpose is no more satisfactory than a digital computer.

where $R_{\mathcal{S}}(e_i \wedge e_j)$ is the curvature of the bundle $\mathcal{S}_{2k}$ written as a 2-form on X with values in $End(\mathcal{S}_{2k})$.

The first term in this formula, $\nabla\nabla^*$ is the *Bochner Laplacian* in the bundle $\mathcal{S}_{2k}$ which a selfadjoint non-strictly positive .

This $\nabla\nabla^*$, regarded as a real operator, is characterized by the integral identity

$$\int_X \langle \nabla\nabla^* \phi(x), \psi(x)\rangle dx - \langle \nabla\phi(x), \nabla\psi(x)\rangle dx = 0$$

which is satisfied, whenever one of the two functions has a compact support.

The proof of this formula, which makes sense and is valid for all vector bundles with orthogonal connections, contains two ingredients, where the first *algebraic* one consists in finding

an invariant representation of the integrant as the differential of an $(n-1)$ form and the second ingredient is, of course, Green's formula.

In fact, all algebra needed in our is the following Leibniz formula for the Laplace Beltrami

$$\Delta\langle\phi(x), \phi(x)\rangle = \langle \nabla\nabla^*\phi(x), \phi(x)\rangle + \langle \phi(x), \nabla\nabla^*\phi(x)\rangle + 2\langle \nabla\phi(x), \nabla\phi(x)\rangle.$$

This implies all positivity of $\nabla\nabla^*$ we need.

Next, turn to the curvature term $\mathcal{R} = \sum_{i<j} e_i \cdot e_j \cdot R_{\mathcal{S}}(e_i \wedge e_j)$ in the above Bochner – Weitzenböck formula for $\mathcal{D}^2$, that is an endomorphism $\mathcal{R} : \mathcal{S}_{2k} \to \mathcal{S}_{2k}$, which, being self adjoint as a real , is represented by a family of *symmetric* linear operators $\mathcal{R}_x : (\mathcal{S}_{2k})_x \to (\mathcal{S}_{2k})_x$, $x \in X$, in the fibers $(\mathcal{S}_{2k})_x \simeq S_{2k} = \mathbb{C}^{2^k} = \mathbb{R}^{2^{k+1}}$, while the curvature operators $R_{\mathcal{S}}(v_1 \wedge v_2)$ themselves are *antisymmetric*, for all bivectors $v_1 \wedge v_2 \in \bigwedge^2 T_x(x) = \bigwedge^2 \mathbb{R}^n$, since they represent the action of the Lie algebra of the group $Spin(n) \subset SO(2^{k+1})$ on $\mathbb{R}^{2^{k+1}}$.

In fact, a closer look shows[104] that

$$R_{\mathcal{S}}(v_1 \wedge v_2) = \frac{1}{2}\sum_{i<j} \langle R(v_1 \wedge v_2)(e_i), e_j\rangle e_i \cdot e_j$$

where $R(e_i \wedge e_j) : T(X) \to T(X)$ is the curvature of our connection ∇ as it acts on the tangent bundle of X.

(The presence of "$\frac{1}{2}$" agrees with the idea of the bundle $\mathcal{S}_{2k}$ being a "the square root" of the tangent bundle $T(X)$, hence having one half of the curvature of X, which is clearly seen for the Hopf complex line bundle $L \to S^2$, where $L \otimes_{\mathbb{C}} L$ is isomorphic to the tangent bundle $T(S^2)$ and, accordingly, $curv(L) = \frac{1}{2}curv(S^2)$.)

Everything up to this point was applicable to an arbitrary Euclidean vector bundle $T \to X$ of rank m with a spin structure, i.e. a double cover of the associate principal $SO(m)$-bundle and the action of bundle of the Clifford algebras $Cl(T)$ on the corresponding spin bundle with the fibers $\simeq \mathbb{C}^{2^l}$, for $m = 2l$ or $m = 2l + 1$, where the Dirac operator defined via an orthogonal connection in T enjoys all formulas we have presented so far.

But in the case of $T = T(X)$ the symmetries of the curvature tensor encoded by Bianchi identities allow the following simplification of $\mathcal{R}$.

[104]See formula 4.37 on p. p110 in [Lawson&Michelsohn(spin geometry) 1989].

Lichnerowitz' Identity.

$$\mathcal{R} = \sum_{i<j} e_i \cdot e_j \cdot R_{\mathcal{S}}(e_i \wedge e_j) = \frac{1}{2}\sum_{i<j,k<l} \langle R(e_k \wedge e_l)(e_i), e_j \rangle e_i \cdot e_j = \frac{Sc}{4};$$

Thus,

$$\mathcal{D}^2\phi(x) = \nabla\nabla^*\phi(x) + \frac{1}{4}Sc(X,s)\phi(x) \text{ for all sections } \phi : X \to \mathcal{S}_{2k}.$$

Why it is so. The action of the linear group $GL(n)$ on the space $RCT \simeq \mathbb{R}^{\frac{n^2(n^2-1)}{12}}$ of (potential) Riemannian curvature tensors splits into three irreducible representations $RCT = Sc \oplus Ri \oplus W$, where Sc is the trivial one dimensional representation, Ri the space of traceless symmetric 2-forms and W the space of Weyl tensors. Accordingly, every smooth n-manifolds X supports three (curvature) differential operators of the second order from the space G_+ of positive definite quadratic differential forms g on X to the space of sections of vector bundles over X associated with the tangent bundle $T(X)$ via one of these representations, such that

$\bullet_{lin}$ these operators are linear in the second derivatives of g;

$\bullet_{inv}$ these operators are equivariant under the action of the diffeomorphism group $diff(X)$ operator and where

these operators and their scalar multiples are the only ones with such quasi-linearity and invariance properties

On the other hand the $\mathcal{R}$ is also constructed in a $diff(X)$-equivariant manner but it operators on the spinor bundle S_{2k}, where the double cover of $GL(n)$ can't act.[105] This suggests that there is *no non-scalar intertwining* from the space of curvature tensors on X to the space of symmetric operators on S_{2k}, but since I didn't figure out how to prove this without a few lines of manipulations with Bianchi identities, let us accept this for a fact.[106]

3.2.4 Dirac Operators with Coefficients in Vector Bundles, Twisted S-L-W-B Formula and K-Area

Let $\mathcal{D}_{\otimes L}$ be the Dirac twisted with a complex vector bundle $L \to X$ with a unitary connection ∇^L on it. Then, as earlier, we have the general Bochner-Weitzenböck formula

$$\mathcal{D}^2_{\otimes L} = \nabla\nabla^* + \sum_{i<j} e_i \cdot e_j \cdot R_{\otimes}(e_i \wedge e_j),$$

where this $\nabla = \nabla^{\otimes}$ is the connection in the tensor product of the spinor bundle with L that is defined by the Leibniz rule,

$$\nabla^{\otimes}(s \otimes l) = \nabla^{\mathcal{S}} \otimes l + s \otimes \nabla^L(l);$$

hence, the curvature $\cdot R_{\otimes}$ of this connection, that is the commutator of the $\nabla^{\otimes}$-differentiations, also behaves by this rule:

$$R_{\otimes}(e_i \wedge e_j)(s \otimes l) = R_{\mathcal{S}}(e_i \wedge e_j)(s) \otimes l + l \otimes R_L(e_i \wedge e_j)(l)$$

[105] Lemma 5.23. p 132 in [Lawson&Michelsohn(spin geometry) 1989].

[106] Or see "Proof of Theorem 8.8" on page 161 in [Lawson&Michelsohn(spin geometry) 1989].

which brings us to the following.

[$\mathcal{D}_{\otimes}$] **Twisted S-L-W-B Formula:**

$$\mathcal{D}^2_{\otimes L}(\sigma\otimes l)=\nabla\nabla^*(\sigma\otimes l)+\frac{Sc(X)}{4}(\sigma\otimes l)+\sum_{i<j}e_i\cdot e_j\cdot\sigma\otimes R_L(e_i\wedge e_j)(l).$$

A basic application of this formula is the bound on the area-size of manifolds with $Sc\geq\sigma>0$ expressed in terms of vector bundles over X.

[$K_{☆}$] **Bound on K-Area by Scalar Curvature.** Let X be compact orientable Riemannian Manifold with positive scalar curvature and let $L\to X$ be a complex vector bundle with the unitary connection.

If the norms of the curvature operators $R_x(e_1\wedge e_2):T_x(X_x\to T_x(X)$ *of this connection are bounded by*

$$||R_x(e_1\wedge e_2):T_x(X_x\to T_x(X)||\leq\kappa_n\cdot Sc(X,x)$$

for all $x\in X$*, all unit bivectors* $e_1\wedge e_2$ *in the tangent spaces* $T_x(X)$ *and a universal strictly positive constant* $\kappa_n>0$*, then,* provided X is spin*, all Chern numbers of the bundle* L *vanish.*

Proof. If some Chern number of L doesn't vanish, then an easy computation with Chern classes and the index formula shows[107] that there exists an associated bundle L', such that the curvature R' of the connectin in L' satisfies

$$||R'_x(e_1\wedge e_2):T_x(X_x\to T_x(X)||\leq const_n\cdot||R_x(e_1\wedge e_2):T_x(X_x\to T_x(X)||$$

and such that the *index of the twisted Dirac operator* on the spinor bundle tensored with L',

$$\mathcal{D}^+_{\otimes L'}:\mathbb{S}^+\otimes L'\to\mathbb{S}^+\otimes L',$$

doesn't vanish.

But if

$$||R'_x(e_1\wedge e_2):T_x(X_x\to T_x(X)||<\frac{1}{4}\cdot\frac{2}{n(n-1)}\cdot Sc(X,x).$$

then, according to [$\otimes$] the $\mathcal{D}^2_{\otimes L'}$ is positive and the poof follows by contradiction.

At first sight this [☆] looks as an artifact of symbolic manipulations with curvatures of vector bundles, an insignificant generalization of the Lichnerowicz theorem, as devoid of an actual geometric information about X as this theorem is.

But, surpassingly, although the proof of [☆] is 90% the same[108] as that by Lichnerowicz, the information contents of the two statements are vastly different – almost nothing in common between them:

[107] For details and further applications see [GL(spin) 1980], §4-5 in chapter IV in [Lawson&Michelsohn(spin geometry) 1989], §4-5 in [G(positive) 1996], [Min-Oo(K-Area) 2002] and sections 3.2.4, **??**, **??**.

[108] The proof of [☆], unlike that of Lichnerowicz' theorem, needs only 10% of the power of the Atiyah-Singer theorem – the easy part of it: non-trivial variability of the index of $\mathcal{D}_{\otimes L}$ with variations of (the Chern classes of) L, rather than a more subtle aspect of the formula which involves $\hat{A}$-genus of X.

Lichnerowicz is 99% about *delicate smooth topological invariants* of manifolds with $Sc > 0$, while [☆] reveals raw geometric essence of $Sc(X) \geq \sigma > 0$, which, as it becomes a *positive curvature* condition, *limits the size* of X.[109]

Below is a specific instance of this.

Rough Area (non)-Contraction Corollary. Given a *compact* Riemannian manifold $\underline{X}$, there exists a positive constant $\kappa = \kappa_{\underline{X}} > 0$, which restricts how much manifolds X with $Sc \geq \frac{1}{\kappa}$ *can be area-wise greater than* $\underline{X}$, which is expressed by *a bound on a possible decrease of areas* of surfaces in X under "topologically significant" maps $X \to \underline{X}$.

In precise language,

[★] *let X be an oriented Riemannian manifold with $Sc(X) > 0$ and $f : X \to \underline{X}$ a smooth map, such that the norm of the second exterior power of the differential* of f,

$$\wedge^2 df : \bigwedge^2 T(X) \to \bigwedge^2 T(\underline{X}),$$

is *bounded by the reciprocal of the scalar curvature of X times $\kappa_{\underline{X}}$,*

$$|| \wedge^2 df(x)|| < \frac{\kappa_{\underline{X}}}{Sc(X,x)}, \text{ for all } x \in X.$$

Then, provided X is spin, *the image h of of the fundamental homology class of X in the homology of $\underline{X}$, that is*

$$h = f_*[X] \in H_n(\underline{X}),\ n = dim(X),$$

is torsion.

Proof. By basic topology (a corollary to a theorem by Serre), an *even dimensional non-torsion* homology class h in $\underline{X}$ is "detected" by a complex vector bundle: that is a $\underline{L} \to \underline{X}$, such that some characteristic cohomology class $\underline{c}$ of $\underline{L}$, doesn't vanish on h

$$\underline{c}(h) \neq 0.$$

If $h = f_*[X]$, then $f^*(\underline{c})[X]$, which serves as a *characteristic number* of the induced bundle $L = f^*(\underline{L}) \to X$, is equal to $\underline{c}(h)$; hence it *doesn't vanish* either.

Now, arguing as in the proof of **[Sc $\not>$ 0]** for profinitely hyperspherical manifolds (see section 3.2), let $\underline{\nabla}$ be a unitary connection in $\underline{L}$ and observe that the norm of the curvature R of the induced connection in L, which is, after all, is a 2-*form*, is bounded by the curvature $\underline{R}$ of $\underline{\nabla}$,

$$||R_x|| \leq || \wedge^2 df(x)|| \cdot ||\underline{R}_x||.$$

Thus, if $n = dim(X)$ is even, the proof follows from [☆] and the odd case reduces to the even one by taking the products of both manifolds with the circle.

Remarks and Exercises. **(a)** We use the word "*K-area*" to express the idea that

if X contains "homologically significant" families of surfaces with *small areas*, then K-cohomology classes of X can't be represented by bundles with connections, which have small curvatures

[109]Positivity of the sectional (and Ricci) curvature, imposes bounds the *first and the second derivatives* of the growths of balls in respective manifolds.

and where

the norm of $\wedge^2 df$ measures by how much f *contracts/expands these areas.*[110]

Yet, we shall eventually switch to an uglier but more appropriate word "K-cowaist$_2$".

(b) Let X and Y be closed oriented surfaces with Riemannian metrics on them and let $f_0 : X \to Y$ be a continuous map of degree d. Show that f_0 is homotopic to a smooth *strictly area decreasing* map f, i.e. where $||\wedge^2 df(x)|| < 1$ for all $x \in X$, if and only if $area(X) > d \cdot area(Y)$.

(c) The principal case in the above corollary, which yields most topological applications, [111] is where $\underline{X}$ is the n-sphere S^n and where the non-torsion condition amounts to *non-vanishing of the degree* of $f : X \to S^n$.

In fact, as one knows by a theorem of Serre, the multiple of every cohomology class h in $\underline{X}$ with $h \smile h = 0$ can be induced from the the fundamental class of S^n by a smooth map $\underline{X} \to S^n$, the general case of this corollary, for all dimensions, can be (with a minor effort) reduced $\underline{X} = S^n$.

(d) We call this corollary "rough", since the (lower) bound on $\kappa_{\underline{X}}$ its proof delivers is far from optimal;

Optimal bounds, however, are available, albeit only in a few cases, including $\underline{X} = S^n$ as we shall see in the following sections.

Questions. **(A)** *Is the spin condition in* $[\star]$ *redundant?*

Or the opposite is true: if an orientable non-spin n-manifold X admits a metric g_0 with $Sc(g_0) > 0$, then it carries metrics g_ε, for all $\varepsilon > 0$, with $Sc(g_\varepsilon) \geq 1$, for which allow smooth maps $f_\varepsilon : (X, g_\varepsilon) \to S^n$ with $deg(f_\varepsilon) \neq 0$, and $|| \wedge^2 df|| \leq \varepsilon$?

(B) *Can the torsion conclusion in* $[\star]$ *be replaced by "p-torsion for some particular p, preferably for $p = 2$ and, in lucky cases, even by just $f_*[X] \neq 0$?*

(It is not even clear if this can be done with a bound on $||df||$ rather than on $\wedge^2 ||df||$, where there is a chance for a successful use of minimal hypersurfaces.)

3.3 Sharp Lower Bounds on *sup*- and *trace*-Norms of Differentials of Maps from Spin manifolds with $Sc > 0$ to Spheres.

There is no single numerical invariant faithfully representing the size of X, but there are several ways of comparison the sizes of different manifolds.

In the case, where two Riemannian metrics are defined on the same background manifold, say g and $\underline{g}$ on $\underline{X}$, one compares these at a point $\underline{x}$ by simultaneously diagonalizing them and recording the ratios of their values on the vectors e_i from the common orthonormal frame $\{e_1, e_2, ..., e_n\} \subset T_{\underline{x}}(\underline{X})$, that are the numbers

$$\lambda_i(\underline{x}) = \lambda_i(\underline{g}/g, \underline{x}) = \frac{||e_i||_{\underline{g}}}{||e_i||_g}.$$

In terms of these numbers, the inequalities $\lambda_i(\underline{x}) \leq 1$, $\underline{x} \in \underline{X}$, say that $g \geq \underline{g}$, while the inequalities $\lambda_i \lambda_j(\underline{x}) \leq 1$ convey that g is (only) *area wise* (non-strictly) *greater* than $\underline{g}$, where, of course, the former implies the latter.

[110] See [G(positive) 1996], [Min-Oo(K-Area) 2002] and sections 3.2.4, **??** for more about this K-area.

[111] See [GL(spin) 1980], [GL(complete) 1983], [Lawson&Michelsohn(spin geometry) 1989].

Another way to compare the metrics is by using the *trace of* $\underline{g}$ *relative to* g, denoted

$$trace(\underline{g}/g) = \sum_1^n \lambda_i, \ \ n = dim(\underline{X}),$$

where the inequality

$$\frac{1}{n} trace(\underline{g}/g) \le 1$$

expresses the idea of g being greater than $\underline{g}$.

This "trace-wise greater" is less restrictive, yet, moderately so, than the "ordinary greater" $g \ge \underline{g}$, for

$$g \ge \underline{g} \Rightarrow g \underset{tr}{\ge} \underline{g} \Rightarrow g \ge \frac{1}{n^2}\underline{g}.$$

(Notice that $\lambda_i(\underline{g}/c^2 g) = \frac{1}{c}\lambda_i(\underline{g}/g)$.)

A more relevant for us is the "area trace"

$$trace_{\wedge^2}(\underline{g}/g) = \sum_{i \ne j} \lambda_i \lambda_j$$

where "trace area-wise greater" inequality reads

$$\frac{1}{n(n-1)} trace_{\wedge^2}(\underline{g}/g) \le 1,$$

which is related to the "untraced area-wise greater" ratio by the relations

$$[g \underset{\wedge^2}{\ge} \underline{g}] \Rightarrow [g \underset{tr_{\wedge^2}}{\ge} \underline{g}] \Rightarrow \left[g \ge \frac{1}{n(n-1)}\underline{g}\right].$$

3.3.1 Area Inequalities for Equidimensional Maps:Extremality and Rigidity

In order to apply the above to Riemannian metrics g and $\underline{g}$ on *different* manifolds X and $\underline{X}$ we relate them by a smooth map, say $f : X \to \underline{X}$, where the principal case is of $dim(X) = dim(\underline{X}) = n$ and where, to make sense of what follows, the map f must be "*homotopically onto*", that is *not homotopic to a map into a proper subset in* $\underline{X}$.

If both manifolds are *orientable* – they are assumed compact without boundaries at this point – this is equivalent to *non-vanishing of the degree* $deg(f)$ of the map, [112]

If non-orientability is easily taken care of by just passing to orientable double covers, what does cause a problem is the *spin condition*, the relevance of which the following two geometric theorems remains problematic.

$[X_{spin}{}^{\to\bigcirc}]$ **Spin-Area Convex Extremality Theorem.** Let $\underline{X} \subset \mathbb{R}^{n+1}$ be a smooth compact convex hypersurface and let $\underline{g}$ be the Riemannian metric on $\underline{X}$ induced from $\mathbb{R}^{n+1}$. Let $X = (X, g)$ be a compact orientable Riemannian n-manifold with $Sc \ge 0$ and let $f : X \to \underline{X}$ be a smooth map of *non-zero degree.*

[112]The implication $[deg(f) \ne 0] \Rightarrow [f$ is homotopically onto$]$, which is obvious by the modern standards, is by no means trivial. For instance, "homotopically onto" for the identity map of the n-sphere is equivalent (one line kindergarten argument) to the Brauer fixed point theorem for the $(n + 1)$-ball.

Let $g^\circ = Sc(g)\cdot g$ and $\underline{g}^\circ = Sc(\underline{g})\cdot\underline{g}$ be the corresponding Sc-normalized metrics

If X is spin *and n is* even*, then the map f can't be strictly area decreasing, that is the metric g° is* not *area-wise greater, than the induced metric $f^*(\underline{g}^\circ)$ on X.*

Put it another way,

there necessarily exists a point $x \in X$, where the norm of the second exterior power of the differential of f is bounded from below by the scalar curvature of X as follows

$$Sc(\underline{X}, f(x))\cdot \|\wedge^2 df(x)\| \geq Sc(X,x),$$

which, in terms of $\lambda_i^\circ = \lambda_i(f^*(\underline{g}^\circ))/g^\circ)$, reads

$$\max_{x\in X, i\neq j} \lambda_i^\circ(x)\lambda_j^\circ(x) \geq 1.$$

In the simplest case, where $\underline{X}$ is the unit sphere $S^n \subset \mathbb{R}^{n+1}$, this theorem can be refined as follows.

$[X_{spin} \to \bigcirc]$ **Spherical Trace Area Extremality Theorem**. Let X be a compact orientable Riemannian spin manifold of dimension n and $f : X \to S^n = \underline{X}$ be a map with $deg(f) \neq 0$.

Then *f can't be* trace *area-wise strictly decreasing with respect to the Sc-normalized metrics $g^\circ = Sc(g)\cdot g$ on X and $\underline{g}^\circ = Sc(\underline{g})\cdot\underline{g}^\circ = n(n-1)ds^2$, which, in terms of the exterior power of f, says that there is a point $x\in X$, where the trace-norm of the second exterior power of the differential of f is bounded from below by the scalar curvature of X as follows*

$$2\|\wedge^2 df(x)\|_{trace} \geq Sc(X,x),$$

that is

$$\frac{1}{2n(n-1)}\sum_{i\neq j}\lambda_i^\circ(x)\lambda_j^\circ(x) \geq 1 \textit{ for } \lambda_i^\circ = \lambda_i(f^*(\underline{g}^\circ))/g^\circ).$$

Remarks (a) Neither $[X_{spin}{}^{\to}\bigcirc]$ nor $[X_{spin} \to \bigcirc]$ seem obvious even, where X is also a convex hypersurface in $\mathbb{R}^{n+1}$.

Question. Are there counterparts of $[X_{spin}{}^{\to}\bigcirc]$ and/or of $[X_{spin} \to \bigcirc]$ for symmetric function s_k of the principal curvatures $\alpha_1, \alpha_2, ..., \alpha_n$ of convex hypersurfaces X and $\underline{X}$? (We shall return to this question in (b) of 3.4.)

(b) The condition $n = 2k$, which is unneeded for $[X_{spin} \to \bigcirc]$, probably is also redundant for $[X_{spin}{}^{\to}\bigcirc]$.

(c) These two theorem will be later generalized in several directions.

(d_1) One may allow non-compact, and sometimes even non-complete manifolds X with suitable conditions on maps f, in order to have their degrees being properly defined.

(e_2) In the case, where $dim(X) = dim(\underline{X}) + 4l$, the condition $deg(f) \neq 0$ can be replaced by $\hat{A}[f^{-1}(x)] \neq 0$ for a generic point $x \in X$ of a smooth map $f : X \to \underline{X}$. (c_3) Instead of a convex hypersurface in $\mathbb{R}^{n+1}$, one may take a more general Riemannian manifold for $\underline{X}$, namely one with *a non-negative curvature operator* and – this is, probably, unnecessary – *with non-zero Euler characteristic.*

(f) Who is extremal? These two extremality theorems can be thought of as *properties of* X, saying that "*large scalar curvature makes* X *small*".

From another perspective, *these theorems are about* $\underline{X}$, saying that $\underline{X}$ *can't be enlarged without making its scalar curvature smaller at some point.*

This suggest two avenues of generalizations that we shall explore in the following sections.

1. Widen the class of manifolds X and maps $f : X \to \underline{X}$, which satisfy the above or similar theorems and, regardless of the scalar curvature, study invariants of manifolds X responsible for existence/non-existence of metrically contracting, yet topologically significant, maps from X to "standard" manifolds $\underline{X}$ such as the spheres, for instance.

2. Find further instances of *extremal* manifolds $\underline{X} = (\underline{X}, \underline{g})$ with $Sc(\underline{g}) > 0$, i.e. where no Sc-normalized metric g can be greater the so normalized $\underline{g}$,

$$Sc(g) \cdot g \not> Sc(\underline{g}) \cdot \underline{g}$$

and study properties of such metrics.[113]

A few Words about the Proofs. [114]. The logic here is the same as in the proof of the rough area (non)-contraction corollary from the previous section, where the sharpness of the bound on $\wedge^2 df$ is achieved by a choice of the bundle $\underline{L} \to \underline{X}$ with a non-zero top Chern class with a connection $\underline{\nabla}$ with minimal possible curvature, that allows the necessary strong bound on the "twisted curvature" term $\sum_{i<j} e_i \cdot e_j \cdot \sigma \otimes R_L(e_i \wedge e_j)(l)$ in the Schrödinger-Lichnerowicz-Weitzenböck-Bochner formula for the Dirac operator on X tensored with induced connection $\nabla = f^*(\underline{\nabla})$ in the bundle $L = f^*(\underline{L}) \to X$,

$$\mathcal{D}^2_{\otimes L}(\sigma \otimes l) = \nabla\nabla^*(\sigma \otimes l) + \frac{Sc(X)}{4}(\sigma \otimes l) + \sum_{i<j} e_i \cdot e_j \cdot \sigma \otimes R_L(e_i \wedge e_j)(l).$$

The natural choice of $\underline{L}$ – this was suggested by Blaine Lawson 40 years ago –[115]) is one of the *Bott generator bundles*, that are the $\frac{1}{2}$-spinor bundles $\underline{L}^\pm = \mathcal{S}^\pm(\underline{X})$ (with $rank_{\mathbb{C}}(\underline{L}) = 2^{k-1}$ for $n = 2k$), which, being the "moral square roots" of the tangent bundle $T(\underline{X})$, have their curvatures equal to the one half of that of $T(\underline{X})$. (This is clearly seen for $n = 2$ where $\underline{L}^+$ is the Hopf complex line bundle over S^2.

What makes $\underline{L}^\pm$ promising candidates for S-L-W-B-extremality, is the fact that $\underline{L}^\pm$-twisted Dirac operator on the manifold $\underline{X}$ itself *does have harmonic spinors* but only *barely so*: these spinors are *parallel* as they correspond to *constant functions* and/or to *constant multiples of the Riemannian volume n-form* on $\underline{X}$.

The extremality property of $\underline{L}^\pm$ was confirmed by Llarull in the case of $\underline{X} = S^n$ and – this was by no means expected – by Goette and Semmelmann

[113]See [Sun-Dai(bi-invariant)2020] for the proof of the extremality of *bi-invariant metrics* on compact Lie groups in the class of *left invariant* metrics.

[114]For detailed poofs the above mentioned results see [Llarull(sharp estimates) 1998], [Min-Oo(Hermitian) 1998], [Min-Oo(K-Area) 2002], [Goette-Semmelmann(symmetric) 2002], [Goette(alternating torsion) 2007], [Listing(symmetric spaces) 2010]; also we say a bit more about this in sections**??**, **??**.

[115]I recall this well, since I was taken by surprise by the properties of this bundle, which has the minimal curvature (one half of that of the tangent bundle of the sphere) among all unitary bundles with non-trivial Euler class.

for manifolds $\underline{X}$ with positive curvature operators, while the possibilities of Sc-normalization and of tracing $\wedge^2 df$, were suggested by Listings. (Although there is *no technical novelties in the proofs* of the Sc-normalised and traced modifications of $[X_{spin}\rightarrow\bigcirc]$ and $[X_{spin}\rightarrow\bigcirc]$ these significantly *widen the range of applications* of these extremality theorems.)

Besides facing algebraic complexity of the "twisted curvature" one has to ensure the existence of non-zero $L^{\pm}$-twisted harmonic spinors on X for $L^{\pm} = f^*\underline{L}^{\pm}$.

The index formula guarantees this for $n = 2k$ and, under an additional condition on f, also for $n = 4l + 1$, but in general the existence of such spinors for all metrics on X and all n remains problematic.

$\bigcirc_{\blacksquare}$. *The Proof of* $[X_{spin}\rightarrow\bigcirc]$ *for odd* $n = dim(X)$. Given a map $X \to S^n \subset S^{n+1}$, radially (and obviously) extend it to the map $X \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to S^{n+1}$ with the bottom and the top of the cylinder $X \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ sent to the poles of S^{n+1}, $X \times \left\{\mp\frac{\pi}{2}\right\} \to \mp 1$.

One can proceed three ways from this point.

1. Endow $X\times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with the (spherical suspension) warped product metric $\hat{g}$ with the same warping factor as that for the spherical cylinder $S^{n+1}\setminus\{-1,+1\}$ and observe that, say in the case of $Sc(X) \geq n(n-1) = Sc(S^n)$, this metric has greater scalar curvature than that of S^{n+1}.

Then, by an easy argument, an ε-small C^0-perturbation of this metric ε-near the boundary extends, for all $\varepsilon > 0$, to a complete metric $\hat{g}_\varepsilon$ on the infinite cylinder $X \times (-\infty, +\infty)$, such that $Sc(\hat{g}_\varepsilon) \geq n(n+1) - \varepsilon$ and such that the geometry of $X \times (-\infty, +\infty), g_\varepsilon) \geq n(n+1) - \varepsilon$ is cylindrical for $|t| \geq \frac{\pi}{2} + \varepsilon$ infinity with the scalar curvature $\geq n(n+1) + 1$.

Thus the untraced inequality $[X_{spin}\rightarrow\bigcirc]$ applies to the product $S^n \times S^1(R)$ $R \geq 2$ obtained by closing this cylinder at infinity and letting $\varepsilon \to 0$.

2. Apply the traced inequality $[X_{spin}\rightarrow\bigcirc]$ to maps $X^n \times S^1(R) \to S^{n+1}$, where $S^n \times S^1(R)$ comes with with the product metric, and let the radius of the circle $R \to \infty$. (This is, essentially, how it was done in [Llarull(sharp estimates) 1998].)

3. Regard a map $X^n \times S^1 \to S^{n+1}$ of non-zero degree as a family of maps $f_s : X \to S^{n+1}$ and use the spectral flow index theorem for the family of operators on $X = X \times s$ parametrized by S^1.[116]

.

Exercise. Fill in the details in (1) and (2).

Question Is there a more direct (K^1-theoretic?) proof of the inequality $[X_{spin}\rightarrow\bigcirc]$ for odd n with no direct reference to S^{n+1} and desirably of $[X_{spin}\rightarrow\bigcirc]$ as well for, odd n, e.g. by a spectral flow argument?

Infinite Dimensional Remark. Both, spherical suspension in 1 and the cylindrical one in 2, when repeated N-times times and can be interpreted in the limits for $N \to \infty$ as properties of

[116]Such argument was used in [Vafa-Witten(fermions) 1984] for lower bounds on spectral gaps for the Dirac operator, succinctly exposed in [Atiyah(eigenvalues) 1984] and applied in §6 in [G(positive) 1996] to spectral bounds for the Laplace operators on odd dimensional Riemannian manifolds.

Also spectral flow for Dirac operators combined with a *refined Kato inequality* is used in [Davaux(spectrum) 2003] for the proof of sharp upper bounds on the scalar curvatures of Riemannian metrics on compact manifolds which admit hyperbolic metrics.

1^∞ infinite dimensional manifolds X^∞ with $Sc(X^\infty) \geq Sc(S^\infty)$;

2^∞ $Sc(X^\infty) \geq Sc(S^n \times \mathbb{R}^{\infty - n}$;

inequalities are implemented in both cases by certain special Fredholm-type maps $X^\infty \to S^\infty$.

Conversely, one can prove an infinite dimensional version of $[X_{spin} \to \bigcirc]$ for limits of the above maps, say for

Fredholm maps from a Hilbertian manifold X to the Hilbertian sphere, $f : X \to S^\infty$,such that $deg(f \neq 0$ and such that there exists a sequence of equatorial spheres

$$S^{N_1} \supset S^{N_2} \supset \ldots \supset S^{N_i} \supset \ldots \supset S^\infty,$$

where the union $\bigcup_i S^{N_i}$ is *dense* in S^∞ and such that the pullbacks $X_i = f^{-1}(S^{N_i}) \subset X$ are smooth submanifolds of dimensions N_i, the scalar curvatures of which with the induced metrics satisfy $Sc(X_i) - N_i(N_i - 1) - 0$ for $i \to \infty$.

Infinite Dimensional Questions. What is the most general/natural infinite dimensional inequality $[X_{spin} \to \bigcirc]$?

Is there a direct proof of such an inequality with no use of finite dimensional approximation?

Are there natural Hilbertian and/or non-Hilbertian spaces X to which such an inequality may apply?

Stability Remark. Probably, (I haven't thought trough this) the reduction argument *even* $\rightsquigarrow$ *odd* implies certain stability of harmonic spinors on $(2m-1)$-manifolds X twisted with spherical spinors, that are section of the induced bundle $f^*(\mathbb{S}(S^{2m-1}))$ by maps $f : X \to S^{2m-1}$ with $deg(f) \neq 0$.

Another (seemingly unrelated) instance of *stability of harmonic spinors (seemingly) independent of the index theorem* is present in

Witten's argument in his proof of the Euclidean positive mass theorem as well in Min-Oo's proof of the hyperbolic one.

Probably, there are many examples of stable (twisted) harmonic spinors on *compact manifolds*, where this stability is not not predicted, at least not directly, by the index theorem.[117]

Area Rigidity Problem: Examples and Counter Examples. Given a smooth convex hypersurface $\underline{X} \subset \mathbb{R}^{n+1}$ and let $\underline{g}$ be the induced Riemannian metric on $\underline{X}$.

Describe (all) Riemannian$n$-manifolds $X = (X, g)$ along with smooth maps $f : X \to \underline{X}$, such that

$$Sc(g, f(x)) \leq Sc(\underline{g}, x) \cdot \| \wedge^2 df(x) \|$$

at all $x \in X$ and also X and f where

$$Sc(g, f(x)) = Sc(\underline{g}, x) \cdot \| \wedge^2 df(x) \|.$$

In the "ideal rigid" case, at least for $Sc(\underline{X}) > 0$, one wants all such maps to be *locally isometric with respect to the Sc-normalised metrics* $g^\circ = Sc(g) \cdot g$ and $\underline{g}^\circ = Sc(\underline{g}) \cdot \underline{g}$. (This, if I am not mistaken, is the same as *local homothety* with respect to the original metrics: the induced Riemannin metrics $f^*(\underline{g})$ on X are constant multiples of g, i.e. $g = \lambda \cdot f^*(\underline{g})$)

[117]To make sense of this one has to properly specify the meaning of "stability" not to run into (counter) a example, see *Harmonic Spinors and Topology* by Christian Bár,`https://link.springer.com/chapter/10.1007/978-94-011-5276-1_3`

But the true picture is more interesting than this "ideal". Here is what one can say in this regard.

(A) If n=2 then the equality $Sc(\underline{g}, f(x)) = Sc(g, x) \cdot || \wedge^2 df(x)||$ says that f is *locally area preserving* with respect to g° and $\underline{g}^\circ$; hence, the space of such maps is (at least) as large as the group of area preserving diffeomorphisms of the disc.

(B) If $n \geq 3$, then locally area preserving maps are locally isometric and, in fact,

"Ideal rigidity", i.e. the implication

$$Sc(\underline{g}, f(x)) \leq Sc(g, x) \cdot || \wedge^2 df(x)|| \Rightarrow g = \lambda \cdot f^*(\underline{g}),$$

was proven by Mario Listing under the following assumptions:[118]

- $\underline{X}$ is a closed *strictly* convex hypersurface of dimension $n \geq 3$, where this "strictly" signifies that all principal curvatures are > 0 (rather than non-existence of straight segments in $\underline{X}$);
- X is a closed connected orientable *spin* manifold and $deg(f) \neq 0$.

Now let us look at *non-strictly convex* hypersurfaces of dimensions $n \geq 3$.

(C) Let a hypersurface $\underline{X} \subset \mathbb{R}^{n_0+m}$ be the product

$$\underline{X} = \underline{X}_0 \times \mathbb{R}^m$$

where $\underline{X}_0 \subset \mathbb{R}^{n_0}$ is a smooth hypersurface. Then all (self) maps

$$f = (f_0, f_1) : \underline{X} \to \underline{X}_0 \times \mathbb{R}^m = \underline{X},$$

such that $||df_1|| \leq 1$, satisfy $Sc(\underline{g}, f(x)) = Sc(g, x) \cdot || \wedge^2 df(x)||$.

If $m \geq 2$, there are no *closed* Euclidean hypersurfaces displaying such non-rigidity (unless I am missing obvious Euclidean examples)[119] but this non-rigidity, of cylinders, i.e. for $m = 1$, can be cast into a compact form; also this can be done to conical hypersurfaces as follows.

(D) Let $C \subset \mathbb{R}^{n+1}$ a smooth convex cone and let $\underline{X} \subset C$ be a smooth closed convex hypersurface, such that the intersection $\underline{X} \cap \partial C$ contains a conical annuls A in the boundary of C pinched between two spheres,

$$A = \{a \in \partial C\}_{R_1 \leq ||a|| \leq R_2}.$$

Thus, the boundary of $\underline{X} \subset C$ consists of three parts:

the *side boundary* that is the intersection $\underline{X} \cap \partial C$;

bottom $\underline{X}_1 \subset \underline{X}$ that lies on the R_1-side in the interior of C, i.e. $||\underline{x}_1|| < R_1$, for $\underline{x}_1 \in \underline{X}_1$,

top of $\underline{X}_2 \subset \underline{X}$ that lies on the R_2-side in the interior of C, i.e. $||\underline{x}|| > R_2$ for $\underline{x}_2 \in \underline{X}_2$.

Scale up the top of $\underline{X}$ and set:

$$X = (\underline{X} \setminus \underline{X}_2) \cup \lambda X_2, \ \lambda > 1.$$

[118] See theorem 1 in [Listing(symmetric spaces) 2010], and compare with Theorem 4.11 in [Llarull(sharp estimates) 1998].

[119] There are these in $\mathbb{R}^{n_0} \times \mathbb{T}^m$.

This X admits an obvious (infinite dimensional) family of diffeomorphisms $f : X \to \underline{X}$, that

fix the bottom,

return back the top by $x \to \lambda^{-1} x$,

send all straight radial segments in the side boundary of X to themselves,

satisfy the equality $Sc(\underline{g}, f(x)) = Sc(g, x) \cdot \| \wedge^2 df(x) \|$.

Probbaly, (C) and (D) give a fair picture of possible kinds of *not-quite-rigid* $\underline{X}$ with $Sc(\underline{X}) > 0$ *in the class of convex* X, but it is not so clear for the class of all X with $Sc(X) > 0$

3.3.2 Area Contracting Maps with Decrease of Dimension

The lower bounds on the norms $\| \wedge^2 df\|$ for equividimensional maps $f : X \to \underline{X}$ with *non-zero degree* generalize to maps, where $dim(X) > dim(\underline{X})$ with an appropriate generalization of the concept of degree.

For example, the proofs of the rough Area (non)-contraction property (section 3.2.4) and of both its above refinements $[X_{spin} \to \bigcirc]$ and $[X_{spin} \to \bigcirc]$, which say that such norms can't be too small at all points in X,

$$\| \wedge^2 df(x)\| \nless \frac{Sc(X,x)}{Sc(\underline{X}, f(x)\|} \text{ and } \| \wedge^2 df(x)\|_{trace} \nless 2\frac{Sc(X,x)}{n(n-1)} \text{ correspondingly,}$$

extend with (almost) no change to maps $f : X^{n+4l} \to \underline{X}^n$ with *non-zero* $\hat{A}$-*degrees*, which means non vanishing of the $\hat{A}$-genera of the pullbacks $f^{-1}(\underline{x}) \subset X^{n+4l}$ of generic points $\underline{x} \in \underline{X}^n$. [120]

For instance:

$[X_{spin} \overset{\hat{A}}{\to} \bigcirc]]$ **$\hat{A}$-Extremality Theorem**. Let X be a compact orientable Riemannian spin manifold of dimension $n + 4l$ and $f : X \to \underline{X} = S^n$ be a smooth map, such that the $\hat{A}$-genus of the f-pullback of a regular point from S^n doesn't vanish,

$$\hat{A}[f^{-1}(\underline{x}_0)] \neq 0, \ \underline{x}_0 \in S^n.$$

.

Then there exists a point $x \in X$, where the the trace-norm of the second exterior power of the differential of f is bounded from below by the scalar curvature of X as follows,

$$2\| \wedge^2 df(x)\|_{trace} \geq Sc(X,x).$$

Since

$$2\|\wedge^2 df(x)\|_{trace} = \sum_{i\neq j} \lambda_i(x)\lambda_j(x) \leq n(n-1) \max_{i\neq j} \lambda_i(x)\lambda_j(x) = n(n-1)\|\wedge^2 df(x)\|,$$

this implies that if $Sc(X) \geq n(n-1) = Sc(S^n)$, then the map f *can't be strictly area decreasing.*

Generalization to $\hat{\alpha}$. The above remains true with $\hat{\alpha}$ instead of $\hat{A}$, e.g. where the pullback of a regular point $f^{-1}(\underline{x}_0) \subset X$ is diffeomorphic to Hitchin's exotic sphere Σ^n for $n = 8k + 1, 8k + 3$.[121]

[120]This is done in [GL(spin)1980], [Llarull(sharp estimates) 1998], [Goette-Semmelmann(symmetric) 2002] and in [Goette(alternating torsion)2007] for bounds on $\|\wedge^2 df\|$, but the corresponding lower bound on $\|\wedge^2 df\|_{trace}$ is missing from [Listing(symmetric spaces) 2010]; however, as I see it, there in no problem with this either.

[121]Such a Σ^n is homeomorphic to the ordinary sphere S^n, but *doesn't bound a spin manifold.*

Question. Does the conclusion of the above theorem remain true if the nonvanishing of $\hat{A}[f^{-1}(\underline{x}_0)]$ is replaced by the following

the pullbacks $(f')^{-1}(\underline{x})$, for all smooth maps $f' : X^{n+m} \to \underline{X}^n$ homotopic to f and

all f'-non-critical $x \in \underline{X}^n$, *admit no metrics with* $Sc > 0$.

This is beyond the present day techniques, already for manifolds X^{n+m} homeomorphic to $S^n \times Y^m$, where Y^m is SYS-manifold.

But if Y^m is *the torus* or, more generally an *enlargeable manifold*, e.g. if it admits *a metric with non-positive sectional curvature*, then Dirac theoretic techniques on *complete manifolds* (see sections **??**, **??**) delivers the proof of the following.

$\times\mathbb{R}^m$**- Stabilized Mapping Theorem.** Let X^{n+m} be a complete orientable Riemannian spin manifold with $Sc(X^{n+m}) \geq \sigma > 0$ [122] and let $\underline{X}^n$ be a smooth convex hypersurface in $\mathbb{R}^{n+1}$. Let $f_1 : X^{n+m} \to \underline{X}^n$ and $f_2 : X^{n+m} \to \mathbb{R}^m$ be smooth maps, where f_2 is a proper[123] *distance decreasing* map and where the "product map",

$$(f_1, f_2) : X^{n+m} \to \underline{X}^n \times \mathbb{R}^m$$

has *non-zero degree.*

Then, if n is even, there exists a point $x \in X$, where

$$|| \wedge^2 df(x)|| \geq \frac{Sc(X,x)}{Sc(\underline{X}, f(x)).}$$

Furthermore, if $\underline{X}^n = S^n$, one can allow odd n *and replace the above inequality by the stronger one:*

$$2|| \wedge^2 df(x)||_{trace} \nleq Sc(X,x).^{124}$$

There is a particularly useful corollary of this theorem, where $X^{n+m} = Y^n \rtimes \mathbb{T}^m$ is *a* $\mathbb{T}^{\rtimes}$*-extension* of a manifold Y^n, that is the product $Y^n \times \mathbb{T}^m$ with a warped product metric $dy^2 + \phi(x)^2 dt^2$ and where the map $f : X \times \mathbb{T}^m$ factors as $Y^n \times \mathbb{T}^m \to Y^n \to \underline{X}$ for the coordinate projection $Y^n \times \mathbb{T}^m \to Y^n$

For instance, such a $\mathbb{T}^{\rtimes}$-stabilized mapping theorem for $m = 1$ together with the *μ-bubble separation theorem* (sections 3.6, **??**), yield *a sharp area mapping inequality* for a class of *manifolds X with boundaries*, e.g. for $X = Y \times [-1, 1]$.

3.3.3 Parametric Area Inequalities for Families of Maps

Introduce parameters wherever possible is a motto of modern mathematics; Grothendieck concept of *topos* – a category of sets parametrized by a "topological site" – is the most general manifestation of this.

The first instance of this in the present context is an application of the index theorem to the

[122] In view of [Zhang(Area Decreasing) 2020], one can, probably, relax this to $Sc(X^{n+m}) \geq 0$.

[123] This is the usual "proper": pullbacks of compact subsets are compact.

[124] See[Cecchini(long neck) 2020], [Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(Scalar&mean) 2021] for more general results applicable to manifolds X^{n+m} with boundaries and to all closed manifolds Y^m, the non-existence of metrics with $Sc > 0$ on which follows from non-vanishing of *Rosenberg index.*

family of flat complex line bundles L_p over the torus parametrized by the dual ($Picard$) torus

and thus showing that the torus $\mathbb{T}^{2m}$ with an *arbitrary Riemannian metric* g supports a *non-zero harmonic spinor* twisted with a *flat* unitary bundle; hence, *no metric* g *on the torus may have* $Sc(g) > 0$ by the (untwisted) S-L-W-B formula. [125]

Today, this idea is expressed in terms of elliptic operators $\mathcal{E}_{\otimes A}$ with *coefficients in* C^**-algebras* A, which, for commutative A, are algebras of continuous functions on toplogical spaces P parametrizing families of operators $\mathcal{E}_p$, $p \in P$.

Closer home, we want to determine a homotopy bound on a *space of maps* $f : X \to \underline{X}$ in terms of $\inf Sc(X)$ and the the norms $|| \wedge^2 df||$ of these maps.

Here is an instance of what we are looking for.

$[X \times P \to \bigcirc]$ **Sharp Parametric Area Contraction Theorem**. Let X be an orientable spin manifold of dimension n, let P be an m-dimensional orientable pseudomanifold, let g_p, $p \in P$, be a C^2-continuous family of smooth Riemannian metrics and let $f : X \times P \to S^{n+m}$ be a continuous map, where all maps $f_p = f_{|X_p} : X = X_p = X \times \{p\} \to S^{n+m}$ are C^1-smooth.

Then there exists a point $(x,p) \in X$*, where the* g_p*-trace norm of the exterior square of the differential of* $f_p(x)$ *is bounded from below by*

$$2||trace(\wedge^2 df_p)|| = \sum_{i \neq j}^{n} \lambda_i(x)\lambda_j(x) \geq Sc(g_p, x)$$

for some $(x,p) \in X \times P$.

Consequently,

the inclusion $\mathcal{I}_{\{g\}}$ *of the space* $\mathcal{F}_{\{g\}}$ *of pairs* (g,f)*, where* g *is a Riemannian metric on* X *and* $f : X \to S^{n+m}$ f *is a smooth map, such that*

$$2||trace(\wedge^2 df_p)|| = \sum_{i \neq j}^{n} \lambda_i(x)\lambda_j(x) < Sc(g_p, x) \text{ for all } (x,p) \in X \times P,$$

to the space of all continuous maps $X \to S^{n+m}$,

$$\mathcal{I}_{\{g\}}\mathcal{F}_{\{g\}} \hookrightarrow \mathcal{F}_{cont}(X, S^{n+m}),$$

is contractible.

Outlines of two Proofs. 1. Apply the parametric index theorem to the Dirac operators on X_p twisted with bundles $L_p \to X_p$ induced from the same bundle $\underline{L} = \mathcal{S}^{\pm}(S^{n+m}) \to S^{n+m}$ that was used in the proof of the area contraction theorems in section 3.3.1 and confirm curvature estimates needed for the twisted S-L-W-B formula.

(If n is odd, one has to argue as in $\bigcirc_{\blacksquare}$ in the proof of $[X_{spin} \to \bigcirc]$ for odd n in section 3.3.1.)

2. Reduce the parametric problem to the non-parametric trace extremality theorem $[X_{spin} \to \bigcirc]$ from section 3.3.1 applied to maps $X^{n+m} \to S^{n+m}$.

To do this, assume P is a manifold[126] and let h_λ, $\lambda \geq 0$, be a family of Riemannian metric on P such that $g_\lambda \geq \lambda \cdot h_0$ and $Sc(h_\lambda) \leq \lambda^{-1}$ and send

[125] This idea goes back to George Lusztig's paper *Novikov's higher signature and families of elliptic s* where it is used for a proof of the homotopy invariance of "torical" Pontryagin classes.

[126] In the general case, by using a Thom's theorem, replace P by a manifold P' mapped to P with non-zero degree

$\lambda \to \infty$. Then, due to *additivity of trace*, application of $[X_{spin} \to \bigcirc]$ yields $[X \times P \to \bigcirc]$.

Remarks.(a) If instead of the trace norm of df we had used the sup-norm, this argument would give you a non-sharp inequality, namely with the extra constant $\frac{(n+m)(n+m-1)}{n(n-1)}$.

(b) *Non-product families.* Let $\{X_p\}$ be a continuous family of compact connected orientable Riemannian n-manifolds parametrized by an orientable N-psedomanifold $P \ni p$, that is $\{X_p\}$ is represented by a fibration $\mathcal{X} = \{X_p\} \to P$ with the fibers X_p.

Let $f : \mathcal{X} \to S^{n+N}$, where $n = dim(X_p)$ and $N = dim(P)$, be continuous map the *restrictions* of which to all X_p are, smooth *area non-decreasing*, e.g. 1-Lipschitz maps, the differentials of which are continuous in $p \in P$, and let the degree of f be *non zero.*

If the fiberwise tangent bundle $\{T(X_p)\}$ of $\mathcal{X}$ is spin, then the above mentioned parametric index theorem to the Dirac operators on X_p implies that

the infimum of the scalar curvatures of all X_p *satisfies*

$$\inf_{x \in X_p, p \in P} Sc(X_p, x) \leq n(n-1).$$

Moreover, in the extremal case of $\inf_{x \in X_p, p \in P} Sc(X_p, x) = n(n-1)$, one can show that *some of* X_p *is isometric to* S^n*.*

(If P is a smooth manifold, such that $\mathcal{X}$ is spin, then all this can be proved with the index theorem on $\mathcal{X}$.)

(c) *Maps to Fibrations.* Let $\underline{\mathcal{X}} \to P$ be a sphere bundle with the fibers $S_p^{n+N} = S^{n+N}$ and $f : \mathcal{X} \to \underline{\mathcal{X}}$ a fiberwise map,

$$f = \{f_p : \mathcal{X}_p \to S_p^{n+N}\}.$$

Then, with a suitable defined condition "$deg(f) \neq 0$",the above inequality on the scalar curvatures of the fibers X_p remains valid.

To see this, reduce (c) to (b) as follows.

Let $\underline{\mathcal{X}}^\perp \to P$ be the complementary S^m-bundle, that is the *join bundle* $\underline{\mathcal{X}} * \underline{\mathcal{X}}^\perp$ with the fibers $S_p^{n+N+m+1} = S_p^{n+N} * S^m$ is *trivial*, and observe that the map f canonically suspends to a fiberwise map

$$X * \underline{\mathcal{X}}^\perp \to \underline{\mathcal{X}} * \underline{\mathcal{X}}^\perp,$$

which, due to the triviality of the fibration $\underline{\mathcal{X}} * \underline{\mathcal{X}}^\perp$, defines a map

$${}^*f : \mathcal{X} * \underline{\mathcal{X}}^\perp \to S^{n+N+m+1}.$$

Since the scalar curvatures of the fibers $\mathcal{X}_p * \underline{\mathcal{X}}_p^\perp$ are bounded from below by the curvature of $S^{n+N+m+1}$ (see exercise [*] in section **??**) one can use (b), where, as in the reduction of the odd dimensional case of maps $X \to S^n$ to n even in $\bigcirc_\blacksquare$ in section 3.3.1, the fibers $\mathcal{X}_p * \underline{\mathcal{X}}_p^\perp$ and thus the space $\mathcal{X} * \underline{\mathcal{X}}^\perp$ must be completed by slightly perturbing the metric and then extending it cylindrically at infinity with (arbitrarily) large scalar curvature.

Exercises. (c_1) Use the trace norm on $\wedge^2 df$ and reduce (c) to (b) with the fiberwise version of $\bigcirc_\blacksquare$.

(c_2) Directly define "$deg(f)$" and prove (c) with the parametric index theorem.

(d) *Families of Non-Compact Manifolds.* The above generalizes to families of complete manifolds X_p and maps $f:\mathcal{X}\to S^{n+N}$, which are (locally) *constant at infinities* of all X_p (degrees are well defined for such maps f), where, the parametric relative index theorem, according to [Zhang(area decreasing) 2020], applies whenever all X_p have (not necessarily uniformly) positive scalar curvatures and where the conclusion concerns the scalar curvatures of X_p on the support of the differential df on the manifolds X_p

$$\inf_{x\in supp(df_{|X_p}),p\in P}\frac{Sc(X_p,x)}{2||\wedge^2 df_{|X_p}(x)||_{trace}}\leq 1,$$

(e) *Foliations.* There is a further generalizations of (b) to smooth foliations n-dimensional leaves on compact orientable $(n+N)$-dimensional manifolds $\mathcal{X}$, with smooth Riemannian metrics on them X.i

Namely, let $\mathcal{X}\to S^{n+N}$ be a smooth map of non-zero degree.

If either the manifold $\mathcal{X}$ is spin or the tangent bundle to the leaves is spin, then there exists a point $x\in\mathcal{X}$,such that the scalar curvature of the leaf $X=X_x\subset\mathcal{X}$ passing trough x at x is related to the differential of f restricted to X by the inequality

$$Sc(X,x)\leq 2||\wedge^2 df_{|X}(x)||_{trace}.$$

This is proven with $n(n-1)||df||^2$ instead of $||\wedge^2 df_{|X}(x)||_{trace}$ by Guangxiang Su [Su(foliations) 2018] and extended to complete manifolds in [Su-Wang-Zhang(area decreasing foliations) 2021] by sharpening the arguments by Alain Connes and Weiping Zhang. (The proofs in these papers, if I red them correctly, allows a use of $||\wedge^2 df_{|X}(x)||_{trace}$ rather than $||df||^2$.

Examples. Most natural (homogeneous) foliations with non-compact leaves support no metrics with $Sc>0$ by Alain Connes' theorem, but their products with spheres S^i, $i\geq 2$ carry lots of such metrics, to which Su's theorem applies.

Questions. Does this theorem remain valid for foliations with smooth fibres but only C^k-continuous in the transversal direction, such for instance, as stable/unstable foliations of Anosov systems?

(Notice in this regard that another Connes' theorem, which generalizes Atiyah L_2-index theorem and applies to foliations with *transversal measures,* needs these foliations to be only C^3-continuous in the transversal direction, compare with discussion in sections $9\frac{2}{3}$, $9\frac{3}{4}$ in [G(positive)1996].)

What is the comprehensive inequality that would include all of the above from (b) to (e)?

Families with Singularities. Is there a meaningful version of the above for families X_p, where some X_p are singular, as it happens, for instance, for Morse functions $\mathcal{X}\to\mathbb{R}$?

Notice in this regard that Morse singularities, are, essentially, conical, where positivity of $Sc(X_p)$ for singular X_p in the sense of section **??** can be enforced by a choice of a Riemannian metric in $\mathcal{X}$. [127]

[127] These are cones over $S^k\times S^{n-k-1}$, $n=dimX_p$, where the scalar curvature of such a cone can be made positive, unless $k\leq 1$ and $n-k-1\leq 1$.

Conversely, positivity of $Sc(X_p)$, for all X_p including the singular ones, probably, yields a smooth metric with $Sc > 0$ on $\mathcal{X}$.

And it must be more difficult (and more interesting) to decide if/when a manifolds with $Sc > 0$ admits a Morse function, where all, including singular, fibers have positive scalar curvatures or, at least, positive operators $-\Delta + \frac{1}{2}Sc$.

3.3.4 Area Multi-Contracting Maps to Product Manifolds and Maps to Symplectic Manifolds

A guiding principle in the scalar curvature geometry reads:

If certain geometric and/or topological properties of Riemannian manifolds X_i, $i = 1, 2, ..., k$ imply that $\inf Sc(X_i) \le \sigma_i$, then such a property of Riemannian manifolds X *homeomorphic the products* $\times_i X_i = X_1 \times ... \times X_k$ implies that $\inf Sc(X) \le \sum_i \sigma_i$.

1. Topological non-Existence Example. If X_1 and X_2 admit no complete metrics with $Sc > 0$, and if X_2 is compact, then in many, probably, not in all cases the product $X_1 \times X_2$ admits no such metric either, (this seems to fail for SYS-manifolds).

A a prominent instance of this – here and everywhere with scalar curvature – is X_2 equal to the N-torus $\mathbb{T}^N$.

2. Length Contraction Example. Let $\underline{X}_i$ $i = 1, ..., k$, be orientable (spin) length extremal Riemannian manifolds with $Sc(\underline{X}_i) \ge 0$, which means that all smooth maps of *non-zero* degrees from orientable (spin) Riemannian n_i-manifolds X_i with $Sc(X_i) > 0$ to $\underline{X}_i$

$$f_i : X_i \to \underline{X}_i,$$

satisfy

$$\inf_{x_i \in X_i} \frac{Sc(X_i, x_i)}{Sc(\underline{X}_i, f(x_i)) \|df_i(x_i)\|^2} \le 1.$$

Then – this is expected in many cases – the Riemannian manifold $\underline{X} = \times_i \underline{X}_i$ is also (spin) length extremal. (This is, probably, true for all *known* examples of *spin* length extremal manifolds $\underline{X}_i$.)

Moreover all smooth maps from orientable (spin) Riemannian manifolds X to the product $\underline{X} = \times_i \underline{X}_i$ defined by a k-tuple of maps $X \to X_i$,

$$\Phi = (\phi_1, ..., \phi_k) : X \to \bigtimes_i^k \underline{X}_i,$$

which have non-zero degree should satisfy the following stronger inequality,

$$\min_{i=1,...,k} \left(\inf_{x \in X} \frac{Sc(X, x)}{Sc(\underline{X}_i, \phi_i(x)) \|d\phi_i(x)\|^2} \right) \le 1.$$

And in the ideal world one expects even more:

$$\left(\inf_{x \in X} \frac{Sc(X_i, x_i)}{Sc(\underline{X}, \Phi(x)) \left(\sum_{i=1}^k \|d\phi_i(x)\| \right)^2} \right) \le k^2.$$

One also expects this product property for *area* rather than *length*, that is with the norm of the exterior power of the differentials, $|| \wedge^2 d\phi_i(x)||$ instead of $||d\phi_i(x)||^2$, which is (partly) justified by what follows.

Rough Multi-Area non-Contraction Inequality. Let $\underline{X}$ be a compact Riemannian manifold decomposed into product of Riemannian manifolds of positive dimensions,

$$\underline{X} = \underline{X}_1 \times ... \times \underline{X}_i \times ... \times \underline{X}_k,\ dim(\underline{X}_i) \geq 1,$$

let X be a compact orientable spin manifold of dimension $n \leq dim(\underline{X})$ and let $X \to \underline{X}$ be a smooth map defined by a k-tuple of maps to $\underline{X}_i$,

$$f = (f_1, ..., f_i, ...f_k) : X \to \underline{X} = \underline{X}_1 \times ... \times \underline{X}_i \times ... \times \underline{X}_k.$$

If the image of the fundamental homology class under f,

$$f_*[X] \in H_n(\underline{X})$$

is non-torsion, *then the scalar curvature of X is bounded by the area contraction by f, as follows*

$$\min_i \inf_{x\in X} \frac{Sc(X,x)}{|| \wedge^2 df_i(x)||} \leq \sigma,$$

where the constant σ depends on $\underline{X}$ but not on X.[128]

Proof. Since $f_*[X]$ is non-torsion, there exist cohomology classes $h_i \in H^{n_i}(\underline{X}_i;\mathbb{Q})$, $\sum_i n_i = n$, such that the cup product $h^* \in H^n(\underline{X})$ of their lifts to $\underline{X}$ doesn't vanish on $f_*[X]_{\mathbb{Q}}$).

By multiplying $\underline{X}_i$, where k_i are odd, by circles and multiplying X by the product of these circles, we reduce the situation to the case, where all k_i as well $n = dim(X) = \sum_i n_i$ are *even.*

Then, by the rational isomorphism between the K-theory and ordinary cohomology,

there exist complex vector bundles $\underline{L}_i \to \underline{X}_i$, such that the Chern character of the tensor product $\underline{L} \to \underline{X}$ of the pull-backs of $\underline{L}_i$ to $\underline{X}$ doesn't vanish on $f_*[X]_{\mathbb{Q}}$ either.

It follows, that the *index of the Dirac operator* on X with values in the f-induced bundle $L^* = f^*(\underline{L})$ – we assume that X is spin and the this is defined – or in some associated bundle $L^\star \to X$ *doesn't vanish.* (This is elementary algebra as in the definition f the K-area.)

Endow the bundles L_i with unitary connections and observe, as we did earlier, that the norm of the curvature of the corresponding connection in $L^\star \to X$ is bounded by a constant C which depends only on $\underline{X}$ and on the norms $|| \wedge^2 df_i||$, but not in any other way on X and on f.

Therefore, by the twisted Schrödinger-Lichnerowicz-Weitzenböck-Bochner formula the index of $\mathcal{D}_{\otimes L^\star}$ would vanish for $Sc(X) >> C$ and the proof follows.

Rank 1 Corollary. *If $Sc(X) > 0$ and (the differentials of) all maps f_i have ranks ≤ 1 then $f_*[X]_{\mathbb{Q}} = 0$.*

This follows from the inequality $\sigma(0,0,...,0) \leq 0$ and the definition of $\underline{\sigma}$.

For instance, this shows again that

[128]If $\underline{X}$ is infinite dimensional, e.g. this is the Grassmann manifold of m-planes in the Hilbert space, then σ may depend on $n = dim(X)$.

continuous maps from orientable Riemannian spin manifolds X with $Sc(X) > 0$ to T^m send the fundamental homology classes $[X] \in H_n(X)$ to zero in $H_n(T^m)$, since tori are products of circles and maps to circles have ranks ≤ 1.

(Maps f with all their components f_i of rank one, may be themselves smooth embeddings $X \to \underline{X}$.)

Sharp Multi-Area Inequalities. Let $\underline{X}_i$, $i = 1, ..., k$, be compact orientable Riemannian manifolds, either with *non-negative curvature s* or Hermitian ones with positive Ricci curvatures. Let X be a compact orientable manifold and let

$$f = (f_1, ..., f_k) : X \to \underline{X} = \bigtimes_{i=1}^{k} \underline{X}_i$$

be a map a positive degree. Let $|| \wedge^2 df_i||$ stands either for the *norm* of the second exterior power of the differential of the map $f_i : X \to \underline{X}_i$ or , in the case where $\underline{X}_i$ is the sphere S^{n_i}, it for the *averaged trace* of $\wedge^2 df_i$ defined as earlier:

$$\frac{1}{n(n-1)} ||trace(\wedge^2 df_i(x))|| = \frac{1}{n(n-1)} \sum_{\mu \neq \nu}^{n} \lambda_\mu(x)\lambda_\nu(x).$$

(The latter is non-greater than the former.)

Conjecture. There exists a point $x \in X$, such that

$$(\bigstar) \qquad Sc(X, x) \le Sc(\underline{X}, f(x)) \cdot \sum_i || \wedge^2 df_i(x)||.$$

1. Start with enumerating the cases, where this conjecture *was proved* for maps from *spin manifolds* X to *unsplit into products* manifolds $\underline{X}$, i.e. for $k = 1$.

1.A. *$\underline{X}$ is the n-sphere S^n.*

The main computation and reduction of the case $n = 2m - 1$ to $n = 2m$ via the map $X \times \mathbb{T}^1 \to S^{2m}$ was performed in [Llarull(sharp estimates) 1998]. Then the scale invariant trace form of Llarull's inequality was established in [Listing(symmetric spaces) 2010] for even n, and as we explained in section 3.3.1 the trace form of the area inequality allows an automatic reduction $n = 2m - 1 \rightsquigarrow n = 2m$.

1.B. *$\underline{X}$ is a Hermitian symmetric space with $Ricci(\underline{X}) > 0$.* This was proved for symmetric $\underline{X}$ in [Min-Oo(Hermitian) 1998] and extended to all Hermitian spaces with $Ricci(\underline{X}) \ge 0$ and $Ricci(\underline{X}, x_0) > 0$ at some point in [Goette-Semmelmann(Hermitian) 2002].

1.C. *$\underline{X}$ has non-zero Euler characteristic.* Proved in [Goette-Semmelmann(symmetric) 2002] and brought to the scale invariant form in [Listing(symmetric spaces) 2010].

2. *Stabilization by $\mathbb{T}^N$*. Whenever the inequality $(\bigstar)$ is established for manifolds X_o of dimension n_o and maps $X_o \to \underline{X}_o$ by confronting the index theorem with the twisted Schrödinger-Lichnerowicz-Weitzenböck-Bochner formula (there is no known alternative for this) then this argument also applies to maps $f : X \to \underline{X}_o \times \mathbb{T}^N$, $dim(X) = n_o + N$.

To show this, recall that the N-tori $\mathbb{T}^N$ for N even, support (almost flat) unitary bundles $\underline{L}_\varepsilon$ for all $\varepsilon > 0$, (and similar families of flat bundles a la Lusztig) with

(a) *non-zero Chern characters* man and, at the same time with

(b) *curvature operators with norms* $\leq \varepsilon$.

Now, suppose that ($\bigstar$) follows with the Dirac $\mathcal{D}$ on X_o twisted with the bundle $L_o \to X_o$ induced from a bundle $\underline{L}_o \to \underline{X}_o$ by a map $f_o : X_o \to \underline{X}_o$.

Then observe that the same argument applies to $\mathcal{D}$ on on X twisted with the bundle $L \to X$ induced by a map $f : X \to \underline{X}_o \times \mathbb{T}^N$ of non-zero degree from the tensor product $\underline{L}_o \otimes \underline{L}_\varepsilon \to \underline{X}_o \times \mathbb{T}^N$ by letting $\varepsilon \to 0$.

Indeed, (a)&($deg(f) \neq 0$) imply non-vanishing of $index(D_{\otimes L})$,while (b) guaranties the same bound on the L-curvature term in the twisted S-L-B-W formula for $\varepsilon \to 0$, as in the L_o-curvature for $D_{\otimes L_o}$.

Remark. As we mentioned above, one can use families of *flat* bundles over $\mathbb{T}^N$, (or more generally, suitable Hilbert moduli over the C^*-algebra of $\pi_1(\mathbb{T}^N)$) which have a advantage of giving (slightly) sleeker proofs of rigidity theorems.

3. The above argument, probably, applies to general manifolds $\underline{X}_1$ with bundles $\underline{L}_1 \to \underline{X}_1$ instead of $\underline{L}_\varepsilon \to \mathbb{T}^N$, where an essential point is checking that the curvature contribution to the S-L-B-W formula from the induced bundle $L = f^*(\underline{L}_0 \otimes \underline{L}_1) \to X$ for maps $f : X \to \underline{X}_o \times \underline{X}_1$ is bounded by the sum of the corresponding contributions from $f_o^*(\underline{L}_o)$ and $f_1^*(\underline{L}_1)$ for maps $f : X_o \to \underline{X}_o$ and $f_1 : X_1 \to \underline{X}_1$.

We suggest the reader will verify this, while we turn ourselves to a special case, where the necessary linear algebraic computation has been already done.

4. *Maps to Products of 2-Spheres and to Symplectic Manifolds.* Let

$$\underline{X} = \bigtimes_{i=1}^{k} S_i^2,$$

$\underline{S}_i^2$ are spheres with smooth Riemannian metrics, let X be a compact orientable Riemannian manifold of dimension $2k$ and let

$$f = (f_1, ..., f_k) : X \to \underline{X}$$

be a smooth map.

Let $\underline{\omega}_i$ be the area forms of S_i^2, thus, $\int_{\underline{S}_i^2} \underline{\omega}_i = area(\underline{S}_i^2)$, and let ω_i be the 2-forms on X induced from $\underline{\omega}_i$ by $f_i \to S_i^2$.

Observe that $\| \wedge^2 df_i(x)\| = \|\omega_i(x)\|$ equals the maximal area dilation by f_i at x of surfaces $S \ni x$ in X.

f has non-zero degree, then there exist a point $x \in X$, where the scalar curvature of X is bounded un terms of $\| \wedge^2 df_i(x)\|$ as follows,

($\bigstar_2$)
$$Sc(X,x) \leq 8\pi \sum_i \frac{\| \wedge^2 df_i(x)\|}{area(\underline{S}_i^2)},$$

where the equality holds if and only if X is the product of Euclidean spheres $X = \times_{i=1}^k S^2(r_i)$ with no restrictions on their radii r_i and on the Riemannian metrics in $\underline{S}_i^2$.

Proof. Start by observing that the right hand side of ($\bigstar_2$) doesn't depend on the choice of Riemannian metrics on $\underline{S}_i^2$ and we may assume all $\underline{S}_i^2$ isometric to the unit sphere $S^2 = S^2(1)$.

Let $\underline{L} \to \underline{X} = (S^2)^k$ be the tensor product of the pullbacks of the Hopf bundle over S^2 under the k projections $\underline{X} \to S^2$ and observe that the curvature form of this (complex unitary line) bundle $\underline{L} \to X$ is:

$$curv(\underline{L}) = \frac{1}{2}\sum_i \omega_i.$$

Therefore, for all $x \in X$, the diagonal decomposition of form ω_x in an orthonormal basis in the tangent space $T_x(X)$, orthonormal basis (τ_i, θ_i), $i = 1, ..., k$,

$$\omega = \sum_i \lambda_i \tau_i, \wedge \theta_i,\ \ \lambda_i \geq 0$$

satisfies

$$\sum_i \lambda_i \leq \sum_i \| \wedge^2 df_i(x) \|.$$

It follows (theorem1.1 in [Hitchin(spinors) 1974]) that if

$$Sc(X, x) > 8\pi \sum_i \frac{\| \wedge^2 df_i(x)\|}{area(\underline{S}_i^2)},$$

then X supports *no non-zero harmonic spinors* twisted with L.

On the other hand the top term in the Chern character of L is *non-zero* and the index theorem says that X *does support* such a spinor, and, as everywhere in this kind of argument, the proof follows by contradiction.

Symplectic Manifolds and $\underline{\omega}$-Extremality. The above argument equally applies to maps of non-zero degree between $2k$-dimensional orientable manifolds, $f : X \to \underline{X}$, where $\underline{X}$ is endowed with a closed 2-form $\underline{\omega}$, such that

• the cohomology class $\underline{c} = \frac{1}{2\pi}[\underline{\omega}] \in H^2(\underline{X}; \mathbb{R}$ is *integral*: $\int_S [\underline{\omega}] \in 2\pi\mathbb{Z}$ for all closed oriented surfaces in $\underline{X}$ (the basic example is one half of the area form on S^2);

• the product of $\exp c = 1 + c + \frac{c^2}{2} + ... + \frac{c^k}{k!}$ where $c = f^*(\underline{c}) \in H^2(X)$ for the cohomology homomorphism $f^* : H^2(\underline{X}) \to H^2(X)$, with the *Todd class* $\hat{A}(X)$ (a polynomial in Pontryagin classes of X, see section **??**) doesn't vanish on the fundamental homology class of X

$$(\exp c) \smile \hat{A}[X] \neq 0.$$

(For instance $c^k \neq 0$ and $\underline{X}$ is *stably parallelizable*, which, by *Hirsch immersion theorem*, is equivalent to the existence of a smooth immersion $\underline{X} \to \mathbb{R}^{2k+1}$, while $c^k \neq 0$.)

$\kappa_\star$-Invariant. Let $\underline{X} = (\underline{X}, \underline{\omega}, \underline{h})$ be a smooth manifold, where:

$\underline{\omega}$ is a *differential 2-form* on $\underline{X}$, e.g. a symplectic one, i.e. where $\underline{\omega}$ is closed, the dimension of $\underline{X}$ is even and $\underline{\omega}^m$, $m = \frac{dim(\underline{X})}{2}$, nowhere vanishes on $\underline{X}$

and where $\underline{h} \in H_n(\underline{X})$ is a *distinguished homology class.*

Define $\kappa_\star(\underline{X})$n as the infimum of the numbers $\kappa > 0$, such that all smooth maps of from all closed orientable Riemannian *spin*[129] manifolds of dimension n to $\underline{X}$,

$$f : X \to \underline{X},$$

which send the fundamental homology class of $X to \underline{h}$,

$$f_*[X] = \underline{h},$$

[129]This can be relaxed to properly formulate *spin*^{c}.

satisfy

$$\inf_{x\in X} Sc(X,x) \le 4\cdot\kappa\cdot trace(\omega(x)),$$

where $\omega = f^*(\underline{\omega}))$ is the f-pullback of the form $\underline{\omega}$ and

$$trace(\omega(x)) = \sum \lambda_i$$

for the above g-diagonalization of ω.

(See §$5\frac{4}{5}$ in [G(positive)1996] and section3.4 in [Min-Oo(scalar) 2020] for integral versions of this invariant.)

A Riemannian manifold X is called *$\underline{\omega}$-extremal* if it admits a smooth map $f : X \to \underline{X}$, such that $f_*[X] = \underline{h}$ and

$$Sc(X,x) = 4\cdot\kappa_\star\cdot trace(\omega(x)), \text{ for all } x\in X.$$

The above proof of ($\bigstar_2$) actually shows that the product of spheres $\underline{X} = (S^2)^k$ is *$\underline{\omega}$-extremal* for the sum of the area forms $\underline{\omega}_i$ of the S^2-factors of $\underline{X}$,

$$\underline{\omega} = \sum_i \underline{\omega}_i,$$

where $\underline{h} \in H_{2k}(\underline{X})$ is the *fundamental* class $[\underline{X}]$, where $\kappa = \frac{1}{2}$ and where any *symplectomorphism* $X = \underline{X} \to \underline{X}$ can be taken for f.

Remarks. (a) The above is a reformulation of a special case of *area extremality*[130] theorems from [Min-Oo(Hermitian) 1998], [Bär-Bleecker(deformed algebraic) 1999] and [Goette-Semmelmann(Hermitian) 1999], where the authors establish the $\underline{\omega}$-extremality of several classes of *Kähler manifolds* including compact Hermitian symmetric spaces, Kähler manifolds X with $Ricci(X) > 0$ and also of certain complex algebraic submanifolds $X \hookrightarrow \underline{X} = \mathbb{C}P^N$, with the Fubini-Study form $\underline{\omega}$ on $\mathbb{C}P^N$.

(b) Besides multi-area contraction inequalities there are similar multi-length inequalities, such as the multi-width $\square^n$-inequality from section **??**, where the (stronger) multi-area contraction inequality doesn't apply.

Conjecture All (most?) $\underline{\omega}$-extremal manifolds are *Kählerian*, or closely associated with with *Kählerian* or similar manifolds, such, e.g. as Kälerian$\times\mathbb{T}^m$.

Admission. I don't even see, why the forms $\underline{\omega}$ in all extremal cases must be closed but not, say, "maximally non-closed", such as generic ones.

Question. Are there further sharp inequalities between (norms of) differentials df_i for maps

$$f = (f_1, ...f_i, ...f_k) : X \to \underline{X} = \bigtimes_{i=1}^{k} S^{n_i}, \quad \sum_i n_i = n,$$

with $deg(f) \ne 0$ and (the lower bound on) $Sc(X)$ besides

$$\inf_{x\in X} \frac{Sc(X,x)}{Sc(\underline{X}, f(x))\cdot\sum_i \|\wedge^2 df_i(x)\|} \le 1$$

[130] *Area extremality* of a Riemannian manifold $X = X(g)$ (essentially) means that all metrics g' with $Sc(g') > Sc(g)$ on X must have $area_{g'}(S) < area(S_g)$ for some surface $S \subset X$. If X is a Kähler manifold then ω-extremality (obviously) implies area extremality for the Kähler form ω of X.

from the above (★) and/or its $\| \wedge^2 df_i(x)\|_{trace}$ counterpart?

Namely, what are conditions on numbers σ and $b_1, ...b_i, ...b_k$, such that there exists a compact orientable (spin) manifold X of dimension $n = \sum_i n_i$ with $Sc(X) \geq \sigma$ and a smooth map $f = (f_1, ...f_i, ...f_k) : X \to \underline{X} = \times_{i=1}^k S^{n_i}$ with $deg(f) \neq 0$, such that $\| \wedge^2 df_i(x)\| \leq b_i$ for all $x \in X$?

3.4 Sharp Bounds on Length Contractions of Maps from Mean Convex Hypersurfaces

The Atiyah-Singer theorem, when applied to the *double* $\mathbb{D}(X)$ *of a compact manifold* X with boundary, delivers a non-trivial geometric information on X as well as on the boundary $Y = \partial X$.

For instance, if $mean.curv(Y) > 0$, then, as we explained in section **??**, the natural, continuous, metric g on $\mathbb{D}(X)$ can be approximated by C^2-metrics g' by smoothing g along the "Y-edge" *without a decrease of the scalar curvature* in a rather canonical manner. Here is an instance of what comes this way.

$[Y_{spin} \to \bigcirc]$ **Mean Curvature Spin-Extremality Theorem.** Let X be a compact Riemannian manifold of dimension n with orientable *mean convex boundary*[131] Y and let $\underline{Y} \subset \mathbb{R}^n$ be a smooth compact convex hypersurface.

Let h and $\underline{h}$ denote the Riemannian metrics in Y and $\underline{Y}$ induced from their ambient manifolds and let $h^\natural$ and $\underline{h}^\natural$ be their MC-normalizations (see section **??**),

$$h^\natural(y) = mean.curv(Y, y)^2 \cdot h(y) \text{ and } \underline{h}^\natural(y) = mean.curv(Y, y)^2 \cdot \underline{h}(y).$$

Then, provided the manifold X *is* spin, *all* λ*-Lipschitz maps* $f : Y \to \underline{Y}$ *with* $\lambda < 1$ *are contractible.*

In other words,

if a smooth (Lipschitz is OK) map $f : Y \to \underline{Y}$ *has a non-zero degree, then there exists a point* $y \in Y$*, where the norm of the differential of* f *is bounded from below as follows:*

$$\|df(y\| \geq \frac{mean.curv(Y, y)}{mean.curv(\underline{Y}, f(y))}.^{132}$$

If $\underline{Y} = S^{n-1}$, this extremality, as in the case of the scalar curvature, can can be sharpened with a use of the trace norm of the differential df ..., except that I have not verified the computation and leave the following "theorem" with a question sign.

$[Y_{spin} \to \bigcirc]$ **Mean Curvature Trace Extremality Theorem(?)**[133]. Let X be a compact orientable Riemannian spin manifold of dimension n with orientable boundary Y and $f : Y \to S^{n-1} = \underline{y}$ be a map with $deg(f) \neq 0$.

Then f *can't be trace-wise strictly decreasing with respect to the MC-normalized metrics* $h^\natural = mean.curv(Y)^2 h$ *for the Riemannian metric* h *on* Y *induced from* $X \supset Y$ *and* $\underline{h}^\natural = (n-1)^2 ds^2$ *on* S^{n-1}, that is there is a point $x \in X$, where *the*

[131] This mean the mean curvature of the boundary is non-negative, where the sign convention is such that boundaries of convex domains in $\mathbb{R}^n$ are mean convex.

[132] Here we agree that $\frac{0}{0} = \infty$.

[133] Probably, the quickest way to remove "?", at least for even n, is by adapting/refining the argument from [Lott(boundary) 2020]and/or from [Bär-Hanke(boundary) 2021].

trace-norm of the differential of f is bounded from below by the mean curvature of X as follows:

$$\frac{1}{(n-1)}\sum_{i=1}^{n-1}\lambda_i^{\natural}(y)\geq 1 \text{ for } \lambda_i^{\natural}=\lambda_i(f^*(\underline{h}^{\natural}))/h^{\natural}),$$

which means that the trace-norm of df with respect to the original (non-normalized) metrics satisfies:

$$\frac{1}{(n-1)}||df(y)||_{trace}\geq\frac{mean.curv(\underline{Y},f(y))}{mean.curv(Y,y)}.$$

The simplest and most interesting common corollary of these two theorems is the following.

❂ (*Seemingly Elementary*) **Example.** If the mean curvature of a smooth hypersurface $Y\subset\mathbb{R}^n$ is *bounded from below by* $n-1$, that is the mean curvature of the unit sphere $S^{n-1}\subset\mathbb{R}^n$, then *all λ-Lipschitz map $f:Y\to\mathbb{R}^n$, where $\lambda<1$, are contractible.*[134]

(If "Lipschitz" is understood with respect to the *Euclidean distance function* on X, rather than the larger one which is associated with the *induced Riemannian metric*, the proof easily follows from *Kirszbaum theorem.*)

About the Proof of ○. Let X lie in a (slightly larger) Riemannian n-manifold $X_+\supset X$ without boundary, let $Y_\varepsilon^{n+l-1}\subset X_+\times\mathbb{R}^l$ be the boundary of the ε-neighbourhood of $X\subset X_+\times\mathbb{R}^l$ and let us similarly, define $\underline{Y}_\varepsilon^{n+l-1}\subset\mathbb{R}^{n+l}=\mathbb{R}^n\times\mathbb{R}^l$ as the boundary of the ε-neighbourhood of $\underline{X}\subset\mathbb{R}^{n+l}=$ for $\underline{X}\subset\mathbb{R}^n$ with boundary Y.

Observe – this needs a little computation as in section 1.4 – that the lower bounds on the scalar curvatures of the "interesting parts"

$$Y_{\varepsilon\varepsilon}^{n+l-1}\subset Y_\varepsilon^{n+l-1}\text{ and }\underline{Y}_{\varepsilon\varepsilon}^{n+l-1}\subset\underline{Y}_\varepsilon^{n+l-1}$$

which are ε-close to the original $Y\subset Y_\varepsilon^{n+l-1}$ and $\underline{Y}\subset\underline{Y}_\varepsilon^{n+l-1}$, are perfectly controlled by their mean curvatures, while their complements, being flat in the ambient manifolds, have the same scalar curvatures as X and $\underline{X}$, where the latter is equal to zero.

Then extend $f:Y\to\underline{Y}$ a map

$$f_\varepsilon:Y_\varepsilon^{n+l-1}\to\underline{Y}_\varepsilon^{n+l-1},$$

such that the "interesting part"of Y_ε^{n+l-1} goes to that of Y_ε^{n+l-1} and the complement of one to the complement of the other and such that the "interesting part" of this extensions is done in a most economical manner along normal geodesics to $Y\subset Y_{\varepsilon\varepsilon}^{n+l-1}$ and to $\underline{Y}\subset\underline{Y}_{\varepsilon\varepsilon}^{n+l-1}$.

If we do it with a proper care then, for a small enough ε and l with the same parity as n, we shall be able to apply the spin-area convex extremality

[134]It is impossible not to ask oneself what happens for $\lambda=1$, i.e. where f is distance non-increasing. You bet, such an f is either contractible, or it is an *isometry.* Indeed (almost) all our extremality theorems are accompanied by rigidity results in the equality cases, as we shall see later on.

But it is non-trivial to formulate and hard to solve the *stability problem*: what happens to geometries of hypersurfaces $X_\varepsilon\subset\mathbb{R}^n$ with $mean.curv(X_\varepsilon)\geq n-1$ and to $(1+\varepsilon)$-Lipschitz maps to S^{n-1} with non-zero degrees, when $\varepsilon\to 0$.

theorem [$X_{spin}$$\rightarrow\bigcirc$] from the section 3.3.1 to the map f_ε, which that would need a preliminary smoothing of the manifolds Y_ε^{n+l-1} and $\underline{Y}_\varepsilon^{n+l-1}$ by tiny C^1-perturbations (these manifolds themselves are only C^1-smooth), where, while while smoothing the hypersurface $\underline{Y}_\varepsilon^{n+l-1}$ convex, smoothing of $\underline{Y}_\varepsilon^{n+l-1}$ must *keep the flat part flat.*

Because of the latter, the point $y_\varepsilon \in \underline{Y}_\varepsilon^{n+l-1}$, where

$$Sc(\underline{Y}_\varepsilon^{n+l-1}, f(y\varepsilon)\cdot \| \wedge^2 df(\varepsilon)\| \geq Sc(Y_\varepsilon^{n+l-1}, y\varepsilon),$$

provided by [$X_{spin}$$\rightarrow\bigcirc$] must be necessary located in the "interesting region" $Y_{\varepsilon\varepsilon}^{n+l-1}$; then the needed inequality for the mean curvature of Y will be satisfied by the point $y \in Y$ nearest to y_ε.

Remark about [Y_{spin} $\rightarrow\bigcirc$]. To carry out the above argument one needs a generalization of of the spherical trace inequality [$X_{spin} \rightarrow \bigcirc$] from the previous section to manifold $\underline{X}$ that don't have full $O(n+1)$-symmetry of S^n.

In the present case the relevant metric $\underline{g}$ is $O(n)$ invariant and one needs a separate bounds on the two parts of the trace norm of $\wedge^2 df$:

the first part comes from $\frac{n-1(n-2)}{2}$ bivectors $e_i \wedge e_j$ with e_i and e_j, $i,j = 1,...,n-1$, tangent the S^{n-1}-spherical $O(n)$-orbits and the second one from the $n-1$ remaining $e_i \wedge e_n$ with the vector e_n normal to these orbits.

This is an instance of a more general principle:

to achieve the sharpest inequality, one should choose the norm for measuring df in accordance with the the symmetries of the manifold $\underline{X}$.

We shall see later on other instances of this "principle", e.g. for maps to products of spheres in section 3.3.4.

On Non-spin Manifolds and on $\sigma < 0$**.** Conjecturally, if the boundary $Y = \partial X$ of a compact orientable Riemannin n-manifold X with $Sc \geq -n(n-1)$ admits a smooth map f with non-zero degree to the boundary of the R-ball in the hyperbolic n-space with sectional curvature -1,

$$f : Y \to \partial B(R) \subset \mathbf{H}^n(-1),$$

and if

$$mean.curv(Y) \geq n-1 \text{ and } \|df\| \leq 1,$$

then the map f is an *isometry*. Moreover, f extends to an isometry $X \to B(R)$.[135]

We shall prove a partial result in this direction with a use of *stable capillary μ-bubbles*, which may also apply to maps to more general hypersurfaces in $\mathbf{H}^n(-1)$ (see section **??**), but it remains unclear how to approach the trace-norm version of this conjecture.

Questions and Exercises. (a) Is there an elementary proof of this inequality for $n \geq 4$?

(b) Besides the lower bound on the mean curvature, that is the sum of the principal curvatures, $\sum_i \alpha_i$, the "size" of a hypersurface Y is bounded by the scalar curvature $\sum_{i\neq j} \alpha_i\alpha_j$ and also - this is obvious by the product of the principal curvatures $\prod_i \alpha_i$.

[135]Granted f *is* an isometry (with respect to the induced Riemannin metrics in $\partial X \subset X$ and $\partial B(R) \subset B(R)$), an isometry $X \to B(R)$ follows from *Min-Oo's hyperbolic rigidity theorem* from section **??**.

Are there similar inequalities for other elementary symmetric functions of α_i.

(If $Y \subset \mathbb{R}^n$ is *convex*, i.e. all $\alpha_i \geq 0$, then $\prod_i \alpha_i$ minorizes the rest of elementary symmetric functions, which gives a trivial proof of ❂ and similar inequalities for other symmetric functions for distance decreasing maps from convex hypersurfaces to S^n.)

the above theorems for *convex* hypersurfaces $Y\mathbb{R}^n$.)

But it is unclear if, for instance, there is a bound on this radius in terms of $\sum_{i>j>k} \alpha_i\alpha_j\alpha_k$ for $n \geq 5$ when this sum is positive.)

(d) Let $Y_0 \subset \mathbb{R}^n$ be a smooth compact cooriented submanifold with boundary $Z = \partial Y_0$, such that

the mean curvature of Y_0 with respect to its coorientation satisfies

$$mean.curv(Y) \geq n - 1 = mean.curv(S^{n-1}).$$

Show that

every distance decreasing map

$$f : Z \to S^{n-2} \subset \mathbb{R}^{n-1}$$

is contractible,

where "distance decreasing"refers to the distance functions on $Z \subset \mathbb{R}^n$ and on $S^{n-2} \subset \mathbb{R}^{n-1}$ coming from the ambient Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$.

Hint. Observe that the maximum of the principal curvatures of Y_0 is ≥ 1 and show that *the filling radius* of $Z \subset \mathbb{R}^n$ is ≤ 1.[136]

(e) *Question.* Does contractibility of f remains valid if the distance decreasing property of f is defined with the (intrinsic) spherical distance in S^{n-2} and with the distance in $Z \subset Y_0$ associated with the *intrinsic metric* in $Y_0 \supset Z$, where $dist_{Y_0}(y_1, y_2)$ is defined as the infimum of length of curves in Y_0 between y_1 and y_2?

(f) Formulate and prove the mean curvature counterparts of the theorems $[X_{spin} \overset{\hat{A}}{\to} \bigcirc]]$, $\times\mathbb{R}^m$ and $[X \times P \to \bigcirc]$ for maps $X^{n+m} \to \underline{X}^n$ and $X^n \to \underline{X}^{n+m}$ from sections 3.3.1 and 3.4, either by the above $Y^{n+l-1}_{\varepsilon\varepsilon}$-construction or by generalizing Lott's argument for manifolds with boundaries.

(h) *Question.* Is there a version (or versions) of the mean curvature extremality theorems for maps to products of convex hypersurfaces in the spirit of area multi-contracting maps in section 3.3.4

3.5 Riemannian Bands with $Sc > 0$ and $\frac{2\pi}{n}$-Inequality.

We saw in the previous sections how a use of twisted Dirac operators leads to geometric bounds, including certain sharp ones, on the size of compact Riemannian spin manifolds. Such bounds usually (always) extend to non-compact complete manifolds, but until recently no such result was available for non-complete manifolds and/or for manifolds with boundaries. [137]

[136] This means that Z is homologous to zero in its 1-neighbourhood.

[137] Several such results have appeared in the papers [Zeidler(bands) 2019], [Zeidler(width) 2020] and [Cecchini(long neck) 2020],[Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(Scalar&mean) 2021], [Guo-Xie-Yu(quantitative K-theory) 2020], which we briefly discuss letter on .

On the other hand minimal hypersurfaces were used in [GL(complete)1983] for obtaining rough bounds for non-complete manifolds; below, we shall see how such hypersurfaces (and μ-bubbles in general) serve for getting *sharp* geometric inequalities of this kind.

Bands, sometime we call them *capacitors*, are manifolds X with two distinguished disjoint non-empty subsets in the boundary $\partial(X)$, denoted

$$\partial_- = \partial_- X \subset \partial X \text{ and } \partial_+ = \partial_+ X \subset \partial X.$$

A band is called *proper* if $\partial_\pm$ are unions of connected components of ∂X and

$$\partial_- \cup \partial_+ = \partial X.$$

The basic instance of such a band is the segment $[-1,1]$, where $\pm\partial = \{\pm 1\}$.

Furthermore, *cylinders* $X = X_0 \times [-1,1]$ are also bands with $\pm\partial = X_0 \times \{\pm 1\}$, where such a band is proper if X_0 has no boundary.

Riemannian bands are those endowed with Riemannian metrics and

the width of a Riemannian band $X = (X, \partial_\pm)$ is defined as

$$width(X) = dist(\partial_-, \partial_+),$$

where this distance is understood as the infimum of length of curves in V between ∂_- and ∂_+.

We are mainly concerned at this point with *compact* Riemannian bands X of dimension n, such that

$\square\!\!\!\!\!\!\!\;\;\|_{Sc\ngtr 0}$ *no* closed embedded hypersurface $Y \subset X$, which *separates* ∂_- from ∂_+, admits a $\mathbb{T}^1$-stabilization $Y^\rtimes$ with positive scalar curvature, i.e. no complete (warped product) metric on the product $Y \times \mathbb{T}^1$ of the form $dy^2 + \phi(y)^2 dt^2$ has $Sc(h^\rtimes) > 0$.

(Since Y is compact, the existence of this (warped product) metic $h^\rtimes$ with $Sc(h^\rtimes) > 0$ is equivalent to the existence of a metric h with $Sc(h) > 0$ on Y itself, since the conformal Laplacian $-\Delta + areaextremality 3n - 24(n-1)Sc$ is more positive that the $-\Delta + \frac{1}{2}Sc$ implied by positivity of $Sc(h^\rtimes)$.)

Representative Examples of compact bands with this property are:

$\bullet_{\mathbb{T}^{n-1}}$ *toric bands* which are homeomorphic to $X = \mathbb{T}^{n-1} \times [-1,1]$;

$\bullet_{SYS}$ manifolds X homeomorphic to a Schoen-Yau-Schick manifolds times $[-1,1]$;

$\bullet_{\hat\alpha}$ these, called $\hat\alpha$ *bands,* are diffeomorphic to $Y \times [-1,1]$, where the Y is a closed spin $(n-1)$-manifold with non-vanishing $\hat\alpha$-invariant (see 3.1 the IV above);

$\bullet_{\mathbb{T}^{n-1}\times\hat\alpha}$ these are bands *diffeomorphic to products* $X_{n-k} \times \mathbb{T}^k$, where $\hat\alpha(X_{n-k}) \neq 0$.

(A characteristic non-compact example with a similar property is

$\bullet_{\times\mathbb{R}\setminus\{\mathbb{Z}\}}$: X is homeomorphic to the product $\mathbb{T}^{n-2} \times \mathbb{R} \times [-1,1]$ minus a discrete subset.)[138]

[138]The property $\square\!\!\!\!\!\!\!\;\;\|_{Sc\ngtr 0}$ for *toric bands* and for SYS-bands follows from the Schoen-Yau codimension 1 descent theorem (see section 2.7), in the case $\bullet_{\hat\alpha}$ this is the Lichnerowicz-Hitchin theorem (section 3.1) and $\bullet_{\mathbb{T}^{n-1}\times\hat\alpha}$ is a corollary to theorem 2.1 in [GL(spin) 1980], while a "complete" version of this property for the non-compact $(\mathbb{T}^{n-2} \times \mathbb{R} \times [-1,1]) \setminus \{\mathbb{Z}\}$ is an example, where theorem 6.12. from [GL(complete) 1983] applies. (See sections **??**, **??** for more about these and more general examples.

$\frac{2\pi}{\mathbf{n}}$**-Inequality.** Let X be a *proper compact Riemannian bands* X *of dimension* n *with* $Sc(X) \geq \sigma > 0$.

If no closed hypersurface in X *which separates* ∂_- *from* ∂_+ *admits a metric with positive scalar curvature, then*

$$\left[\circledcirc_\pm \leq \tfrac{2\pi}{n}\right] \qquad width(X) = dist(\partial_-, \partial_+) \leq 2\pi\sqrt{\frac{(n-1)}{n\sigma}} = \frac{2\pi}{n} \cdot \sqrt{\frac{n(n-1)}{\sigma}}$$

In particular if $Sc(X) \geq Sc(S^n) = n(n-1)$, *then*

$$width(X) \leq 2\pi\sqrt{\frac{(n-1)}{n\sigma}} = \frac{2\pi}{n}.$$

Moreover, the equality holds in this case only for warped products $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$[139] *with metrics* $\varphi^2 h + dt^2$, *where the metric* h *on* Y *has* $Sc(h) = 0$ *and where*

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

as in section 2.4.

About the Proof. If a hypersurface $Y \subset X$, which separates ∂_- from ∂_+ contains a descending chain (flag) of closed oriented hypersurfaces,

$$Y \supset Y_{-1} \supset \ldots \supset Y_{-i} \supset \ldots,$$

where where each $Y_{-i} \subset X$ is equal to a transversal intersection of $Y_{-(i-1)}$ with a smooth closed oriented sub-band $H_i \subset X$, of codimension one,

$$H_i \cap X_{-(i-1)} = X_{-i}$$

and where Y_{-i} represent *non-zero* classes in the homology $H_{n-1-i}(X)$, then one can proceed by the inductive Schoen-Yau's kind of descent method (see sections 2.7) with minimal hypersurfaces

$$\ldots X_{-i} \subset X_{-(i-1)} \subset \ldots \subset X_{-1} \subset X,$$

where these X_{-i} are $\mathbb{T}^{\rtimes}$-symmetrised as in the $[\rtimes_\varphi]^N$-symmetrization theorem in section 2.8 where X_{-i} in our band X have "free" (pairs of) boundaries contained in $\partial_\mp(X_{-(i-1)})$, and such that the intersections $X_{-i} \cup Y$ are homologous to Y_{-i}.

This argument delivers the sharp version of $\frac{2\pi}{\mathbf{n}}$ for *over-toric bands*, i.e. those which admit maps $X \to \mathbb{T}^{n-1}$, $n = dim(X)$, with non-zero degrees of their restriction to $\partial_\mp$, but when it comes to SYS-bands, one gets only a weaker lower bound on $width(X)$, that is by $\frac{4\pi}{n}$, instead of $\frac{2\pi}{n}$.

The same weakening of $\frac{2\pi}{\mathbf{n}}$ takes place if separating hypersurfaces $Y \subset X$, are *enlargeable*, e.g. if the interior of X, assumed compact, admits a complete metric with non-positive sectional curvature. And if separating Y are *SYS times enlargeable*, one has to be content with $\frac{8\pi}{n}$.[140]

In section 3.6, we present a more efficient argument, where, instead of working with chains of minimal hypersurfaces, we show in one step that if

[139] Here, since X is non-compact, the width is understood as the distance between the two ends of X.

[140] This is worked out in §2-6 of [G(inequalities) 2018].

$width(X) \geq 2\pi\sqrt{\frac{(n-1)}{n\sigma}}$, then a certain *stable* μ*-bubble* $Y_{st} \subset X$*, which separates* Y_- *from* Y_+*, supports a metric with* $Sc > 0$.

Besides, the sharp $\frac{2\pi}{\mathbf{n}}$ for wide class of *spin* bands was recently proven by Zeidler, Cecchini and Guo-Xie-Yu with new index/vanishing theorems on Dirac operators with potentials on manifolds with boundaries. [141]

Remarks.(a) If hypersurfaces separating ∂_- from ∂_+ in X are enlargeable, e.g. if X is homeomorphic to $\mathbb{T}^{n-1} \times [0,1]$, then a non-sharp version of $\frac{2\pi}{n}$-inequality,

$$dist(\partial_-, \partial_+) \leq 2^n\pi\sqrt{\frac{(n-1)}{n\sigma}}$$

follows from theorem 12.1 in[GL (complete)1983].

(b) One might think that the sharp $\frac{2\pi}{n}$-inequality, must be obvious for domains in the unit sphere S^n homeomorphic to $\mathbb{T}^{n-1} \times [-1,1]$ and for bands with *constant sectional curvatures* in general; to my surprise, I couldn't find a direct proof of it even for X is homeomorphic to $\mathbb{T}^{n-1} \times [0,1]$.

3.5.1 Quadratic Decay of Scalar Curvature on Complete Manifolds with $Sc > 0$.

QD-Exercise. *Quadratic Decay Property.* Let X be a complete non-compact Riemannian n-manifold and $X_0 \subset X$ a compact subset, such that *there is no domain* $X_1 \subset X$, which contains X_0 and the boundary ∂X_1 of which (assumed smooth) *admits a metric* with $Sc > 0$, e.g. X is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$.

Show that there exists a constant $R_0 = R_0(X, x_0)$, such that

the minima of the scalar curvature of X *on concentric balls* $B(R) = B_{x_0}(R) \subset X$ *around a point* $x_0 \in X$*, satisfy*

$$\min_{x \in B(R)} Sc(X, x) \leq \frac{4\pi^2}{(R - R_0)^2} \text{ for all } R \geq R_0.$$

Hint. Apply $\frac{2\pi}{n}$-inequality to the annuli between the spheres or radii R and R for a suitable constant c.

(Compare this with the quadratic decay theorem in section 1 of [G(inequalities) 2018] and see [Wang-Xie-Yu(decay) 2021] for estimates of the scalar curvature decay rates by contractibility radius and the diameter control of the asymptotic dimension and observe that, if X is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$, than the quadratic decay with the constant $2^{n+1}\pi^2$ follows from [GL(complete 1983].)

Critical Rate of Decay Conjecture. There exists a universal *critical constant* c_n, conceivably, $c_n = \frac{4\pi^2(n-1)}{n}$, such that:

[a] if a smooth manifold X admits a complete metric g_0 with $Sc(g_0) > 0$, then, for all $c < c_n$, it admits a complete metric g_ε, with $Sc(g_\varepsilon) > 0$ and at most c*-sub-quadratic* scalar curvature decay,

$$Sc(g_\varepsilon, x) \geq \frac{c}{dist(x, x_0)^2} \text{ for a fixed } x_0 \in X \text{ and all } x \in X \text{ with } dist(x, x_0) \geq 1;$$

[141] See [Zeidler(bands) 2019], [Zeidler(width) 2020], [Cecchini(long neck) 2020] and the most recent [Guo-Xie-Yu(quantitative K-theory) 2020],[Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(Scalar&mean) 2021].

and

[b] if X admits a complete metric g_0 with $Sc(g_0) > 0$ and c-sub-quadratic for $c > c_n$ scalar curvature decay,

$$Sc(g_\varepsilon, x) \geq \frac{c_n}{dist(x, x_0)^2} \text{ for } dist(x, x_0) \geq 1,$$

then it admits a complete metric with $Sc \geq \sigma > 0$.

Moreover,

for all continuous functions $\omega = \omega(d)$, there exists a complete metric g_ω on X, such that

$$Sc(g_\omega, x) \geq \omega(dist(x, x_0)) \text{ for a fixed point } x_0 \text{ and all } x \in X.$$

Here is a related ***compactness conjecture***, which expresses the following idea:

The existence of a complete metric with $Sc \geq \sigma > 0$ on an X is detectible by topologies of compacts parts V of X:

if, for all compact subsets $V \subset X$ and all constants $\rho > 0$, there exists a (non-complete) metric on X with $Sc \geq 1$, such that the closed ρ-neighbourhood $U_\rho(V) \subset X$ is *compact*, then X *admits a complete Riemannian metric with* $Sc \geq 1$.

3.6 Separating Hypersurfaces and the Second Proof of the $\frac{2\pi}{n}$-Inequality

The main ingredient in the proof of the general $\frac{2\pi}{n}$-Inequality is the following.

||| **μ-Bubble Separation Theorem.** Let X be an n-dimensional, Riemannian band, possibly non-compact and non-complete.

Let

$$Sc(X, x) \geq \sigma(x) + \sigma_1, \ ,$$

for a continuous function $\sigma = \sigma(x) \geq 0$ on X and a constant $\sigma_1 > 0$, where σ_1 is related to $d = width(X) = dist_X(\partial_-, \partial_+)$ by the inequality

$$\sigma_1 d^2 > \frac{4(n-1)\pi^2}{n}.$$

(If scaled to $\sigma_1 = n(n-1)$, this becomes $d > \frac{2\pi}{n}$.)

Then there exists a smooth hypersurface $Y \subset X$, which separates ∂_- from ∂_+, and a smooth positive function ϕ on Y, such that the scalar curvature of the metric $g_\phi = g_\phi^{\rtimes} = g_{Y_{-1}} + \phi^2 dt^2$ on $Y \times \mathbb{R}$ is bounded from below by

$$Sc(g_\phi, x) \geq \sigma(x).$$

Derivation of $\frac{2\pi}{n}$-Inequality from |||. If a band X with $Sc \geq \sigma > 0$ has $width(X) = dist(\partial_-, \partial_+) > 2\pi\sqrt{\frac{(n-1)}{n\sigma}}$, then ||| implies the existence of a separating hypersurface Y and a function $\phi(y)$, such that $Sc(g_\phi^{\rtimes}) \geq \varepsilon$ for a *small* $\varepsilon > 0$.

About the Proof of |||. If X is compact and $n \leq 7$, we take a *μ-bubble* Y_{min} for Y, that is the minimum of the functional

$$Y \mapsto vol_{n-1}(Y) - \mu[Y, \partial_-]$$

defined in the space of separating hypersurfaces $Y \subset X$, where $[Y, \partial_-] \subset X$ denotes the region in X between Y and $\partial_- \subset \partial X$ and where the key point is to choose μ suitable for this purpose.

What is required of μ is that

• the boundaries $\partial_\pm$ must serve as barriers for our variational problem and thus ensure the existence of Y_{min};

• positivity of the second variation should imply the positivity of the $\Delta + Sc(Y_{min}) - \sigma$ on Y.

This is achieved with μ, that is modeled after the measure $\underline{\mu}$ on $\mathbb{T}^{n-1} \times [-1, 1]$, (the density of) which is equal the mean curvatures of the hypersurfaces $\mathbb{T}^{n-1} \times \{t\}$ with respect to the warped product metric $\varphi^2 h + dt^2$ for

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}.$$[142]

$|||_{\circlearrowleft}$ **Separation with Symmetry.** *If the Riemannian band is isometrically acted upon by a compact group G, then the separating hypersurface $Y \subset X$ and the function ϕ on Y can be chosen invariant under this action.*

Proof. Use the multi-dimensional Morse lemma (see section 2.9); alternatively, apply more elementary uniqueness/symmetry property of the lowest eigenfunction of the (linear elliptic) second order variational (linear elliptic) $\Delta_Y + s$ on a hypersurface Y, which minimizes the functional $vol_{n-1}(Y) - \mu[Y, \partial_-]$ among G-invariant separating hypersurfaces $Y \subset X$.

Remark. In our case, the group G is the torus $\mathbb{T}^k$, which freely acts on X, and the equivariant μ-bubble problem (trivially) reduces to the ordinary one on the quotient space $X/\mathbb{T}^k$.

To make use of this for the next step of $\mathbb{T}^\rtimes$-symmetrization, one only needs to check – this is an exercise to the reader – that the corresponding warped product with $\mathbb{T}^{k+1}$ will have the same scalar curvature as one gets by doing this in X itself.

Compact/Non-compact. If X is non-compact, then, as usual, we exhaust X by compact submanifolds with boundaries, proceed as in the compact case (these compact bands an not proper, part of their boundary is not contained in $\partial X = \partial_- \cup \partial_+$, but this causes no problem) and then pass to the limit. This is routine.

Example of Corollary. Let $X = (X, g)$ be an n-dimensional manifold with uniformly positive scalar curvature, $Sc(X) \geq \sigma > 0$, and let $f : X \to \underline{X} = \mathbb{R}^{n-m}$ be a smooth proper (infinity to infinity) 1-Lipschitz (i.e. distance non-increasing) map.

Then the homology class of the pullback of the generic point, $f^{-1}(\underline{x}) \subset X$, is representable by a compact submanifold $Y \subset X$, such that the product $Y \times \mathbb{T}^m$ admits a $\mathbb{T}^m$-invariant (warped product) *metric $h^\rtimes$ $(h = g_{|Y})$ with $Sc(h^\rtimes) > 0$.*

[142] This is the same $\varphi(t)$ that was used in section 12 in [GL(complete) 1983] for proving a rough lower bound on the norms of the differentials of smooth maps of non-zero degrees from *non-complete* Riemannian manifolds X with $Sc(X) \geq 1$ to S^n for $n = dim(X) \leq 7$.

Consequently, Y itself admits a metric with $Sc > 0$.

Singularity Problem for $dim(X) > 7$ and the Second Proof of the μ-Bubble Separation Theorem. By the standard theorems of the geometric measure theory, the minimizing μ-bubble $Y \subset X$ exists for all n but it may have singularities of codimension 7.[143]

(The first instance of this is the vertex of the famous cone from the origin over $S^3 \times S^3 \subset S^7 \subset \mathbb{R}^8$.)

If $n = 8$, then (a minor generalization of) Natan Smale's *generic regularity theorem* takes care of things, but if $n \geq 9$ one needs to adapt *Lohkamp's minimal smoothing* results and/or techniques to our case. My, rather superficial, understanding of Lohkamp works suggests that this is possible, but it can't be safely applied unless everything is written out in full detail.

I feel more comfortable at this point with generalizing theorem 4.6 from the Schoen-Yau paper [SY(singularities) 2017], where it used in the inductive descent method with *singular minimal* hypersurfaces, to our minimizing μ-bubbles.

Such a generalization feels plausible and, if it's true, this must be obvious to Schoen and Yau. (I guess, the same can be said about what Lohkamp thinks about generalization of his theorem to μ-bubbles.)

Granted this, one gets the sharp $\frac{2\pi}{\mathbf{n}}$-inequality for SYS-bands, and in fact for all bands X which satisfy ❑, where the poof of non-existence of metrics of positive scalar curvatures on separating hypersurfaces $Y \subset X$ is obtained by exclusively by inductive decent with no appeal to Dirac operators and related invariants, such as the $\hat{A}$-genus and the $\hat{\alpha}$-invariant.[144]

Minimal Hypersurfaces in Non-compact Bands. An essential advantage of μ-bubbles over minimal hypersurfaces is that the former are easier to "trap" them and prevent from fully sliding away to infinity than the former.

For instance if X is a complete non-flat manifold with positive sectional curvature which is conical at infinity then it contains no complete (even locally) volume minimizing hypersurfaces, but it contains lots of stable complete (and compact) μ-bubbles.

However, a version of the $\frac{2\pi}{\mathbf{n}}$ can be proven for non-compact complete bands by reduction to "large" compact *non-proper* bands X, where the boundary is divided into three parts

$$\partial X = \partial_+ \cup \partial_- \cup \partial_{side}$$

where $\partial_+ = \partial_+(X)$ and $\partial_- = \partial_-(X)$ are disjoint with controlled lower bound on the distance between them, while $\partial_{side} = \partial_{side}(X)$, which intersects both ∂_+ and ∂_- is supposed to be far away from the bulk of the intended minimal hypersurfaces in X.

Example. Let $\underline{X}$ be the cylinder $B^{n-1}(R) \times [-1, +1]$, where $B^{n-1}(R)$ the Euclidean R-ball of dimension $n - 1 \geq 2$ and where $\partial_{side}(\underline{X}) = S^{n-3}(R)^{n-1}(R) \times$

---

[143]The most general existence theorem of this type applicable to all codimensions is in the technically difficult Almgren's 1986 paper "*Optimal Isoperimetric Inequalities*".

The existence and regularity theorem we need in codimension one are easier, they follow by the usual technique of integer currents and regularity theorems, see [Ros(isoperimetric) 2001]), [Morgan(isoperimetric)(2003); the arguments from this papers which are applied there to the more traditional formulation of the isoperimetric problem, can be carried over to our μ-bubble setting with no problem; alternatively, one can use the language of Caccippoli sets.

[144]This is complementary to what can obtained by Dirac operators methods of Zeidler and Cecchini.

$[-1,+1]$ for the equatorial sphere $S^{n-3}(r) \subset S^{n-2}(r) = \partial B^{n-1}(r)$.

Let $\underline{\partial}_r \subset \underline{X}$, $r > R$ be the r-cylinder concentric to $\partial_{side}(X)$, that is

$$\underline{\partial}_{side(r)} = S^{n-2}(R)^{n-1}(R) \times [-1,+1].$$

Minimal hypersurfaces $Y = Y_r \subset X$ we shall meet in X will be similar to those $\underline{Y} \subset \underline{X}$, which have their boundaries contained in $\underline{Z}_r = \partial_+(\underline{X}) \cup \partial_-(\underline{X}) \cup \underline{\partial}_{side(r)}$ and which represent non-zero homology classes in $H_{n-1}(\underline{X}, \underline{Z}_r$.

Namely, let X be a compact orientable non-proper n-dimensional band. Let $f = (f_1, f_2) : X \to \underline{X} = B^{n-1}(R) \times [-1,+1$ be a smooth map which sends $\partial_\mp(X) \to \partial_\mp(\underline{X})$ and $\partial_{side}(X) \to \partial_{side}(\underline{X})$ and such that

$\bullet_1$ the map $f_1 : X \to B^{n-1}$ is 1-Lipschitz;

$\bullet_2$ the map $f_2 : X \to [-1,+1]$ is λ-Lipshitz for $\lambda > 0$

$\bullet_3$ the map f has *non-zero* degree.

Observe that

$\bullet_1$ implies that

$$dist(\partial_{side}(X), \partial_{side(r)}(X) \geq R - r;$$

$\bullet_2$ makes

$$width(X) = dist(\partial_-(X), \partial_+(X)) \geq d = d_\lambda = \frac{2}{\lambda};$$

$\bullet_3$ shows that if an oriented hypersurface $Y \subset X$ with $\partial Y \subset Z_r$, represents a *non-zero* homology class in $H_{n-1}(X, Z_r)$, then it necessarily intersects $\partial_{side(r)}(X)$.

In fact, Y intersects every $(n-3)$-dimensional submanifold $Z' \subset Z_r$ (observe that $dim(Z_r) = n-2$ for generic maps f) which separates $\partial_-(Z_r) = Z_r \cap \partial_-(X)$ from $\partial_+(Z_r) = Z_r \cap \partial_+(X)$.

3.6.1 Paradox with Singularities

Singularities must enhance the power of minimal hypersurfaces and stable μ-bubbles rather than to reduce it, since the large curvatures of hypersurfaces $Y \subset X$ (these curvatures are infinite at singularities) *add* to the positivity of the second variation .

Thus, for instance, if a $Y \subset X$, where $dim(X) = 8$ and $Sc(X) \geq -1$, is a stable minimal hypersurface with a singularity at $y_0 \in Y$ and if no smooth submanifold in the homology class of Y admits a metric with $Sc > 0$, e.g. X is homeomorphic to the torus and Y is non-homologous to zero, then scalar curvature of X can't be non-negative outside a small neighbourhood of $y_0 \in X$.

Yet, there is no known argument for $dim(X) \geq 9$ fully implementing this idea.

On $n = dim(X) = 8$. If $dim(X) = 8$ then stable minimal hypersurfaces and μ-bubbles $Y \subset X$ have isolated singularities which can be removed by small generic perturbation as in [Smale(generic regularity) 2003] as follows.

Theorem. Let $Y_0 \subset X$ be a cooriented compact isolated volume minimizing hypersurface and let et $X_t = [X_0, Y_t] \subset X$ be the bands between Y_0 and hypersurfaces Y_t, which are positioned close to Y_0 on their "right sides" in X, and which minimize the function $Y \mapsto vol_{n-1}(Y) - t \cdot vol[Y_0, Y]$ for $0 \leq t \leq \delta$ for a small $\delta > 0$.

If $n = 8$, then submanifolds Y_t are non-singular for an open dense set of $t \in [0, \delta]$.

Outline of the Proof. The key (standard) facts one needs here are as follows.

1. *Monotonicity.* If the sectional curvature of X is bounded by $\bar{\kappa}^2$, then the volume of intersections of m-dimensional minimal subvarieties $Y \subset X$ with r-balls $B_{y_0}(r) \subset X$ centered at $y_0 \in Y$ satisfy

$$\frac{dr^{-m}vol_m(Y \cap B_x(r)}{dr} \le const_n r\bar{\kappa}. \text{ for all } r \le r_0 = r_0(X, y_0) > 0.$$

2. *Corollary.* The densities of (singularities of) minimal $Y \subset X$ are *semicontinuous:*

if be a sequence of pointed manifolds with uniformly bounded geometries, (X_i, x_i), Haussdorf converges to (X, x) and if minimal subvarieties $Y_i \subset X_i$, which contain the points x_i, current-converge to $Y \subset X$, then

$$\limsup dens(Y_i, x_i) \le dens(X, x),$$

where, recall,

$$dens(Y, x) = \lim_{r \to 0} r^{-m} vol_m(Y \cap B_x(r)), mboxm = dim(Y).$$

3. *Weak Compactness*: The set $\mathcal{Y}_A$ of minimal subvaraities $Y \subset X$ with volumes bound by a constant A is compact in the current topology for all $A < \infty$.

4. *Codimension one Intersection Property.* Minimal codimension one cones $C_1, C_2 \subset \mathbb{R}^n$ *necessarily intersect* by the *maximum principle.*

5. *Split Cone Property.* Let $C \subset \mathbb{R}^n$ be a minimal cone. Then either the density of this cone at the apex $0 \in C$ is maximal $+\varepsilon$,

$$dens(C, 0) \ge dens(C, c) + \varepsilon \text{ for all } 0 \ne c \in C \text{ and some } \varepsilon = \varepsilon(C) > 0,$$

or the cone split, i.e. $C = C_{-1} \times \mathbb{R}^1$ for a minimal cone $C_{-1} \subset \mathbb{R}^{n-1}$.

Now, turning to the proof, let all Y_t have singularities at some points $y_t \to y_0 \in Y_0$, $t \to 0$, and assume without loss of generality, this is possible due to 2, that

$$dens(Y_t, y_t) = dens(Y_0, y_0).^{145}$$

Let λ-scale these Y_t at y_0, thus making λY_0, converge to a minimal cone, call it $Y_0' \subset T_{y_0}(X)\mathbb{R}^n$, and let Y_t' be what remains of the limits of other Y_t.

Since these Y_t' don't intersect Y_0', none of Y_t' is conical, which is only possible if the singularities of Y_t slide tangentially along Y_0 for $t \sim \lambda^{-1}$ by the distance $c(t)$, such that $c(t)/\lambda \to \infty$ for $\lambda \to \infty$. It follows that if all Y_t were singular, these singularities would accumulate in the limit to a (one dimensional or larger) singularity of Y_0' of constant density equal to that of $dens(Y_0, y_0)$. Therefore, the cone Y_0' splits, since $n = 8$, it is non-singular and the proof follows by contradiction.

On $n = dim(X) \ge 8$. (a) Schoen-Yau in their desingularization argument apply descent by warped $\mathbb{T}^{\rtimes}$-symmetrised/stabilized minimal hypersurfaces

$$X = X^n \supset Y^{n-1} \supset ... \supset Y^{n-i} \supset .... \supset Y^2,$$

[145] Our Y_t are μ-bubble rather than minimal, but this makes no difference at this point.

where minimization and $\mathbb{T}^{\rtimes}$-stabilization (essentially) apply to non-singular parts of these Y and where the main difficulty, as far as I can see, is to show that Y^{n-i} can't be eventually sucked in the singularity of Y^{n-1},[146] and where the outcome of this process - the surface Y^2 – is non-singular.[147]

(b) The main desingularization result by Lohkamp in [Lohkamp(smoothing) 2018], is

approximation theorem of volume minimizing codimension one cones $C^{n-1} \subset \mathbb{R}^n$ by smooth minimal hypersurfaces (generalizing Smale's result in the case of cones) with the following

Splitting Corollary. sf Let X be a compact orientable Riemannian manifold with $Sc(X) > 0$. Then all homology classes in $H_{n-1}(X)$ are representable by hypersurfaces $Y \subset X$, which support metrics with $Sc > 0$.

Remarks (a) As far as the topology of compact manifolds with $Sc > 0$ this result is more general than that by Schoen and Yau.

For instance it implies that

products of Hitchin's spheres and connected sums of tori with non-spin manifolds admit no metrics with $Sc > 0$.

Nor alternative proof of this kind of results is available.

(b) As far as I understand,[148] Lohkamp's smoothing allows applications of our μ-bubble arguments to manifolds of all dimensions n, with possible exceptions for *rigidity theorems* for *non-compact* manifolds.

(c) The above 1-5 seems to suffice for smoothing conical singularities (am I missing hidden subtleties?) but it is unclear to me how Lohkamp's splitting corollary for $n \geq 9$ follows from it.

3.6.2 $\mathbb{T}^{\rtimes}$-Stabilized Scalar Curvature and Geometry of Submanifolds of Codimensions One, Two and Three

Besides $\frac{2\pi}{\mathbf{n}}$, there are other immediate applications of the separation theorem |||.

[1] Compact Exhaustion Corollary. Let X be a complete Riemannian manifold with $Sc(X) \geq \sigma > 0$.

Then X can be exhausted by compact domains U_i with smooth boundaries $Y_i = \partial U_i$

$$U_1 \subset U_2 \subset \ldots \subset U_i \subset \ldots \subset X, \ \bigcup_i U_i = X,$$

such that U_{i+1} is contained in the ρ-neighbourhood of U_i for all $i = 1, 2, \ldots$ and and where all Y_i admit $\mathbb{T}^{\rtimes}$-extension $Y_i \rtimes \mathbb{T}^1$ with

$$Sc(Y_i \rtimes \mathbb{T}^1) \geq \frac{\sigma}{2}.$$

[146] If $n \leq 9$, this problem for overtorical X can be handled with Dirac operators, as in section 5.3 in [G(billiards) 2014].

[147] Schoen and Yau articulate their main results (theorems 4.5 and 4.6 in [SY(singularities) 2017]) for compact SYS-manifolds, although the basic arguments of their paper are essentially local and and apply to a wider class of manifolds.

[148] My understanding of the results by Lohkamp as well as those by Schoen and Yau is limited, since I haven't mastered the proofs from [SY(singularities) 2017]) and from [Lohkamp(smoothing) 2018].

Poof. Let $S(10), S(20), ...S(10i), ... \subset X$ be concentric spheres around a point $x_0 \in X$, let Y_i be hypersurfaces in the annuli $[S(10i), S(10(i+1))]$ between these spheres, which separate $S(10i)$ from $S(10(i+1))$ and which enjoy the properties supplied by III. Then take the domains in X bounded by Y_i for U_i.

[2] **Codimension 2 Corollary.** Let X be a (possibly non-compact) connected orientable n-dimensional Riemannian manifold with boundary, let $\underline{X}$ be a compact connected orientable surface with boundary and with an arbitrary metric compatible with topology and let $\Psi : X \to \underline{X}$ be a smooth *distance decreasing* map which sends the boundary ∂X to $\partial \underline{X}$.

If $Sc(X) \geq \sigma + \sigma_1$, $\sigma, \sigma_1 > 0$ and the inradius of $\underline{X}$ is bounded from below by

$$inrad(\underline{X}) = \sup_{\underline{x} \in \underline{X}} dist(\underline{x}, \partial \underline{X}) > \frac{2\pi}{\sqrt{\sigma}},$$

then X contains an oriented codimension two (possibly disconnected) submanifold $Y \subset X$, which, if X is non-compact, is *properly* embedded to X *and which is homologous* for the homology group $H^{ncpt}_{n-2}(X)$ with *infinite* supports in the case of non-compact X) *to the pullback $\Psi^{-1}(\underline{x}) \subset X$ of a generic point $\underline{x} \in \underline{X}$, and such that Y with the induced Riemannian metric from X admits a $\mathbb{T}^2$-extension, that is the product $Y \times \mathbb{T}^2$ with the metric $g_\phi = dy^2 + \phi^2(dt_1^2 + dt_2^2)$, such that*

$$Sc(g_\phi) \geq \sigma_1.$$

Proof. Let $X_1 \subset X$ be the I-hypersurface that, according to III, separates the boundary of X from the f-pullback of the (small disc around) the point $\underline{x} \in \underline{X}$ furthest from the boundary (as in the proof of $\mathbb{T}^\rtimes$-stabilized Bonnet-Myers diameter inequality [BMD] in section 2.8 and apply $\frac{2\pi}{\mathbf{n}}$ to the infinite cyclic covering of $X_1 \rtimes \mathbb{T}^1$ induced by the natural cyclic covering of $\underline{X}$ minus this point.

[2′] **Codimension 2 Sub-Corollary.** Let X be a closed orientable n-dimensional Riemannian manifold with $Sc(X) \geq \sigma > 0$, let $\underline{X}$ be a closed surface with an arbitrary metric compatible with topology and let $\Psi : X \to \underline{X}$ be a smooth *distance decreasing* map.

If no *closed oriented codimension two submanifold $Y \subset X$* homologous to the pullback $\Psi^{-1}(\underline{x}) \subset X$ *of a generic point $\underline{x} \in \underline{X}$ admits a metric with $Sc > 0$, then the diameter of the surface $\underline{X}$ is bounded in terms of σ as follows.*

$$diam(\underline{X}) < \frac{2\pi}{\sqrt{\sigma}}.$$

Proof. Let $\underline{x}_0, \underline{x}_1 \in \underline{X}$ be mutually furthest points and apply the above to the pullback X_- of the complement $\underline{X}_-$ to a small disc in $\underline{X}$ around $\underline{x}_0$.

[3] **Area non-Contraction Corollary.** Let X be a proper compact orientable Riemannian band of dimension $n+1$, let $\underline{X} \subset \mathbb{R}^{n+1}$ be a smooth convex hypersurface and let $f : X \to \underline{X}$ be a smooth map the restriction of which to $\partial_- \subset \partial X$ (hence, to ∂_+ as well) has *non-zero* degree.

If X is spin and if n is even,[149] *then there exists a point $x \in X$, where the exterior square of the differential of f is bounded from below in terms of*

[149] As we have said already several times, these conditions must be redundant.

$d = width(X) = dist(\partial_-, \partial_+)$ and the scalar curvature $Sc(X, x)$ as follows.

$$Sc(\underline{X}, f(x)) \cdot || \wedge^2 df(x)|| \geq Sc(X, x) - \frac{4(n-1)\pi^2}{nd^2}.$$

Furthermore, *if $\underline{X} = S^n$, then, now for odd as well as for even n, the trace norm of $\wedge^2 df$ satisfies:*

$$2|| \wedge^2 df(x)||_{trace} \geq Sc(X, x) - \frac{4(n-1)\pi^2}{nd^2}.$$

Proof. Apply the $\mathbb{T}^m$*-stabilized area/mapping extremality theorem* (3.3.1, 3.3.4) for $m = 1$ to $Y \rtimes \mathbb{T}^1$ where $Y \subset X$ is the separating hypersurface from III.

Exercises. (**a**) *Codimension 3 Linking Inequality.* Let X be a closed orientable n-dimensional Riemannian manifold with $Sc(X) \geq \sigma > 0$, let $\underline{X}$ be the 3-sphere with an arbitrary metric compatible with topology and let $f : X \to \underline{X}$ be a smooth *distance decreasing* map. Show that

if no *closed oriented codimension three submanifold $Y \subset X$* homologous to the pullback $f^{-1}(\underline{x}) \subset X$ *of a generic point $\underline{x} \in \underline{X}$ admits a metric with $Sc > 0$, then the distances between all pairs of embedded circles $S_1, S_2 \subset \underline{X}$ with* non zero linking numbers *between them satisfy:*

$$dist(S_1, S_2) < \frac{2\pi}{\sqrt{\sigma}}.$$

Hint. Use the argument from the proof of the codimension 2 corollary **[2]** and consult [Richard(2-systoles) 2020][150]

(**b**) *Area non-Contraction in Codimension 3.* Let X, $\underline{X}$ and $f : X \to \underline{X}$ be as in (**a**), let $\underline{X}_1 \subset \mathbb{R}^{n-2}$ be a smooth closed convex hypersurface and let $f_1 : X \to \underline{X}_1$ be a smooth map, such that the "product" of the two maps,

$$(f, f_1) : X \to \underline{X} \times \underline{X}_1,$$

has *non-zero degree.* Show that

if X is spin and n is odd (thus, $dim(\underline{X}_1)$ even) then there exists a point $x \in X$, where the exterior square of the differential of f is bounded from below in terms of $d = width(X) = dist(\partial_-, \partial_+)$ and the scalar curvature $Sc(X, x)$ as follows.

$$Sc(\underline{X}, f(x)) \cdot || \wedge^2 df(x)|| \geq Sc(X, x) - \frac{4(n-1)\pi^2}{nd^2},$$

for d equal the supremum of the distances between pairs of linked circles in $\underline{X}$.

3.6.3 On Curvatures of Submanifolds in the unit Ball $B^N \subset \mathbb{R}^N$

(The earlier versions of this section contained errors.)

Here is our

[150]Our codimension 2 area bounds, including this exercise, are motivated by Richard's bound on systoles of 4-manifolds with $Sc > \sigma$ proved in this paper.

Problem. Given a closed smooth n-manifold X and a number $N > n$, evaluate the minimum of the curvatures of smooth immersion of X to the unit N-ball,

$$f : X \hookrightarrow B^N = B^N(1) \subset \mathbb{R}^N.$$

We shall briefly describe in this section what is known and and what is unknown about this problem and refer to section 3 and 7 in [G(inequalities) 2018] and to [G(growth of curvature) 2021] for more general discussion and for the proofs.

Six Examples of Immersed and Embedded Manifolds with Small Curvatures

Just to clear the terminology, we agree that a smooth map $f : X \to Y$ is an *immersion* if the differential $df : T(X) \to T(Y)$ is *injective* on all tangent spaces $T_x(X) \subset T(X)$.

An immersion f of a compact manifold is an *embedding* if it has no double points, $f(x) \neq f(y)$ for $x \neq y$.

If Y is a *Riemannian* manifold, e.g. $Y = \mathbb{R}^N$, then the curvature of this f, denoted

$$curv_f(X) = curv_f(X \hookrightarrow Y) = curv(X \hookrightarrow Y) = curv(X),$$

is the *supremum of the Y-curvatures of all geodesics* in X, where "geodesic" is understood with respect to the Riemannian metric in X induced from Y.

1. *Clifford Embeddings.* Here, $X = X^n$ is the product of m spheres of dimensions n_i, $\sum_{i=1}^m n_i = n$, all of the radius $r = \frac{1}{\sqrt{m}}$,

$$X = S^{n_1}\left(\frac{1}{\sqrt{m}}\right) \times \ldots \times S^{n_i}\left(\frac{1}{\sqrt{m}}\right) \times \ldots \times S^{n_m}\left(\frac{1}{\sqrt{m}}\right)$$

and

$$f_{Cl} : X \hookrightarrow S^{N-1} \subset B^N(1) \subset \mathbb{R}^N, \ N = m + \sum_i n_i,$$

is the obvious embedding, that is the $\frac{1}{\sqrt{m}}$-scaled Cartesian product of the imbeddings $S^{n_i}(1) \subset \mathbb{R}^{n_i+1}$.

Clearly,

$$curv_{f_{Cl}}(X \hookrightarrow B^N) = \sqrt{m}$$

and the curvature of X in the unit sphere is

$$curv_{f_{Cl}}(X \hookrightarrow S^{N-1}) = \sqrt{m-1}.$$

Two natural questions arise:

Can the products of spheres be immersed to the unit ball with smaller curvatures?

Are there non-spherical, immersed or embedded, submanifolds $X \hookrightarrow B^N(1)$ with $curv(X) < \sqrt{2}$?

A definite answer is available only for immersions $X^n \to S^{n+1}$ by a *theorem of Jian Ge.* [151]

[151] See [Ge(linking) 2021] and ◊ in this section.

[$\bigcirc \times \bigcirc$] *Clifford's are the only codimension one immersed non-spherical submanifolds* X *in the spheres with curvatures* $curv(X \hookrightarrow S^{n+1}) \leq 1$.

But if $m \geq 3$ then there are immersions of *non-spherical* n-manifolds to S^{n+m-1} with smaller curvature.

2. *Veronese embeddings* of projective spaces to spheres,

$$f_{Ver} : \mathbb{R}P^n \to S^{\frac{(n+1)(n+2)}{2}-2} \subset B^{\frac{(n+1)(n+2)}{2}-1} = B^{\frac{(n+1)(n+2)}{2}-1}(1)$$

satisfy

$\left[\sqrt{\frac{n-1}{n+1}}\right]$
$$curv_{f_{Ver}}\left(\mathbb{R}P^n \hookrightarrow S^{\frac{(n+1)(n+2)}{2}-2}\right) = \sqrt{\frac{n-1}{n+1}} < 1.$$

and

$$curv_{f_{Ver}}\left(\mathbb{R}P^n \hookrightarrow B^{\frac{(n+1)(n+2)}{2}-1}\right) = \sqrt{\frac{n-1}{n+1}+1} < \sqrt{2}.$$

Conjecturally, these have the minimal curvatures among *all non-spherical* n-submanifolds in the unit spheres and unit balls, where the minimum for all n is achieved (only conjecturally) by Veronese's projective plane in unit 4-sphere, where

$\left[\frac{1}{\sqrt{3}}\right]$,
$$curv_{f_{Ver}}(\mathbb{R}P^2 \hookrightarrow S^4) = \frac{1}{\sqrt{3}} = 0.577350...$$

and

$$curv_{f_{Ver_2}}(\mathbb{R}P^2 \hookrightarrow B^5) = \frac{2}{\sqrt{3}} = 1.15470... \ .$$

3. The $\frac{1}{\sqrt{l}}$-scaled Cartesian power of the Veronese map

$$F = \frac{1}{\sqrt{l}} \cdot f_{Ver}^{\times l} : X^{2l} = \underbrace{\mathbb{R}P^2 \times ... \times \mathbb{R}P^2}_{l} \to S^{4l-1} \subset B^{4l}(1)$$

competes with the Clifford embedding, for

$$curv_F(X^{2l} \hookrightarrow B^{4l}) = \sqrt{l} \cdot \sqrt{\frac{l}{3}+1} < \sqrt{2l}.$$

4. If $N \geq (1+\Delta)^n$, say for $\Delta > 10$ then *all* n*-manifolds* X admits immersions

$$f : X \hookrightarrow S^N$$

with

$$curv_f(X) \leq C_\Delta,$$

where $C_\Delta < \sqrt{2}$ for all n and where

$$C_\Delta \to \sqrt{2 - \frac{6}{n+2}} \text{ for } \Delta \to \infty$$

with the rate of convergence which, a priori, may depend on n.

It is unclear if the "true" C_∞ is, actually, *smaller* than $\sqrt{2-\frac{6}{n+2}}$ and it is also unclear what happens to C_Δ for Δ close to zero.

5. It easily follows from the above that

if the dimension n_m of the last factor in a product of spheres

$$X^n = \bigtimes_{i=1}^{m} S^{n_i}, \ \sum_{i=1}^{m} n_i = n,$$

is much greater then the remaining ones, say, roughly,

$$n_m \geq \exp\exp \sum_{i-1}^{m-1} n_i,$$

then X^n admits an immersion

$$f : X^n \hookrightarrow B^{n+1}(1)$$

such that

$$curv_f(X^n) < 2\sqrt{3}.$$

This is *smaller* than Clifford's $\sqrt{m}$ starting from $m = 12$.

It is unclear, however, if these X^n admit *embeddings* to the unit ball with $curv(X^n \hookrightarrow B^{n+1}) \leq 100$, for example.

6. There are *no topological bounds on curvatures* of *immersed* submanifolds of a *given dimension* n:

if an X^n admits a smooth immersion to $\mathbb{R}^N$, then it also admits an immersion to the unit ball with $curv(X^n \hookrightarrow B^N) < const_n$.

But all we can say about this constant is, roughly, that

$$0.1n < const_n < 10n^{\frac{3}{2}}.$$

Imbeddings, at least these with codimension one, are different from immersions in this regard.

For instance, if $X = X^n$ is disconnected and contains m *mutually non-diffeomorphic* components, then, clearly,

$$curv_f(X \hookrightarrow B^{n+1}) \geq const_n m, \ const_n \geq \frac{1}{(10n)^n},$$

for all embeddings $f : X \hookrightarrow B^{n+1}(1)$.

It is also not hard to construct similar *connected* X for $n \geq 6$ and, probably, for all $n \geq 3$.

Conceivably the same is possible for imbeddings with higher codimensions k, at least for $k << n$, where one expects that, say for $k < \frac{n}{3}$ and a given, *arbitrarily large*, constant $C > 0$, there exists

a connected n-dimensional submanifold $X \subset \mathbb{R}^{n+k}$, such that all imbeddings $X \hookrightarrow B^{n+k}(1)$ satisfy

$$curv(X \hookrightarrow B^{n+k}) \geq C.$$

But it should be noted that

all connected orientable surfaces embed to the unit ball B^3 with curvatures ≤ 100

and

the *connected sums* X of copies of products of spheres with any number of summands admit *embeddings*

$$f : X \hookrightarrow B^{n+1}(1),\ n = dim(X),$$

with

$$curv_f(X) \le 100n^{\frac{3}{2}}.$$

Questions. Do all smooth n-manifold admit embeddings to the unit $2n$-ball with

$$curv(X^n \hookrightarrow B^{2n}) \le 100?$$

Do the products of spheres

$$X = \bigtimes_{i=1}^{m} S^{n_i},\ \text{where all } n_i \ge 2, \text{ e.g. } X = (S^2)^m,$$

embed to $B^N(1)$, $N = 1 + \sum_i n_i$ with $curv(X) \le 100$?

LOWER BOUNDS ON $curv(X)$.

A. It is obvious that

immersed n-manifolds $X \hookrightarrow B^N(1)$ with $curv(X) \le 1 + \delta$ for a small $\delta > 0$ keep close to an equatorial N-sphere in $S^n \subset S^{N-1} = \partial B^N$; thus, they are diffeomorphic to S^m.

In fact, it is is not hard to show, that

$\delta = 0.01$, *is small enough for this purpose,*

while, conjecturally, this must hold for

$$\delta < \frac{2}{\sqrt{3}} = 1.15470...$$

with the Veronese surface being the extremal one.

B. Also conjecturally,

$[\bigcirc \times \bigcirc]_?$ the inequality $curv_f(X) < \sqrt{2}$ for codimension one immersions $f : X \to B^{n+1}$ must imply that X is diffeomorphic to S^n (with the equality for non-spherical X achieved by the Clifford embeddings).

This is apparently unknown even for $n = 2$..

C. Let X be an n-dimensional $\nexists$-PSC manifold, i.e. *admitting no metric with* $Sc > 0$, e.g. Hitchin's sphere or a connected sum of n-tori.

Then a simple application of Gauss's Theorema Egregium,[152] shows that

immersions $f : X \to S^N$ satisfy

$$curv_f(X) \ge \sqrt{\frac{n-1}{N-n}}$$

and

$$curv_f(X) \ge \sqrt{1 - \frac{1}{n}}.$$

for all n and N.

[152] Compare with [Guijarro-Wilhelm(focal radius) 2017].

Here, observe, it is as it should be: no contradiction with the above 4, for

$$1-\frac{1}{n}\le 2-\frac{6}{n+2}$$

for all $n\ge 2$ with the the equality for $n=2$.

D. If $X=X^n$ is $\nexists$-PSC, then all immersions $f:X\to B^N=B^N(1)$ satisfy

$$curv_f(X)\ge \frac{1}{C_\circ}\sqrt{\frac{n-1}{N-n}+1}$$

where $C_\circ>0$ is a universal constant that is defined as

the minimal possible increase of curvatures of curves

under smooth immersions $B^N\to S^n=S^N(1)$. More precisely, $C_\circ$ is the infimum of the numbers $C>0$, for which

there exits an immersion $g:B^N\subset S^N$, such that all curves $S\subset B^N$ with curvatures

$$curv_{B^N}(S)\le\sqrt{1+\kappa^2}$$

are sent to curves with curvatures

$$curv_{S^N}(g(S))\le C\kappa.$$

This $C_\circ$, most probably, is assumed by *a radial* (i.e. $O(n)$-equivariant) map g and then it must be easily computable; without computation, one can get

$$C_\circ<4.$$[153]

E. **Conjecture + Theorem.** If If $X=X^n$ is $\nexists$-PSC, then conjecturally all immersions $f:X\to B^N=B^N(1)$ satisfy

$$\left[\tfrac{n}{N-n}\right]\qquad curv_f(X)\ge const\frac{n}{N-n}.$$

E$_1$. *It is esay to see* in this regard that the $\frac{2\pi}{n}$-inequality yields this conjecture for $N=n+1,n+2$:

if $N=n+1$, then

$$curv_f(X)\ge\frac{N}{2\pi}=\frac{n+1}{2\pi}.$$

and if $N=n+2$, then

$$curv_f(X)\ge\frac{N}{4\pi}=\frac{n+2}{4\pi}.$$

Here we must recall that our proof of the $\frac{2\pi}{n}$-inequality in section 3.5 is unconditional only for $N\le 8$, where these inequalities are not especially informative. And if $N\ge 9$, our proof relies on not formally published "desingularization" results by Lohkamp and by Schoen-Yau.

[153] A natural candidate for g is a *projective map*, where $curv_{S^n}(g(S)\le const_g curv_{B^n}(S)$ for *all* curves $S\subset B^n$. But since we are essentially concerned only with what happens to curves with $curv>1$, the best g doesn't have to be projective – it might be conformal, for example.

Fortunately, there are now two Dirac theoretic proofs for a large class of $\nexists$-PSC manifolds of all dimensions, including n-tori $\mathbb{T}^n$ and connected sums of these for, example.[154]

E$_2$. If X is *enlargeable* e.g. the connected sum of the n-torus with another closed manifold, then a minor generalization of the Schoen-Yau "desingularization" theorem allows a proof of the following version of $\left[\frac{n}{N-n}\right]$ for $N = n+3$:

$$curv(X \hookrightarrow B^N) \geq const_3 N,$$

where, roughly, $const_3 > \frac{1}{16\pi}$.

Also, granted a more serious (but realistic) generalization of the Schoen-Yau result or a version of Lohkamp's theorem, one can prove a similar inequality for $N = n+4$.

$$curv(X \hookrightarrow B^N) \geq const_4 N$$

with $const_4 > \frac{1}{400\pi}$.

Finally, assuming that one can "go around singularities" of stable μ-bubbles, and that (this is more serious)

the filling radii of n-manifolds Y with $Sc(Y) \geq \sigma > 0$ satisfy

$$filrad(X) \leq 100\frac{n}{\sqrt{\sigma}},$$

one can show for all n and $k = N-n$ that

$$curv(X \hookrightarrow B^N) \geq const_k N$$

where one needs $const_k$ about $\frac{1}{500^{500}k}$.

F. All of the above equally applies to immersions of *products of enlargeable manifolds X_0 with spheres*, say to

$$f : X = X_0^{n_0} \times S^{n_1} \to B^{n_0+n_1+k},$$

where we conjecture that

$$\left[\frac{n_0}{n_1+k}\right] \qquad curv_f(X \subset B^{n_0+n_1+k}) \geq const\frac{n_0}{n_1+k}$$

and where the case $n_1 + k \leq 4$ is within reach. (Notice that $\left[\frac{n_0}{n_1+k}\right]$ implies $\left[\frac{n}{N-n}\right]$.)

Four Questions

I. Are there lower bound on $curv_f(X)$ unrelated to the scalar curvature?

II. What is the minimal dimension $N = N(n)$ such that all n-manifold can be immersed to the unit N-ball with curvatures $\leq$ 1 000?

III What is the minimal $C = C(n)$ such that the n-torus can be immersed to the unit $(n+1)$-ball with

$$curv(\mathbb{T}^n \hookrightarrow B^{n+1}) \leq C?$$

[154]See [Cecchini-Zeidler(generalized Callias) 2021] and [Guo-Xie-Yu(quantitative K-theory) 2020].

IV Can the Cartesian n-th power of the 2-sphere be immersed to the unit $(2n+1)$-ball

$$X = \underbrace{S^2 \times \ldots \times S^2}_{n} \hookrightarrow B^{2n+1}$$

with

$$curv(X \hookrightarrow B^{2n+1}) \leq 100?$$

Looking back on the above examples, questions and conjectures, one may be disconcerted by their chaotic irregularity. But this only highlights the patchiness of our present-day knowledge of the basic geometry of submanifolds in Euclidean spaces.

$\diamondsuit$ *Wide bands with sectional curvatures* ≥ 1. Let a proper compact Riemannian band Y (see 3.5) of dimension $n+1$ admit an immersion to a complete $(n+1)$-dimensional Riemannian manifold Y_+ with sectional curvature

$$sect.curv(Y_+) \geq 1,$$

and let the width of Y with respect to the induced Riemannian metric satisfy

$$width(Y) = dist(\partial_- Y, \partial_+ Y) > \frac{\pi}{2}.$$

Then

Y contains a subband $Y_- \subset Y$ of width $d = width(Y) > \frac{\pi}{2}$, which is homeomorphic to the spherical cylinder $S^n \times [0,1]$.

Acknowledgement. A similar result for $n = 3$ is proved in [Zhu(width) 2020], while our argument below follows that of Jian Ge from [Ge(linking) 2021], who sent me his preprint prior to publication.

Proof. Let Y_- be the intersection of the d-neighbourhoods of the $\partial_\mp$-boundaries of Y,

$$Y_- = U_d(\partial_-) \cap U_d(\partial_+),$$

and observe that the $\partial_\mp$-boundaries of this Y_- are *concave* for $\kappa \geq 1$ and $d > \frac{\pi}{2}$. Therefore, $\partial_\mp$ are diffeomorphic to S^{n-1} and the immersions

$$\partial_\mp \to Y_+$$

extend to immersions of n-balls, such that the *locally convex* boundaries of these are equal to $\partial_\mp$ (with their coorientations opposite to those in Y_-). [155]

It follows, that if Y_+ is simply connected, then then the immersion $Y_- \to Y_+$ is one-to-one and the complement $Y_+ \setminus Y_-$ consists of two convex balls with distance $> \frac{\pi}{2}$ between them.

Hence, $diam(Y_+) > \frac{\pi}{2}$ and Y_+ is homeomorphic to S^{n+1} by the *Grove-Shiohama diameter theorem*; consequently, Y_- is homeomorphic to $S^n \times [0,1]$. QED.

[155]Recall that *a closed immersed locally convex* hypersurface in a complete Riemannian manifold of dimension $n \geq 3$ with sectional curvatures > 0 bounds an immersed ball.

Remark. (a) The conclusion of the theorem, probbaly, holds if $sect.curv(Y_-) \geq 1$ and $sect.curv(Y_-) \geq 0$, since the proof of the diameter theorem seems to work in this case.

(b) It also doesn't seem difficult to prove the rigidity theorem a la *Berger-Gromoll-Grove* in case of an *open band with* $width(Y) = \frac{\pi}{2}$, where the only alternatives to the homeomorphism of Y to $S^n \times (0,1)$ should be as follows:

• Y is isometric the open $\frac{\pi}{4}$-neighbourhood of a Clifford submanifold

$$S^{n_1} \times S^{n_2} \subset S^{n+1} \ n_1 + n_2 = n;$$

•• Y_+ is isometric to the *projective space* over complex numbers, quaternion numbers or Cayley numbers and Y is isometric to the open $\frac{\pi}{2}$-ball minus the center in such an Y_+.

In fact, the poof of this rigidity seems quite easy in the case of the interest (the above $[\bigcirc \times \bigcirc]$), where Y is equal to the (normal) $\frac{\pi}{4}$-neighbourhood of a hypersurface $X \subset S^{n+1}$ with $curv(X) \leq 1$.

Questions. (i) Is the manifold $Y_+ \supset Y$ indispensable? Do there exist "non-obvious" bands with $sect.curv \geq 1$ and with $width \geq \frac{\pi}{2}$?

(ii) Given a closed n-manifold X, e.g. a product of spheres, $X = \bigtimes_i S^{n+1}$, what is the supremum of the widths of the Riemannian bands Y homeomorphic to $X \times [0,1]$ with $sect.curv(Y) \geq 1$?