Formulas from "Four Lectures"

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Contents

0.1 Definition and Basic properties of Scalar Curvature

The scalar curvature of a C^2-smooth Riemannian manifold $X = (X, g)$, denoted

$$Sc = Sc(X, x) = Sc(X, g) = Sc(g) = Sc_g(x)$$

is a continuous function on X, which is traditionally defined as

the sum of the values of the sectional curvatures at the $n(n-1)$ ordered bivectors of an orthonormal frame in X,

$$Sc(X, x) = Sc(X)(x) = \sum_{i,j} \kappa_{ij}(x),\ i \neq j = 1, ..., n,$$

where this sum doesn't depend on the choice of this frame by the Pythagorean theorem.

Algebraically, this formula defines a *second order differential*

$$g \mapsto Sc(g)$$

from the space G_+ of positive definite quadratic differential forms on X to the space S of functions on X, that is characterised uniquely, up to a scalar multiple, by two properties.

⋆ the $g \mapsto Sc(g)$ is *equivariant* under the natural actions of diffeomorphisms of X in the spaces G_+ and S.

⋆ the $g \mapsto Sc(g)$ is *linear in the second derivatives* of g.

To make geometric sense of this, let us summarize basic properties of $Sc(X)$.

$\bullet_1$ *Additivity under Cartesian-Riemannian Products.*

$$Sc(X_1 \times X_2, g_1 + g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).$$

$\bullet_2$ *Quadratic Scaling.*

$$Sc(\lambda \cdot X) = \lambda^{-2} Sc(X),\ \text{for all } \lambda > 0,$$

where

$$\lambda \cdot X = \lambda \cdot (X, dist_X) =_{def} (X, dist_{\lambda \cdot X}) \text{ for } dist_{\lambda \cdot X} = \lambda \cdot dist(X)$$

for all metric spaces $X = (X, dist_X)$ and where $dist \mapsto \lambda \cdot dist(X)$ corresponds to $g \mapsto \lambda^2 \cdot g$ for the Riemannian quadratic form g.

Example. The Euclidean spaces are scalar-flat, $Sc(\mathbb{R}^n)=0$, since $\lambda\cdot\mathbb{R}^n$ is *isometric* to $\mathbb{R}^n$.

$\bullet_3$ *Volume Comparison.* If the scalar curvatures of n-dimensional manifolds X and X' at some points $x\in X$ and $x'\in X'$ are related by the strict inequality

$$Sc(X)(x) < Sc(X')(x'),$$

then the Riemannian volumes of the ε-balls around these points satisfy

$$vol(B_x(X,\varepsilon)) > vol(B_{x'}(X',\varepsilon))$$

for all sufficiently small $\varepsilon>0$.

Observe that this volume inequality is *additive under Riemannian products*: if

$$vol(B_{x_i}(X,\varepsilon)) > vol(B_{x'_i}(X'_i,\varepsilon)), \text{ for } \varepsilon\leq\varepsilon_0,$$

and *for all points* $x_i\in X_i$ and $x'_l\in X'_i$, $i=1,2$, then

$$vol_n(B_{(x_1,x_2)}(X_1\times X_2,\varepsilon_0)) > vol_n(B_{(x'_1,x'_2)}(X'_1\times X'_2,\varepsilon_0)$$

for all $(x_1,x_2)\in X_i\times X_2$ and $(x'_1,x'_2)\in X'_1\times X'_2$.

This follows from the Pythagorean formula

$$dist_{X_1\times X_2}=\sqrt{dist^2_{X_1}+dist^2_{X_2}}.$$

and the Fubini theorem applied to the "fibrations" of balls over balls:

$$B_{(x_1,x_2)}(X_1\times X_2,\varepsilon_0))\to B_{x_1}(X_1,\varepsilon_0) \text{ and } B_{(x'_1,x'_2)}(X'_1\times X'_2,\varepsilon_0))\to B_{x_1}(X'_1,\varepsilon_0),$$

where the fibers are balls of radii $\varepsilon\in[0,\varepsilon_0]$ in X_2 and X'_2.

$\bullet_4$ *Normalisation/Convention for Surfaces with Constant Sectional Curvatures.* The unit spheres $S^2(1)$ have constant scalar curvature 2 and the hyperbolic plane $H^2(-1)$ with the sectional curvature -1 has scalar curvature -2 [1]

It is an elementary exercise to prove the following.

$\star_1$ *The function $Sc(X,g)(x)$ which satisfies $\bullet_1$-$\bullet_4$ exists and unique*;

$\star_2$ *The unit spheres d the hyperbolic spaces with sect.curv $=-1$ satisfy*

$$Sc(S^n(1))=n(n-1) \text{ and } Sc(H^n(-1))=-n(n-1).$$

Thus,

$$Sc(S^n(1)\times H^n(-1))=0=Sc(\mathbb{R}^{2n}),$$

which implies that

the volumes of the small ε-balls in $S^n(1)\times\mathbf{H}^n(-1)$ are "very close" to the volumes of the ε-balls in the Euclidean space $\mathbb{R}^{2n}$.

Also it is elementary to show that the definition of the scalar curvature via volumes of balls agrees with the traditional $Sc=\sum\kappa_{ij}$, where the definition via volumes seem to have an advantage of being geometrically more usable.

[1]The equality $Sc(H^2)=-2$ follows from $Sc(S^2)=2$ by comparing the volumes of small balls in $S^2\times H^2$ and in $\mathbb{R}^4$.

But this is an illusion:

there is no single known (are there unknown?)
geometric argument, which would make use of this definition.

The immediate reason for this is *the infinitesimal* nature of the volume comparison property: it *doesn't integrate* to the corresponding property of balls of specified, let them be small, radii $r \leq \varepsilon > 0$. [2]

The following *alternative, let it be also only infinitesimal, property* of the scalar curvature seems more promising:

⊛ the inequality $Sc(X,x) < Sc(X',x')$ is equivalent to the following relation between the average mean curvatures of the (very) small ε-spheres $S^{n-1}_x(\varepsilon) \subset X$ and $S^{n-1}_{x'}(\varepsilon) \subset X'$:

$$\frac{\int_{S^{n-1}_x(\varepsilon)} mean.curv(S^{n-1}_x(\varepsilon), s) ds}{vol_{n-1}(S^{n-1}_x(\varepsilon))} > \frac{\int_{S^{n-1}_{x'}(\varepsilon)} mean.curv(S^{n-1}_{x'}(\varepsilon), s') ds'}{vol_{n-1}(S^{n-1}_{x'}(\varepsilon))}.$$

There are also several *non-local inequalities* for the mean curvatures of manifolds B with boundaries S, in terms of the scalar curvatures of B (and sometimes of sizes of B) that we shall see in these lectures, e.g. ⬬ and ■ in section **??**, but we are still far from the ultimate inequality of this kind.

[∗] *Exercise: Spherical Suspension.* Compute the scalar curvature of the *spherical join* of two Riemannian manifolds X_1 and X_2, that is the unit sphere in the product of the Euclidean cones over these manifolds:

$$X_1 * X_2 \subset \mathsf{C}X_1 \times \mathsf{C}X_2,$$

where $\mathsf{C}X = (X \times \mathbb{R}^\times_+, r^2dx^2 + dr^2)$, accordingly

$$\mathsf{C}X_1 \times \mathsf{C}X_2 = (X_1 \times X_2 \times \mathbb{R}^\times_+ \times \mathbb{R}^\times_+, r_1^2 dx_1^2 + r_2^2 dx_2^2 + dr_1^2 + dr_2^2)$$

and where the hypersurface $X_1 * X_2 \subset \mathsf{C}X_1 \times \mathsf{C}X_2$ is defined by the equation

$$r_1^2 + r_2^2 = 1.$$

(The manifold $X_1 * X_2$ with this metric, which is defined for $r_1, r_2 > 0$, is incomplete; if completed, it becomes singular, unless X_1 and X_2 are isometric to the unit spheres S^{n_1} and S^{n_2}.)

Show, in particular, that if $Sc(X_i) \geq n_i(n_i - 1) = Sc(S^{n_i})$, $n_i = dim(X_i)$, $i = 1,2$, then

$$Sc(X_1 * X_2) \geq (n_1 + n_2)(n_1 + n_2 - 1).$$

Hint. Use the formula for the curvature of warped products from section **??**.

0.2 Fundamental Examples of Manifolds with $Sc \geq 0$

Symmetric and homogeneous spaces. Since compact symmetric spaces X have non-negative sectional curvatures κ, they satisfy $Sc(X) \geq 0$, where the equality holds only for flat tori.

[2] An attractive conjecture to the contrary appears in [Guth(volumes of balls-large) 2011], also see [Guth(volumes of balls-width)) 2011].

Since the bi-variant metrics on Lie groups have $\kappa \geq 0$ and since the inequality $\kappa \geq 0$ is preserved under dividing spaces by isometry groups, all compact homogeneous spaces G/H carry such metrics, .[3]

Furthermore,

quotients of compact homogeneous spaces by compact freely acting isometry groups carry metrics with $Sc \geq 0$*,*

where prominent examples of these are

spheres divided by finite free isometry groups.

Thus, in particular,

all homology classes in the classifying spaces $\mathsf{B}(G)$ *of finite cyclic groups* G *are representable by compact manifolds with* $Sc > 0$ *mapped to these spaces.*

But, at the present moment, it is unknown if this remains true for all finite groups G.[4]

On the other extreme, there are no known examples of "$Sc > 0$ *representable*" *non-torsion* homology classes in the classifying spaces of *infinite countable* groups or of (possibly torsion) homology classes in the classifying spaces of groups *without torsion.*

(We shall see in the following sections that majority of known topological obstructions to metrics with $Sc \geq 0$ come from *the rational homology* and *K-theory* of classifying spaces of infinite groups.

Also we shall meet examples – we call these ***S**choen-**Y**au-**S**chick -manifolds* – where non-trivial obstructions to $Sc \geq 0$, which reside in *the integer* homology classes in $\mathsf{B}(\mathbb{Z}^n \times \mathbb{Z}/p\mathbb{Z})$, *vanish for non-zero multiples* of these classes.)

Fibrations. Since the scalar curvature is additive, fibered spaces $X \to Y$ with compact non-flat homogeneous fiberes carry metrics with $Sc > 0$.

(This is seen by scaling metrics in Y by large constants.)

Convex Hypersurfaces. Since convex hypersurfaces in $\mathbb{R}^n$ as well as in general spaces with sectional curvatures $\kappa \geq 0$, their scalar curvatures are also non-negative.

1 Curvature Formulas for Manifolds and Submanifolds.

We enlist in this section several classical formulas of Riemannian geometry and indicate their (more or less) immediate applications.

1.1 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) Riemannian Variation Formula. Let h_t, $t \in [0, \varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0, \varepsilon]$.

[3]This is also true for non-compact homogeneous spaces the isometry groups of which contain compact semisimple factors.

[4]This was pointed out to me by Bernhard Hanke.

Notice that an arbitrary Riemannian metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0, \varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = 2A^*,$$

where

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1, ..., n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y, y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

Sign Convention. The first derivative of h changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ of a hypersurface $Y \subset X$ depends on the *coorientation* of Y in X, where our convention is such that

the boundaries of *convex* domains have *positive (semi)definite* second fundamental forms A^*, also denoted II_Y, hence, *positive* mean curvatures, with respect to *the outward* normal vector fields.[5]

(2.1.B) **First Variation Formula.** This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t$$ [6]

[5] At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.

[6] This come with the *minus* sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal(2019].

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for C^1-smooth functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t$$ [7]

Now comes the first formula with the Riemannian curvature in it.

1.2 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannian manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y,\tau) = \kappa(X,\tau) + \wedge^2 A^*(\tau),$$

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y,y) = Sc(X,y) + \sum_{i \neq j} \alpha_i^*(y)\alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

• $\alpha_i^*(y)$, $i = 1,...,n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;

• α^*-sum is taken over all ordered pairs (i,j) with $j \neq i$;

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν,τ_i) for ν being a unit (defined up to $\pm$-sign) normal vector to Y;

• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu,\nu).$$

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i \neq j} \alpha_i \alpha_j = \left(\sum_i \alpha_i\right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

$$(\text{Sc} \geq -2\text{Ric}) \qquad Sc(Y) \geq -2Ricci(\nu,\nu).$$

[7] This remains true for Lipschitz functions but if ψ is (badly) non-differentiable, e.g. it is equal to the characteristic function of a domain $U \subset Y$, then the derivative $\partial_{\psi\nu} vol_{n-1}(Y_t)$ may become (much) larger than this integral.

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n \max_i (c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape* that is the symmetric on $T(Y)$ associated with A^* via the Riemannian scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau, \tau) = \langle A(\tau), \tau \rangle_g \text{ for all } \tau \in T(Y).$$

1.3 Variation of the Curvature of Equidistant Hypersurfaces and Weyl's Tube Formula

(2.3.A) Second Main Formula of Riemannian Geometry.[8] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape operators $A_t = A(Y_t)$ satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where B_t is the symmetric associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $\left(\tau, \nu = \frac{d}{dt}\right)$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry. [9]

Tracing this formula yields

(2.3.B) Hermann Weyl's Tube Formula.

$$trace\left(\frac{dA_t}{dt}\right) = -||A^*||^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -||A^*||^2 - Ricci(\nu, \nu),$$

[8]The first main formula is *Gauss' Theorema Egregium.*

[9]Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darmos(Gravitation einsteinienne) 1927] with the reference to Darboux' textbook of 1897.

where

$$||A^*||^2 = ||A||^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y,y) = Sc(X,y) + \sum_{i\neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu,\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i\neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + ||A^*||^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1,...,n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $||A^*||^2 = \sum_i (\alpha_i^*)^2$.

1.4 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called *umbilic* if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $Y = (Y,h)$ be a smooth Riemannian (n-1)-manifold and $\varphi = \varphi(t) > 0$, $t \in [0,\varepsilon]$ be a smooth positive function. Let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0,\varepsilon]$.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'(t)}{\varphi(t)}$, $i = 1,...,n-1$ for

$$A_t^* = \frac{\varphi'(t)}{\varphi(t)} h_t \text{ for } \varphi' = \frac{d\varphi(t)}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}.$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2}Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

(c) What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[10]

Cylindrical Extension Exercise. Let Y be a smooth manifold, $X = Y \times \mathbb{R}_+$, let g_0 be a Riemannian metric in a neighbourhood of the boundary $Y = Y \times \{0\} = \partial X$, let h denote the Riemannian metric in Y induced from g_0 and let Y has *constant mean curvature* in X with respect to g_0.

Let X' be a (convex if you wish) ball in the standard (i.e complete simply connected) space with constant sectional curvature and of the same dimension n as X, let $Y' = \partial X'$ be its boundary sphere, let, let $Sc(h) > 0$ and let the mean and the scalar curvatures of Y and Y' are related by the following (comparison) inequality.

$$[<] \qquad \frac{|mean.curv_{g_0}(Y)|^2}{Sc(h,y)} < \frac{|mean.curv(Y')|^2}{Sc(Y')} \text{ for all } y \in Y.$$

Show that

if Y is compact, there exists a smooth positive function $\varphi(t)$, $0 \leq t < \infty$, which is constant at infinity and such that the the warped product metric $g = \varphi^2h + dt^2$ has

[10]See §12 in [GL(complete) 1983].