Kolmogorov's Diameter, Hilbert's Rational Designs and Curvature

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1 Introduction

Immersions are C^1-maps $f : X \to Y$ between smooth manifolds, such that their differentials $df : T(X) \to T(Y)$ nowhere vanish,[1]

$$df(\tau) = 0 \implies \tau = 0, \tau \in T(X).$$

The (maximal normal bundle) ***curvature*** of an immersed X in a Riemannian Y,

$$f : X \hookrightarrow Y,$$

is *the supremum of the* Y*-curvatures* of geodesics $\gamma \subset X$, for the induced Riemannian metric in X,

$$curv^{\perp}(X) = curv^{\perp}(f(X)) = curv^{\perp}_f(X) = curv^{\perp}(X \overset{f}{\hookrightarrow} Y) = curv^{\perp}(X \hookrightarrow Y),$$

Minimal Curvature Problem. What is the infimum of curvatures of immersions $f : X \hookrightarrow Y$,

$$inf.curv^{\perp}(X, Y) = inf.curv^{\perp}(X \hookrightarrow Y)?$$

Remark: min *or* inf*?* Probbaly, the infimum $\inf_f curv^{\perp}_f(X, Y)$ is rarely achieved, where the (only) established examples of immersions of closed manifolds with minimal possible curvatures are listed below.

[1]Immersions are locally one-to-one but globally they may have self intersections. Immersions without self intersections are called *embeddings*, where, if X is non-compact, one may require the induced topology in X to be equal the original one.

- Spheres in the unit balls with curvatures $curv^\perp = 1$.

$$S^m \subset \mathbb{R}^N,\ N \geq m.$$

- Tori with curvatures $\sqrt{3m/(m+2)}$,

$$\mathbb{T}^m \subset B^N(1)\ N > m^{m^4}.$$

(See 2.1.B, Remark(a) in 5.3. and [Pet2023].)[2]

- Veronese surfaces in the unit spheres with curvatures $1/\sqrt{3}$,

$$\mathbb{R}P^2 \subset S^n(1)\ n \geq 4.$$

(See 5.2(b).)

- Product of Spheres in $S^{m_1+m_2+1}$ with curvatures 1.

$$S^{m_1} \times S^{m_2} \subset S^{m_1+m_2+1}(1).$$

(See below, section 1.1 and [Ge2021].)

Product Example. If X is a product of spheres,

$$X = \bigtimes_{i=1}^{l} S^{m_i},$$

and Y is the unit ball $B^N(1) \subset \mathbb{R}^N$ then (apart from the trivial case of $l = 1$) we know the exact value of $inf.curv^\perp(\times_{i=1}^l S^{m_i}, B^N(1)$ only where all $m_i = 1$, i.e. for the torus $\mathbb{T}^l$, and where N is large:

$$[\sqrt{3}]_{\mathbb{T}} \qquad inf.curv^\perp(\mathbb{T}^l, B^N(1)) = \sqrt{3\frac{l}{l+2}},\ N >> l^2.$$

(See sections 3, 5 and [Pet2023].)

But if all $m_i = 2$, for instance, i.e. $X = (S^2)^l$ we neither can show that

$$inf.curv^\perp((S^2)^l, B^{2l+1}) \to \infty \text{ for } l \to \infty$$

nor that

$$\frac{inf.curv^\perp((S^2)^l, B^{10l})}{\sqrt{l}} \to 0 \text{ for } l \to \infty.$$

Clifford Embeddings. The product X of spheres $S^{m_i}(r_i) \subset \mathbb{R}^{m_i+1}$, $i = 1, ..., l$, for $\sum_{i=1}^l r_i^2 = 1$ naturally isometrically imbeds to the boundary of the unit N-ball for $N = k + \sum_i m_i$:

$$\mathsf{Cl} : X = S^{m_1}(r_1) \times ... \times S^{m_l}(r_l) \to S^{N-1}(1) \subset B^N(1) \subset \mathbb{R}^{m_i+1} \times .. \times \mathbb{R}^{m_i+1}$$

where, clearly,

$$curv^\perp(X \overset{\mathsf{Cl}}{\subset} B^N) = \max_i 1/r_i.$$

[2] We shall show in section 2.1.D that $inf.curv(\mathbb{T}^m \subset B^N(1)) \leq \sqrt{3m/(m+2)}$ for $n \geq 8m^2+8$ but, at the present moment, the existence of immersions with curvatures $\sqrt{3m/(m+2)}$ is proven only for much larger n.

This, for $r_1 = r_2 = ... = r_l$, delivers a codimension l-embedding with curvature $\sqrt{l}$. Thus,

$$inf.curv^\perp\left(\bigtimes_{i=1}^{l} S^{m_i}, B^N(1)\right) \le \sqrt{l},\ \ N = l + \sum_i m_i.$$

If $l = 1$, then this is optimal. In fact, it is obvious that

$$curv\left(X \hookrightarrow B^m(1) \times \mathbb{R}^N\right) \ge 1,\ \text{for } n \ge 2.$$

for all smoothly immersed closed m-manifolds X in the "unit band" $B^m(1)\times\mathbb{R}^N$.

But, for instance, the *equality*

$$inf.curv^\perp(\mathbb{T}^m \hookrightarrow B^{2m}) = \sqrt{m}$$

is problematic for all $m \ge 2$.

Round m-Tori in the Unit (m + 1)-Balls.

$$inf.curv^\perp(\mathbb{T}^2 \hookrightarrow B^3) \le 3:$$

the boundary of the $\frac{1}{3}$*-neighbourhood of the circle of radius* $\frac{2}{3}$ *in the space has* $curv^\perp(\mathbb{T}^2 \subset \mathbb{R}^3) = 3$.

Similarly (see section 4.1)

$$inf.curv^\perp(\mathbb{T}^3 \hookrightarrow B^4) \le 2\sqrt{2} + 1 < 4$$

$$inf.curv^\perp(\mathbb{T}^7 = \mathbb{T}^3 \times \mathbb{T}^3 \times \mathbb{T}^1 \hookrightarrow B^8) \le 8 + 2\sqrt{2} + 1 < 12$$

....

$$inf.curv^\perp(\mathbb{T}^m, B^{m+1}) < m^{\frac{3}{2}},\ \ m = 2^k - 1.$$

Veronese embeddings[3] of the real projective spaces satisfy (see 5.1),

$$curv\left(\mathbb{R}P^m \hookrightarrow B^{\frac{m(m+3)}{2}}\right) = \sqrt{\frac{2m}{m+1}},\ \text{e.g.}$$

$$curv\left(\mathbb{R}P^2 \hookrightarrow B^5\right) = 2\sqrt{\frac{1}{3}} < 1.155.$$

Conjecture.

$$min.cirv(X^m, B^N) < \sqrt{\frac{2m}{m+1}} \implies X =_{diffeo} S^m.$$

(maximal) ***curvature*** of an immersion between Riemannian manifolds,

$$(X, g) \hookrightarrow (Y, h)$$

is *the supremum of* h*-curvatures in* Y, of g-geodesics $\gamma \subset X$,

$$curv(f) = curv^X(f) = curv^X_Y(f) = curv^g_h(f) = \sup_{\gamma \subset X} curv_h(f(\gamma)).$$

[3]These are flashes from a superior world.

If $g = f^*(h)$ is the induced Riemannian metric in X, this is our curvature of X in Y,

$$curv_h^g(f) = curv^\perp(X \overset{f}{\hookrightarrow} Y).$$

(This $curv^\perp(X)$ unlike $curv(f)$ is defined for immersions of smooth manifolds with no metrics on them.)

Normal Immersions, where $\mathbf{curv}^\perp_{\mathbf{F}}(\mathbf{X}) = \mathbf{curv}^{\mathbf{X}}(\mathbf{f})$. Call an immersion between Riemannian manifolds $f : X(g) \hookrightarrow Y(h)$ *normal* if for all normal vectors to X in Y,

$$\nu \in T_x^\perp(X) = T_f(x)(Y) \ominus df(T_x(X))$$

the second quadratic form II_ν of the immersed $X \overset{f}{\hookrightarrow}$ is *simultaneously diagonalizable* with the quadratic forms $g(x)$ and $f^*(h)$ on the tangent space $T_x(X)$. For instance, isometric immersions are normal.

Clearly, $curv^\perp_f(X) = curv^X(f)$ for *isometric* immersions f

Curvature in Spheres. If an immersion $X \to S^{N-1}(1)$ is normal then so is the corresponding immersion to $\mathbb{R}^N \supset S^{N-1}(1)$, where the spherical curvature of X is related to the Euclidean one by the Pythagorean theorem:

$$(curv^\perp(X \hookrightarrow S^{N-1}(1))^2 = (curv^\perp(X \hookrightarrow \mathbb{R}^N)^2 - 1.$$

Notice that the Clifford embeddings to the unit sphere are known to be *optimal* for $l = 2$,

$$inf.curv^\perp(S^{m_1} \times S^{m_2}, S^{m_1+m_2+1}(1)) = 1, \ m_1, m_2 \geq 1,^4$$

but the corresponding Euclidean equality

$$inf.curv^\perp(S^{m_1} \times S^{m_2}, B^{m_1+m_2+2}(1)) = \sqrt{2},$$

remains conjectural for all $m_1, m_2 \geq 1$, except for $m_1 = m_2 = 1$ [Pet].

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Curvature in Codimension 1. This curvature of $X^m \hookrightarrow Y^{m+1}$ is the supremum of the principal curvatures of X in Y over all points $x \in X$.

Here normality means that the induced quadratic form $f^*(g)(x)$ on the tangent space $T_x(X)$ is, at all $\in X$, diagonalizabel in the same basis as the second fundamental form II of X.

Example. the immersion $S^m(r) \times S^1 \to \mathbb{R}^{m+2}$ obtained by rotating $S^m(r) \hookrightarrow \mathbb{R}^{m+1}$ around a line in $\mathbb{R}^{m+1}$ within distance $R > r$ from the origin is normal with curvature $\max\left(\frac{1}{R}, \frac{1}{R-r}\right)$.

***Remark* I**. If $f = 0$, and X immerses to $\mathbb{R}^n$, then the above delivers an immersion f_1 of X to the unit ball $B^N = B^N(1)$ with a bound on the curvature of f_1 depending on the dimensions n and N, e.g. with

$$curv^\perp(X, B^N) \leq \sqrt{\frac{3n}{n+2}} + \varepsilon \text{ for } N \geq 20n^2 \text{ and all } \varepsilon > 0$$

This and the Whitney embedding theorem implies the following.

1.1.D. If $N \geq 100m^2$, then $inf.curv^\perp(X, B^N) < \sqrt{\frac{3(2m-1)}{2m+1}}$.

[4] See [Ge2021], section 3.7.3 in [Gr2022] and section 5.5 in the present paper.

In fact,

[**N** >>] $$\lim_{N\to\infty} inf.curv^\perp(X,B^N)\leq\sqrt{\frac{3m}{m+2}}.$$

Petrunin Theorem. There exist m-dimensional manifolds X for $m\geq 2$, such that

$$inf.curv^\perp(X,S^N)>\sqrt{\frac{2m-1}{m+2}} \text{ for all } N,$$

i.e. all immersions from X to the unit sphere $S^N(1)$ have with curvatures

$$curv^\perp(X\hookrightarrow S^N(1))\geq\sqrt{\frac{2m-1}{m+2}}+\varepsilon\left(=\sqrt{\frac{3m}{m+2}-1}+\varepsilon\right).$$

for some $\varepsilon=\varepsilon(X,N)>0$.

(Surfaces of genera ≥ 2 are examples for such X with $m=2$.)

1.1.F. Codim 1 Theorem/Example.(See section 4.2) Let

$$X=S^k\times\underbrace{S^1\times...\times S^1}_{l-1}.$$

If $k\geq l^{l^4}$,[5] then there exists an immersion

$$F:X\hookrightarrow B^{k+l}(1)$$

with

$$curv^\perp_F(X)\leq 1+2\sqrt{\frac{3l-3}{l+1}}<4.5.$$

Remark II. The proof of the remark **I** doesn't apply to immersions to $\mathbb{R}^n$ without passing to $\mathbb{R}^{n+1}$ but this is taken care of by the following (see

1.1.H. Remarks/Questions. We don't know how close this inequality to the minimal values of the curvatures of codim1 immersions of products of spheres is.

(a) For instance let P^{l-1} be an $(l-1)$-dimensional manifold diffeomorphic to a product of spheres where some of these have dimensions ≥ 2. Then, if $k>>l$, there exist immersions

$$F_\varepsilon:S^k\times P^{l-1}\hookrightarrow B^{k+l}(1)$$

with

$$curv^\perp_{F_\varepsilon}(S^k\times P^{l-1})\leq 1+2\sqrt{\frac{3l-3}{l+1}}+\varepsilon$$

for all $\varepsilon>0$.

But this is *unclear* for $\varepsilon=0$, even for the product $S^1\times S^k$, which embeds to the ball $B^{k+2}(1)$ with curvature 3 for all k and where we *don't know* if there are immersions of $S^1\times S^{k+2}$ (or other closed non-spherical manifolds of dimension $k+1$) to the unit ball $B^{k+2}(1)$ with curvatures <3.

[5]The hugeness of this number is the product of my perfunctory interpretation of Hilbert's argument in [H1909].

(b) It is not impossible according to what we know, that m-dimensional products of spheres of dimensions ≥ 2 admit immersions to $B^{m+1}(1)$ with curvature <100.

But the best we can do (see section 4.1) are immersions with curvatures $\lesssim m^{\frac{4}{3}}$.

2 Kolmogorov's $D = D(m, N, p)$, Hilbert's Theorem and Spherical Designs

K-Diameter $\sqrt[p]{\mathbf{D(m, N, p)}}$. Let $||y||_{L_p}$, $y = (y_1, ..., y_N) \in \mathbb{R}^N$ denote the normalized norm l_p,

$$||y||_{L_p} = \left(\frac{1}{N}\sum_{i=1}^{N}|y_i|^p\right)^{\frac{1}{p}}$$

Let $D(m, N, p)$ denotes the infimum of the numbers $D > 0$ such that $\mathbb{R}^N$ contains an m-dimensional linear subspace X, such that

$$||x||^p_{L_p} \leq D||x||^p_{L_2}, \text{ for all } x \in X.$$

Observe that $D(1, N, p) = 1$, $D(m, m, p) = m^{\frac{p}{2}-1}$, that $D(m, N, p)$ is monotone increasing in m and decreasing in N and let

$$D(m, p) = D(m, \infty, p) = \lim_{N\to\infty} D(m, N, p).$$

2.1.A. Gamma Function Design Formula. If $p = 4, 6, 8...$, then a simple $O(m)$-averaging argument, shows that

$$[\Gamma/\Gamma] \qquad D(m, p) = \frac{\int_{S^{m-1}}|l(s)|^p ds}{\left(\int_{S^{m-1}}|l(s)|^2 ds\right)^{\frac{p}{2}}} = \frac{m^{\frac{p}{2}-1}\cdot 3\cdot 5\cdots(p-1)}{(m+2)\cdot(m+4)\cdots(m+p-2)},$$

where $l(s)$ is a non-zero linear function on on the sphere.

2.1.B. Hilbert Connection. In his proof of the Waring problem, Hilbert shows the existence of $M = \binom{m+p-1}{m-1} + 1$ *rational points* $s_i \in S^{m-1}$ and of positive *rational* weight $w_i > 0$, $\sum_1^M w_i = 1$, such that $\sum_i w_i l^d(s_i) = \int_{S^{m-1}} l^d(s)d$ *for all* linear functions on he sphere.

This, after partitioning each s_i into Δ atoms for Δ being the smallest common denominator $\mathcal{N}$ of w_i, becomes what is no-a-days called *spherical design* of cardinality $N = \mathcal{N}M$ of w_i, which yields (this is nearly obvious, see **2.1.C** below) the following.

$\mathbf{D(m, N)}$***-Stabilization*****:** $D(m, N, p) = D(m, \infty, p)$ for all sufficiently large $N \geq= N_{Hilb}(m, p)(\leq \mathcal{N}M)$, where – to be safe let it be rough– $N_{Hilb} \leq m^{m^p}$.

Design Rationality**:** If $N \geq N_{Hilb}$ then the space l_p^N contains a *rational* linear subspace X of dimension m, such that

$$||x||^p_{L_p} = D(m, p)||x||^p_{L_2} \text{ for all } x \in X.$$

2.1.C. Spherical Designs and the Equality $D(m, N) = D(m, \infty)$

A *design of even degree* $p = 2, 4, ...$ *and cardinality* N on the sphere S^{m-1} is a map from a set Σ of cardinality N to the sphere, written as $\sigma \mapsto s(\sigma)$, such that the linear functions $l(s)$ on the sphere $S^{m-1} \subset \mathbb{R}^m$ satisfy

$$\frac{1}{N} \sum_{\sigma \in \Sigma} l^d(s(\sigma)) = \int_{S^{m-1}} l^d(s) ds, \ d = 2, ..., p,$$

where ds is the $O(m)$ *invariant probability* measure on the sphere.

Hence, the linear map from the space $\mathbb{R}^{m\perp}(=\mathbb{R}^m)$ of linear functions on the sphere $S^{m-1} \subset \mathbb{R}^m$ to $\mathbb{R}^N = \mathbb{R}^\Sigma$ preserves both, the L_2 and the L_p-norms and, by the above $[\Gamma/\Gamma]$,

the existence a design of cardinality N *implies that* $D(m, N, p) = D(m, p)$.[6]

Non-rational designs, at least for $p = 4$, are known to exit for $N << N_{Hilb}$.

2.1.D $2m^2$**-Design Construction.** *If* $p = 4$, *and if* m *is a power of 2, then there exists a spherical designs of cardinality* $N = 2m^2 + 4m$. [7]

This, now for all m, shows that

(i) $$D(m, N, 4) = \frac{3m}{m+2} \text{ for } N \geq 8(m^2 + m).$$

$[\mathbb{R}^2$ in $l_4^3]$-*Example.* $D(2, N, 4) = \frac{3}{2}$ for $N \geq 3$, with four (rational) planes $X \subset \mathbb{R}^3 = l_4^3$, where $||x|||_{L_4}^4 = \frac{3}{2}||x|||_{L_2}^4$: these are the normals to the vectors (1,1,1,), (1,1, -1), (1,-1,1), (1,-1,-1).

2.1.E. $D(m, N)$**-Inequalities.** If $N \lesssim m^2$, then upper bounds on $D^4(m, N, 4)$ follow from the corresponding estimates in the randomization proofs of the Dvoretzky theorem for the l_p-spaces, where the following inequality follow from (the argument in) [PVZ2017].

(ii) $D(m, N, 4) \leq 3 + const_{(\mathbf{ii})} \frac{m^2}{N}$ for $N \geq m^2$;[8]

(iii) $D(m, N, 4) \leq const_{(\mathbf{iii})} \frac{m^2}{N}$ for $2m \leq N \leq m^2$.[9]

2.1.F. $D(m, N)$ **Concentration Property**. The existence of m-subspaces $X \in l_4^N$ in [FLM1977] and [PVZ2017], such that

$[D]$ $$||x||_{L_4}^4 \leq D||x||_{L_2}^4, \ x \in X,$$

is derived from a *lower bound the measure* of those m-subspaces $X \subset \mathbb{R}^N$, where this inequality fails for some $x \in X$.

In particular, the argument used in [FLM1977] implies that the measure μ_D of those $X \subset \mathbb{R}^N$ with respect to the $O(N)$-invariant probability measure in the Grassmanian $Gr_m(\mathbb{R}^N$ where $||x||_{L_4}^4 \geq D||x||_{L_2}^4$, for some $x \in X$ satisfies:

If ,

$$D > \frac{3m}{m+2}$$

then

$$\mu_D \to 0, \ for \ N \to \infty.$$

[6]See [BB2009], [LW1993] for more about it.

[7]The Kerdock code used in [K1995] yields designs for $m = 4^k$ and $N = \frac{m(m+2)}{2}$.

[8]This follows from **(i)** for $N \geq 8(m^2 + m)$ and, if $const_1$ is large, also for (some) $N \leq 8(m^2 + m)$. Besides, the inequality $D^4(m, m^2, 4) \leq const$ follows from (the proof of) example 3.1 in [FLM1977].

[9]Since $D(m, N, 4) \leq D(m, m, 4) = m$ for all m and N, the significance of this inequality for $N \sim m$ depends on the value of $const_2$.

3 Equivariant Immersions $\mathbb{R}^m \to S^{2N-1}$

3.A. Curvatures of the Clifford Tori. Let

$$\mathbb{T}^N \subset S^{2N-1} \subset B^{2N}(1) \subset (B^2(1))^N \subset \mathbb{R}^{2N}$$

be the Clifford torus and observe that the second quadratic form of this torus in the the ambient Euclidean space $\mathbb{R}^{2N} \supset S^{2N-1} \supset \mathbb{T}^N$, regarded as a quadratic form with values in the normal bundle, is

$$\mathrm{II}{=}\sqrt{N}\sum_{i=1}^N \nu_i dt_i^2,$$

where t_i are the cyclic coordinates on the torus and $\{\nu_i \in T^\perp(\mathbb{T}^N \subset \mathbb{R}^{2N})\}$ is the corresponding orthonormal frame of *normal* vectors to $\mathbb{T}^N$.

This, in terms of the orthonormal *tangent* frame $\{e_i = \frac{\partial}{\partial t_i} \in T(\mathbb{T}^N)\}$, means that

$$\mathrm{II}\colon e_i \otimes e_i \mapsto \sqrt{N}\nu_i \text{ and } \mathrm{II}\colon e_i \otimes e_j \mapsto 0 \text{ for } i \neq j.$$

Thus, the curvature of $\mathbb{T}^N$ in B^N along a unit tangent vector $\bar{x} \in T(\mathbb{T}^N)$,

$$\bar{x} = \sum_i x_i e_i, \text{ where } \sum_i x_i^2 = 1,$$

is

$$curv^\perp(\mathbb{T}^N, \bar{x}) = ||\mathrm{II}(\bar{x} \otimes \bar{x})|| = ||\mathrm{II}(\sum_i x_i e_i \otimes \sum_i x_i e_i)|| =$$

$$||\mathrm{II}(\sum_{ij} x_i x_j (e_i \otimes e_i)|| = \sqrt{N}||\sum_i x_i^2 \nu_i|| = \sqrt{N}\sqrt{\sum_i x_i^4} = \sqrt{N}\frac{\sqrt{\sum_i x_i^4}}{||\bar{x}||^2} =$$

where $||\bar{x}||^2 = |\bar{x}||_{l_2}^2 = \sum_{i=1}^N x_i^2$.

Hence,

$$(\bigstar) \qquad curv^\perp(\mathbb{T}^N, \bar{x}) = \left(\sqrt[4]{N}\frac{||\bar{x}||_{l_4}}{||\bar{x}||_{l_2}}\right)^2 = \left(\frac{||\bar{x}||_{L_4}}{||\bar{x}||_{L_2}}\right)^2,$$

where, recall, the L_p-norms refer to the finite probability spaces with N equal atoms,

$$||\bar{x}||_{L_p} = \frac{||\bar{x}||_{l_p}}{\sqrt[p]{N}}.$$

The Euclidean Small Curvature Theorem. The above ($\bigstar$) implies the existence of an equivariant isometric immersion from the Euclidean m-space to the Clifford N-torus,

$$f^\odot : \mathbb{R}^m \to \mathbb{T}^N \subset S^{2N} \subset \mathbb{R}^{2N}$$

with the relative curvature $curv_{\mathbf{E}}^{\mathbf{e}}(f^\odot)$ equal to $\sqrt{D(m,N)} = \sqrt{D(m,N,4)}$.

3.1 Veronese Maps.

Besides invariant tori, there are other submanifolds in the unit sphere S^{N-1}, which have small curvatures and which are transitively acted upon by subgroups in the orthogonal group $O(N)$.

The generalized Veronese maps are a *minimal equivariant isometric* immersions of spheres to spheres, with respect to certain homomorphisms (representations) between the orthogonal groups $O(m+1) \to O(m+1)$,

$$ver = ver_s = ver_s^m : S^m(R_s) \to S^m = S^{m_s} = S^{m_s}(1),$$

where

$$m_s = (2s+m-1)\frac{s+m-2)!}{s!(m-1!} < 2^{s+m} \text{ and } R_s = R_s(m) = \sqrt{\frac{s(s+m-1)}{m}},$$

for example,

$$m_2 = \frac{m(m+3)}{2} - 1,\ R_2(m) = \sqrt{\frac{2(m+1)}{m}} \text{ and } R_2(1) = 2,$$

(see [DW1971]If $s = 2$ these, called *classical Veronese maps*, are defined by taking squares of linear functions (forms) $l = l(x) = \sum_i l_i x_i$ om $\mathbb{R}^{m+1}$,

$$Ver : \mathbb{R}^{m+1} \to \mathbb{R}^{M_m},\ M_m = \frac{(m+1)(m+2)}{2},$$

where tis $\mathbb{R}^{M_m}$ is represented by the space $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^{m+1})$ of quadratic functions (forms) om $\mathbb{R}^{m+1}$,

$$Q = \sum_{i=1,j=1}^{m+1,m+1} q_{ij} x_i x_j.$$

The Veronese map, which is (obviously) equivariant for the natural action of the orthogonal group group $O(n+1)$ on $\mathcal{Q}$, where, observe, this action fixes the line $\mathcal{Q}_\circ$ spanned by the form $Q_\circ = \sum_i x^2$ as well as the complementary subspace $\mathcal{Q}_\diamond$ of the *traceless forms* Q, where the action of $O(n+1)$ is irreducible and, thus, it has a *unique, up to scaling* Euclidean/Hilbertian structure.

Then the normal projection[10] defines an equivariant map to the sphere in $\mathcal{Q}_\diamond$

$$ver : S^m \to S^{M_m-2}(r) \subset \mathcal{Q}_\diamond,$$

where the radius of this sphere, a priori, depends on the normalization of the $O(m+1)$-invariant metric in $\mathcal{Q}_\diamond$.

Since we want the map to be isometric, we either take $r = \frac{1}{R_2(m)} = \sqrt{\frac{m}{2(m+1)}}$ and keep $S^m = S^m(1)$ or if we let $r = 1$ and $S^m = S^m(R_2(m))$ for $R_2(m) = \sqrt{\frac{2(m+1)}{m}}$.

Also observe that the Veronese maps, which are not embeddings themselves, factor via embeddings of projective spaces to spheres

$$S^m \to \mathbb{R}P^m \subset S^{M_m-2} \subset \mathbb{R}^{M_m-1} = \mathcal{Q}_\diamond,\ M_m = \frac{(m+1)(m+2)}{2}.$$

Curvature of Veronese. Let is show that

$$curv^\perp_{ver}\left(S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)\right) = \sqrt{\frac{R_2(1)}{R_2(m)} - 1} = \sqrt{\frac{m-1}{m+1}}.$$

Indeed, the Veronese map sends equatorial circles from $S^m(R_2(m))$ to planar circles of radii $R_2(m)/R_2(1)$, the curvatures of which in the ball B^{M_m-1} is $R_2(1)/R_2(m) = 2\sqrt{\frac{m}{m+1}}$ and the curvatures of these in the sphere,

$$curv^\perp(S^1 \subset S^{M_m-2}(1)) = \sqrt{curv(S^1 \subset B^{M_m-1}(1))^2 - 1} = \sqrt{\frac{4m}{m+1} - 1} = \sqrt{\frac{3m-1}{m+1}}$$

[10]The splitting $\mathcal{Q} = \mathcal{Q}_\circ \oplus \mathcal{Q}_\diamond$ is necessarily normal for all $O(m+1)$-invariant Euclidean metrics in $\mathcal{Q}$.

is equal to the curvature of the Veronese $S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)$ itself $\sqrt{R_2(1)/R_2(m)} = \sqrt{\frac{2m}{m+1}}$, and the curvatures of these in the sphere,

$$curv^\perp(S^1 \subset S^{M_m-2}(1)) = \sqrt{curv(S^1 \subset B^{M_m-1}(1))^2 - 1},$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)$itself. QED.

It may be hard to prove (conjecture in section 1) that Veronese manifolds have the smallest possible curvatures among non-spherical m-manifold in the unit ball: *if a smooth compact m-manifold X admits a smooth immersion to the unit ball $B^N = B^N(1)$ with curvature $curv^\perp(X \hookrightarrow B^N) < \sqrt{\frac{2m}{m+1}}$, then X is diffeomorphic to S^m.*

It is more realistic to show that the Veronese have smallest curvatures among submanifolds $X \subset B^N$ *invariant under subgroups in $O(N)$, which transitively act on X.*

Remark. Manifolds X^m immersed to S^{m+1} with curvatures < 1 are diffeomorphic to S^n, see 5.5, but, apart from Veronese's, we can't rule out such X in S^N for $N \geq m+2$ [11] and, even less so, non-spherical X immersible with curvatures $< \sqrt{2}$ to $B^N(1)$, even for $N = m+1$.

It seems hard to decide this way or another, but it may be realistic to try to prove *sphericity of simply connected* manifolds immersed with curvatures < 1 to $S^N(1)$ for all N.

The curvatures of Veronese maps can be also evaluated with the *Gauss formula,* (teorema egregium), which also gives the following formula for curvatures of all ver_s:

$m = 2$ $1 - 2c^2 = 1/3$, $2c^2 = 2/3$ $c\sqrt{1/3}$

$C = \sqrt{1 + 1/3} = 2/\sqrt{3}$

From Veronese to Tori. The restriction of the map $ver_s : S^{2m-1}(R_s) \to S^{N_s}$ to the Clifford torus $\mathbb{T}^m \subset S^{2m-1}(R_s)$ obviously satisfies

$$curv^\perp_{ver_s}(\mathbb{T}^m) \leq A_{2m-1,s} + \frac{\sqrt{m}}{R_s} = \sqrt{3 - \frac{5}{2}m + \varepsilon(m,s)}$$

for

$$\varepsilon(m,s) = \frac{2}{4m^2} - \frac{4m-2}{s(s+2m-2)} + \frac{5(2m-1)}{2ms(s+2m-2)} - \frac{2m-1}{(ms(s+2m-2))^2}.$$

This, for $s >> m^2$, makes $\varepsilon(m,s) = O\frac{1}{m^2}$

Since $N_s < 2^{s+2m}$,

starting from $N = 2^{10m^3}$

$$curv^\perp_{ver_s}(\mathbb{T}^m) < \sqrt{3 - \frac{5}{2}m}.$$

where it should be noted that

the Veronese maps restricted to the Clifford tori are $\mathbb{T}^m$-equivariant

and that

[11] Hermitian Veronese maps from the complex projective spaces $\mathbb{C}P^m$ to the spaces $\mathcal{H}_n$ of Hermitian forms on $\mathbb{C}^{m+1}$ are among the prime suspects in this regard.

this bound is *weaker than the optimal one* $\frac{||y||^2_{l_4}}{||y||^2} \geq \sqrt{3-\frac{3}{m+2}}+\varepsilon$ from the previous section.

Remarks. (a) It is not hard to go to the (ultra)limit for $s \to \infty$ and thus obtain an

equivariant isometric immersion ver_∞ of the Euclidean space $\mathbb{R}^m$ to the unit sphere in the Hilbert space, such that

$$curv^\perp_{ver_\infty}(\mathbb{R}^m \hookrightarrow S^\infty) = \sqrt{\frac{(m-1)(2m+1)}{(m+1)^2}} = \sqrt{2-\frac{5}{m+1}+\frac{2}{(m+1)^2}},$$

where equivariance is understood with respect to a certain unitary representation of the isometry group of $\mathbb{R}^m$.

Probably, one can show that this ver_∞ realizes the *minimum* of the curvatures among all equivariant maps $\mathbb{R}^m \to S^\infty$.

(b) Instead of ver_s, one could achieve (essentially) the same result with a use of compositions of the classical Veronese maps, $ver : S^{m_i} \to S^{m_{i+1}}$, $_{i+1} = \frac{(m_i+1)(m_i+2)}{2} - 2$,

$$S^{m_1} \hookrightarrow S^{m_2} \hookrightarrow ... \hookrightarrow S^{m_i},$$

starting with $m_1 = 2m-1$ and going up to $i = m$. (Actually, $i \sim \log m$ will do.)

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