Curvature, Kolmogorov Diameter, Hilbert Rational Designs and Overtwisted Immersions

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Abstract

We prove the existence of *locally distance increasing maps* with *controllably small curvatures* between Riemannian manifolds, where our main construction depends on the presence of *particular spherical and almost spherical sections* of the unit balls in the $l_{p=4}$ spaces.

In the part II we prove similar results for families of maps and also for C^∞-smooth *isometric immersions* $X^m \to Y^N$, where our approach allows an improvement of the present-day bounds on the dimension N of the ambient manifold Y in certain cases.

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1 Introduction

Immersions are C^1-maps $f : X \to Y$ between smooth manifolds, such that their differentials $df : T(X) \to T(Y)$ nowhere vanish,[1]

$$df(\tau) = 0 \implies \tau = 0, \tau \in T(X).$$

The curvature of an immersed X in a Riemannian Y is *the supremum of the Y-curvatures* of geodesics $\gamma \subset X$, for the induced Riemannin metric in X,

$$curv(f(X)) = curv_f(X) = curv(X \overset{f}{\hookrightarrow} Y) = curv(X \hookrightarrow Y),$$

First Minimal Curvature Problem. What is the infimum of curvatures of immersions $f : X \hookrightarrow Y$,

$$min.curv(X, Y) = min.curv(X \hookrightarrow Y),$$

e.g. where Y is a unit ball $B^N = B^N(1) \subset \mathbb{R}^N$ and X is a product of spheres $X = S^{m_1} \times .... \times S^{m_l}$.

Two Extremes: Possibilities or Impossibilities?

A. $min.curv(X^m \hookrightarrow B^n) < 100$ for all $X^m \hookrightarrow \mathbb{R}^n$: if X can be immersed to $\mathbb{R}^n$ then it can be can be immersed to he unit ball with curvature <100.

B. $min.curv(X^m \hookrightarrow B^n) > C$ for all C and almost all (depending on C) compact manifolds X^m: the number of different m-manifolds which can be immersed to $B^n(1)$ is finite or at least, there is a very special, albeit infinite, class of such X depending on C.

HAND-MADE IMMERSIONS

Clifford Embeddings. The product X of spheres $S^{m_i}(r_i) \subset \mathbb{R}^{m_i+1}$, $i = 1, ..., l$, for $\sum_{i=1}^{l} r_i^2 = 1$ naturally isometrically imbedsto the boundary of the unit N-ball for $N = k + \sum_i m_i$:

$$\mathsf{Cl} : X = S^{m_1}(r_1) \times ... \times S^{m_l}(r_l) \to S^{N-1}(1) \subset B^N(1) \subset \mathbb{R}^{m_i+1} \times .. \times \mathbb{R}^{m_i+1}$$

where, clearly,

$$curv(X \overset{\mathsf{Cl}}{\subset} B^N) = \max_i 1/r_i.$$

This, for $r_1 = r_2 = ... = r_l$, delivers a codimension l-embedding with curvature $\sqrt{l}$. Thus,

$$min.curv\left(\bigtimes_{i=1}^{l} S^{m_i}, B^N(1)\right) \le \sqrt{l},\ N = l + \sum_i m_i.$$

[1] Immersions are locally one-to-one but globally they may have self intersections. Immersions without self intersections are called *embeddings*, where, if X is non-compact, one may require the induced topology in X to be equal the original one.

If $l=1$, then this is optimal. In fact, it is obvious that

$$curv\left(X \hookrightarrow B^m(1)\times\mathbb{R}^N\right)\geq 1, \text{ for } n\geq 2.$$

for all smoothly immersed closed m-manifolds X in the "unit band" $B^m(1)\times\mathbb{R}^N$.

But the sharpness of the inequality $min.curv(\mathbb{T}^2\hookrightarrow B^4)\leq\sqrt{2}$ is problematic.

Round m-Tori in the Unit $(\mathbf{m+1})$-Balls.

$$min.curv(\mathbb{T}^2\hookrightarrow B^3)\leq 3:$$

the boundary of the $\frac{1}{3}$-neighbourhood of the circle of radius $\frac{2}{3}$ in the space has $curv(\mathbb{T}^2\subset\mathbb{R}^3)=3$.

Similarly (see section 4.1)

$$min.curv(\mathbb{T}^3\hookrightarrow B^4)\leq 2\sqrt{2}+1<4$$

$$min.curv(\mathbb{T}^7=\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{T}^1\hookrightarrow B^8)\leq 8+2\sqrt{2}+1<12$$

....

$$min.curv(\mathbb{T}^m,B^{m+1})<m^{\frac{3}{2}},\ m=2^k-1.$$

Veronese embeddings[2] of the real projective spaces satisfy (see 5.1),

$$curv\left(\mathbb{R}P^m\hookrightarrow B^{\frac{m(m+3)}{2}}\right)=\sqrt{\frac{2m}{m+1}},\text{ e.g.}$$

$$curv\left(\mathbb{R}P^2\hookrightarrow B^5\right)=2\sqrt{\frac{1}{3}}<1.155.$$

Conjecture.

$$min.cirv(X^m,B^N)<\sqrt{\frac{2m}{m+1}}\implies X=_{diffeo}S^m.$$

1.1 Immersions with Small Curvature and $\mathcal{D}(m,N)$-Approximation

Expansion. A map between metric spaces,

$$f:X\to Y,$$

is λ-*expanding,* $\lambda>0$, if it increases the the length of curves $\xi:[0:1]\to X$ by a factor $\geq\lambda$,

$$length(f\circ\xi)\geq\lambda\cdot length(\xi)\text{ for all continuous maps }\xi:[0:1]\to X.$$

continuous maps.

Expanding is an abbreviation for "1-expanding".

Riemannian Example. A C^1-*smooth* map f between Riemannian manifolds, e.g. open subsets in Euclidean spaces, is λ-expanding if and only if $||df(\tau)||\geq ||\lambda\tau||$ for all tangent vectors $\tau\in T(X)$.

[2]These are flashes from a superior world.

Thus, smooth expanding maps are immersion and every immersion f expands with respect to some Riemannian metrics $g = g(f)$ in X and $h = h(f)$ in Y.

Equidimensional example. If $dim(X) = dim(Y)$ then smooth immersions $X \hookrightarrow Y$ are local diffeomorphisms and C^1-smooth expanding maps are locally distance increasing. [3]

The relative (maximal) ***curvature*** of an immersion between Riemannian manifolds,

$$(X, g) \hookrightarrow (Y, h)$$

is *the supremum of h-curvatures in Y*, of g-geodesics $\gamma \subset X$,

$$curv(f) = curv^X(f) = curv^X_Y(f) = curv^g_h(f) = \sup_{\gamma \subset X} curv_h(f(\gamma)).$$

If $g = f^*(h)$ is the induced Riemannin metric in X, this is our curvature of X in Y,

$$curv(f(X)) = curv_f(X) = curv(X \overset{f}{\hookrightarrow} Y) = curv(X \hookrightarrow Y),$$

(This $curv(X)$ unlike $curv(f)$ is defined for immersions of smooth manifolds with no metrics on them.)

Equidimensional example. If $dim(X) = dim(Y)$, then $curv(X \overset{f}{\hookrightarrow} Y) = 0$, while $curv^X(f)$ measures by how much f deviates from a projective map.

Curvature in Spheres. If an immersion $X \to S^{N-1}(1)$ is normal then so is the corresponding immersion to $\mathbb{R}^N \supset S^{N-1}(1)$, where the spherical curvature of X is related to the Euclidean one by the Pythagorean theorem:

$$(curv(X \hookrightarrow S^{N-1}(1))^2 = (curv(X \hookrightarrow \mathbb{R}^N)^2 - 1.$$

Also, the Clifford embeddings to $S^{N-1}(1)$ are known to be *optimal* for $l = 2$,[4]

Normal Immersions: When $\mathbf{curv_f(X) = curv^X(f)}$. Call an immersion between Riemannian manifolds $f : X(g) \hookrightarrow Y(h)$ *normal* if for all normal vectors to X in Y,

$$\nu \in T^\perp_x(X) = T_f(x)(Y) \ominus df(T_x(X))$$

the second quadratic form II_ν of the immersed $X \overset{f}{\hookrightarrow}$ is *simultaneously diagonalizable* with the quadratic forms $g(x)$ and $f^*(h)$ on the tangent space $T_x(X)$. For instance, isometric immersions are normal.

Clearly, $curv_f(X) = curv^X(f)$ for *isometric* immersions f.

Curvature in Codimension 1. This curvature of $X^m \hookrightarrow Y^{m+1}$ is the supremum of the principal curvatures of X in Y over all points $x \in X$.

Here normality means that the induced quadratic form $f^*(g)(x)$ on the tangent space $T_x(X)$ is, at all $\in X$, diagonalizabel in the same basis as the second fundamental form II of X.

Example. the immersion $\mathbb{S}^m(r) \times S^1 \to \mathbb{R}^{m+2}$ obtained by rotating $S^m(r) \hookrightarrow \mathbb{R}^{m+1}$ around a line in $\mathbb{R}^{m+1}$ within distance $R > r$ from the origin is normal with curvature $\max\left(\frac{1}{R}, \frac{1}{R-r}\right)$.

[3]Expanding locally homeomorphic maps are also locally distance increasing, but the absolute value map $x \mapsto |x|$, for example, is 1-expanding but not locally homeomorphic.

[4]See [Ge2021], section 3.7.3 in [Gr2022] and section 5.5 in the present paper.

Curvature in Spheres. If an immersion $X \to S^{N-1}(1)$ is normal then so is the corresponding immersion to $\mathbb{R}^N \supset S^{N-1}(1)$, where the spherical curvature of X is related to the Euclidean one by the Pythagorean theorem:

$$(curv(X \hookrightarrow S^{N-1}(1))^2 = (curv(X \hookrightarrow \mathbb{R}^N)^2 - 1.$$

Second and Third Minimal Curvature Problems. What is the minimal curvature in a given homotopy or regular homotopy [5] class of immersions ?

What is the minimal curvature of *expanding immersions* between given *Riemannian* manifolds?

Below are partial answers to these questions.

$\boldsymbol{\mathcal{D}}(\mathbf{m},\mathbf{N})$: ***Curvature of Euclidean Expanding Maps****. Let* $\mathcal{D}(m,N)$ be the infimum of the relative curvatures of the smooth expanding maps f from the Euclidean m-space to the unit N-ball,

$$\mathcal{D}(m,N) = \inf_f curv^{\mathbf{e}_m}_{\mathbf{e}_N}(f),$$

where $\mathbf{e}_m$ and $\mathbf{e}_N$ denote the Euclidean metrics in $\mathbb{R}^m$ and $\mathbb{R}^N \supset B^N(1)$.

Example. The composition of the toral Clifford embedding $\mathbb{T}^m \to B^m(1)$ with the universal covering $\mathbb{R}^m \to \mathbb{T}^m$ followed the Euclidean homothety $x \mapsto n\sqrt{n}x$ is an isometric immersion $\mathbb{R}^m \hookrightarrow B^m(1)$ with curvature $\sqrt{m}$. Hence,

$$\mathcal{D}(m,N) \le \sqrt{m}$$

Question. Is $\mathcal{D}(m,2m)$ equal to $\sqrt{m}$?

1.1.A. Euclidean $\boldsymbol{\mathcal{D}}(\mathbf{m},\mathbf{N})$-Theorem.

$\bullet_{\ge 2m}$ If $N \ge 2m$, then

$$\mathcal{D}(m,N) \le \sqrt{\frac{3m}{m+2}} + C_o\frac{m}{\sqrt{N}},$$

where C_o is a universal constant (see section 3). Moreover, if $N \ge 100m^2$ then

$$\mathcal{D}(m,N) \le \sqrt{\frac{3m}{m+2}}.$$

$\bullet_{<2m}$ If $m+1 \le N < 2m$, then

$$\mathcal{D}(m,N) \le 6\frac{m^{\frac{3}{2}}}{N-m}$$

See section 3 for the proof.

Questions. Is $\mathcal{D}(m,N)$ equal to $\sqrt{\frac{3m}{m+2}}$ for $N \ge m^2$?

Is $\mathcal{D}(m,m+1)$ bounded by $2m$?

1.1.B. δ-Approximation Corollary. Let $X = X^m$ be a smooth manifold and $f : X \to \mathbb{R}^N$ a continuous map.

$\bullet^{\ge}$ If $N \ge 2m-1$ then f can be δ-approximated by smooth immersions

$$f_\delta : X \hookrightarrow \mathbb{R}^N, \delta > 0,$$

[5] A C^1-continuous homotopy f_t of smooth maps is *regular* if the maps f_t are *immersions* for all t.

with curvatures

$$curv_{f_\delta}(X) \leq \frac{1}{\delta}\left(\sqrt{\frac{6m-2}{2m+1}} + C_o\frac{m}{\sqrt{N}}\right) + o\left(\frac{1}{\delta}\right),\ \delta \to 0,$$

where "δ-approximated" means that

$$dist_{\mathbb{R}^N}(f_\delta(x), f_0(x)) \leq \delta,\ x \in X.$$

$\bullet^{\lesssim}$ If X admits an immersion to $\mathbb{R}^n$, $n < N$, and $N \leq 2m$, then f can be δ-approximated by smooth immersions

$$f_\delta : X \hookrightarrow \mathbb{R}^N, \delta > 0,$$

with curvatures

$$curv_{f_\delta}(X) \leq \frac{1}{\delta}\frac{6n^{\frac{3}{2}}}{N-n} + o\left(\frac{1}{\delta}\right).$$

Proof. Let $\phi : X = X^m \to \mathbb{R}^n$ be a smooth immersion [6] and observe the following.

1.1.C. Stretching Lemma. If $n \geq m+1$, then, for all Riemannin metrics g on X and all positive functions $\varepsilon(x)$, there exists an a g-expanding immersion $\psi : X \to \mathbb{R}^n$ *regularly homotopic* to ϕ, i.e. it can be joined with ϕ by a C^1-continuous homotopy of smooth immersion, and such that $curv_\psi(X,x) \leq \varepsilon(x)$.

Proof. If X is compact, scale $\phi \to \psi = \lambda\phi$ and send $\lambda \to \infty$.

If X is non-compact and $n < m$ regularly homotop ϕ it to a *proper* (infinity goes to infinity) immersion with a use of Hirsch' immersion theorem and let $\psi_\lambda : X \to \mathbb{R}^n$ be the composition of ψ with a $\lambda(y)$-expanding map : $\mathbb{R}^n \to \mathbb{R}^n$, $y \in \mathbb{R}^n$, for a large and fast growing function $\lambda(y)$.

Now, ε-approximate f by a smooth map f'_ε and add to it the composed map of $\delta^{-1}\psi_\lambda = \psi\delta^{-1}\lambda$ with an expanding map $f_\odot : \mathbb{R}^n \to \mathbb{R}^N$ times δ. It is clear that if the function $\lambda(x) = \lambda_{f_\varepsilon}(x)$ is sufficiently large, depending on the norms of the fist and the second differentials $||df'_\varepsilon(x)||$ and $||d^2f'(x)||$, then the curvature of this sum

$$f_{\delta,\lambda}(x) = f'_\varepsilon(x) + \delta \cdot f^\odot \circ \psi_{\delta^{-1}\lambda}(\delta^{-1}x)$$

is bounded by

$$\frac{curv(f^\odot)}{\delta} + o\left(\frac{1}{\delta}\right)$$

and the proof follows with $\varepsilon \to 0$.

***Remark* I**. If $f = 0$, and X immerses to $\mathbb{R}^n$, then the above delivers an immersion f_1 of X to the unit ball $B^{n+1} = B^{n+1}(1)$ with a bound on the curvature of f_1 depending only on the dimension m of X, e.g.

$$min.curv(X, B^N) \leq \sqrt{\frac{3(2m-1)}{2m+1}} = \sqrt{3 - \frac{6}{2m+1}} \text{ for } N \geq 100m^2.$$

Moreover, we show in section 3)the following.

[6] All X^m immerse to $\mathbb{R}^{2m-1}$, if $m \geq 2$, by the Whitney theorem.

1.1.E. *As* N *becomes very large depending on the topology of* X*, then* $min.curv(X, B^N) < \sqrt{\frac{3(2m-1)}{2m+1}}$. *In fact,*

$$[\mathbf{N} >>] \qquad \lim_{N\to\infty} min.curv(X, B^N) \le \sqrt{\frac{3m}{m+2}} = \sqrt{3 - \frac{6}{m+1}}.$$

1.1.F. Conjecture. If $N \ge 100m^2$ then all m-manifolds X admit immersions to the unit sphere $S^N(1)$ with curvatures

$$curv(X \hookrightarrow S^N(1)) \le \sqrt{\frac{3m}{m+2} - 1} = \sqrt{\frac{2m-1}{m+2}}.$$

The bound $[\mathbf{N} >>]$, albeit unlikely, may be optimal but our bounds on on $curv_{f_1}(X)$ for small N are far from optimal. For instance, Clifford embeddings of products of l spheres to the unit balls have curvatures $l^{\frac{1}{2}} << l^{\frac{3}{2}}$.

But the Clifford embeddings are not optimal either: there are products of l spheres, which admit codimension 1 (not l!) immersions with curvatures bounded by a universal constant, where the best available – we *don't know* if this is optimal – such a constant is $1 + 2\sqrt{\frac{3l-3}{l+1}}$ according to the following.

1.1.G. Codim 1 Theorem/Example.(See section 4.2) Let

$$X = S^k \times \underbrace{S^1 \times ... \times S^1}_{l-1}.$$

If $k \ge l^{l^4}$,[7] then there exists an immersion

$$F : X \hookrightarrow B^{k+l}(1)$$

with

$$curv_F(X) \le 1 + 2\sqrt{\frac{3l-3}{l+1}} < 4.5.$$

Remark II. The proof of the remark **I** doesn't apply to immersions to $\mathbb{R}^n$ withut passing to $\mathbb{R}^{n+1}$ but this is taken care of by the following (see section 4.3).

1.1.H. Regular Homotopy/Approximation Theorem. Let $f : X = X^m \to \mathbb{R}^n$ be an immersion. If $n > m$, then f can be δ-approximated by immersions $f_\delta : X \hookrightarrow \mathbb{R}^n$ which are regularly homotopic to f and such that

$$curv_{f_\delta}(X) \le \frac{500}{\delta} m^{\frac{3}{2}} + o\left(\frac{1}{\delta}\right).$$

1.1.I. Remarks/Questions. We don't know how close this inequality to the minimal values of the curvatures of codim1 immersions of products of spheres is.

[7]The hugeness of this number is the product of my perfunctory interpretation of Hilbert's argument in [H1909].

(a) For instance let P^{l-1} be an $(l-1)$-dimensional manifold diffeomorphic to a product of spheres where some of these have dimensions ≥ 2. Then, if $k >> l$, there exist immersions

$$F_\varepsilon : S^k \times P^{l-1} \hookrightarrow B^{k+l}(1)$$

with

$$curv_{F_\varepsilon}(S^k \times P^{l-1}) \leq 1 + 2\sqrt{\frac{3l-3}{l+1}} + \varepsilon$$

for all $\varepsilon > 0$.

But this is *unclear* for $\varepsilon = 0$, even for the product $S^1 \times S^k$, which embeds to the ball $B^{k+2}(1)$ with curvature 3 for all k and where we *don't know* if there are immersions of $S^1 \times S^{k+2}$ (or other closed non-spherical manifolds of dimension $k+1$) to the unit ball $B^{k+2}(1)$ with curvatures < 3.

(b) It is not impossible according to what we know, that m-dimensional products of spheres of dimensions ≥ 2 admit immersions to $B^{m+1}(1)$ with curvature <100.

But the best we can do (see section 4.1) are immersions with curvatures $\lesssim m^{\frac{4}{3}}$.

1.2 Equidimensional Expanding Maps

Affine Expanding Maps. The product of r_i-balls admits an *affine* equidimensional expanding map to the R-ball

$$f : \bigtimes_{i=1}^{k} B^{n_i}(r_i) \to B^N(R),\ N = \sum_i n_i,$$

if and only if

[$\sum r_i^2$]
$$\sum_i r_i^2 \leq R^2,$$

where – all this is, of course, obvious – in the case of equality $\sum_i r_i^2 = R^2$, such an f is an *isometric embedding.*

But – this was pointed out to me by Roman Karasev– it is unlikely that there is a simple criterion for the existence of such embeddings to cubes, not even for rectangular solids,

$$\bigtimes_{i=1}^{n} B^1(r_i) = \bigtimes_{i=1}^{n}[-r_i, r_i] \to [-\underline{r}, \underline{r}]^n.$$

1.2.A. Rolled Band Example. What is more interesting from our perspective is *a $(1-\varepsilon)$-expanding map, for a given $\varepsilon > 0$, from the infinite cylinder $X = B^{n-1}(r) \times \mathbb{R}^1$ to the ball $B^n(2r)$,*

$$f_\varepsilon : B^{n-1}(r) \times \mathbb{R}^1 \to B^n(2r),$$

where this f_ε comes as the composition of two maps.

(1) The first map is the universal covering map from the cylinder $B^{n-1}(r-\varepsilon) \times \mathbb{R}^1$ to the *round solid torus* embedded to the ball,

$$f_1 : B^{n-1}(r) \times \mathbb{R}^1 \to \mathbb{T}_{sld}(r, r-\varepsilon) \subset B^n(2r),$$

where this torus is equal to the $(r-\varepsilon)$-*neighbourhood* of a *planar circle*

$$S^1(r) \subset B^n(2r)$$

of *radius* r, where the center of $S^1(r+\varepsilon)$ is positioned at the center of the ball $B^n(2r)$.

Observe that the map f_1 is *isometric* on the $(n-1)$-balls

$$B^{n-1}(r-\varepsilon) \times t \subset B^{n-1}(r-\varepsilon) \times \mathbb{R}^1, \ t \in \mathbb{R}^1.$$

(2) The second map f_2 is the linear (scaling) diffeomorphism

$$f_2 : B^{n-1}(r) \times \mathbb{R}^1 \to B^{n-1}(r-\varepsilon) \times \mathbb{R}^1 \text{ for } f_2 : (s,t) \mapsto \left(\frac{s}{1-\varepsilon}, \varepsilon^{-1} t\right);$$

where, clearly, the composition

$$B^{n-1}(r) \times \mathbb{R}^1 \overset{f_2}{\to} B^{n-1}(r-\varepsilon) \times \mathbb{R}^1 \overset{f_1}{\to} \mathbb{T}_{sld}(r, r-\varepsilon) \subset B^n(2r)$$

is the required $(1-\varepsilon)$-expanding map $B^{n-1}(r) \times \mathbb{R}^1 \overset{f_\varepsilon}{\to} B^n(2r)$.

1.2.B $[f \times f]$**-Corollary.** The Cartesian powers of

$$f_\varepsilon : [-r, +r] \times \mathbb{R}^1 \to B^2(2r) \subset \mathbb{R}^2$$

deliver expanding maps

$$B^m(r) \times \mathbb{R}^m \subset [-r, +r]^m \times \mathbb{R}^m \to B^{2m}\left(1 + \frac{1}{\sqrt{m}}\right)$$

for all $m = 1, 2, \ldots$ and $r < \frac{1}{\sqrt{m}}$.

1.2.C. $\frac{1}{2}$***-Exercise.*** Show that if $\underline{r} \leq 2r$, then the cylinder $B^{n-1}(r) \times \mathbb{R}^1$ admits *no expanding map* f to the ball $B^n(\underline{r})$.

Hint. (i) *The axes* – the central line $0 \times \mathbb{R}^1$ of the cylinder – must go by f to the *concentric ball* $B^n(\underline{r} - r) \subset B^n(\underline{r})$.

(ii) The longest straight segment with respect to the f-induced flat metric between pairs of points on this axes must have length $> 2\underline{r} - r$.

The above 1.2.B is generalized in section 4.2 as follows.

1.2.D. Rolled Band into Ball Theorem. *If* $M \geq 100m^2$*, and*

$$r < \frac{\sqrt{m+2}}{\sqrt{3m} + \sqrt{m+2}} \left(> \frac{1}{3}\right),$$

then the product $B^M(r) \times \mathbb{R}^m$ *admits an equidimensional expanding map to the unit ball,*

$$F_r : B^M(r) \times \mathbb{R}^m \to B^{m+M}(1).$$

Remark/Question. If $m = 1$, then, by the above $\frac{1}{2}$-exercise, the bound $r < 1/2$ is optimal, but it is not clear for $m = 2$.

Here the above inequality for $m = 2$, which allows expanding maps from $B^4(r) \times \mathbb{R}^2$ to the unit ball $B^{m+M}(1)$, where the supremum of the possible r is

$$\sup r = \frac{2}{\sqrt{6} + 2} - \varepsilon (\approx 0.45),$$

is implemented with $M = 4$ by means of the normal exponential map for the 2-*subtorus in Clifford torus* $\mathbb{T}^3 \subset B^6(1)$, *which is is normal to the principal diagonal in* $\mathbb{T}^3$.

Similarly the normal exponential map for the Clifford torus $\mathbb{T}^2 \subset B^4(1)$ leads to such maps $B^2(r) \times \mathbb{R}^2 \subset B^4(1)$ with

$$\sup = \frac{1}{1+\sqrt{2}} \approx 0.41 < 0.45,$$

while the best $B^1(r) \times \mathbb{R}^2 \subset B^3$, where

$$\sup r = \frac{1}{3} < 0.41,$$

is obtained with the normal exponential map for the standard round torus in $\mathbb{R}^3$.

And the only known upper bound on r is for $M = 1$:

$$r \leq \frac{\pi}{2\sqrt{\lambda_1(B^3(1))}} = \frac{\pi}{2j_{1/2}} = \frac{1}{2} > \frac{2}{\sqrt{6}+2} \approx 0.45,$$

where this λ_1 is the first Dirichlet eigenvalue of the Laplacian in the unit 3-ball, and $j_{1/2} = \pi$ is the first Bessel function zero(see next section).

None of these four inequalities is known to be (*or not to be*) *optimal.*

1.3 Obstructions to Expansion and Lower Bounds on Curvatures of Immersions

To get a perspective on our existence theorems for expanding maps and maps with small curvatures, we summarize below the known upper bounds on expansion and lower bounds on curvatures of immersions, which are derived from geometric and topological properties of Riemannian manifolds with lower bounds on their scalar curvatures. [8]

1.3.A. Gaussian ($\nexists Sc{>}0$)-Obstruction for Immersion to S^{m+k}. If X is $\nexists Sc{>}0$, i.e, it admits no metric with *positive scalar curvature* (see 5.8 in [Gr2022] and examples below) then

$$min.curv(X^m \hookrightarrow S^{m+k}) \geq \max\left(\sqrt{\frac{m-1}{k}}, \sqrt{\frac{m-1}{m}}\right)$$

The proof of this follows from the Gauss theorema egregium.

Remark. The inequality $curv(X^m \hookrightarrow S^{m+k}) \geq \sqrt{\frac{m-1}{k}}$ is sharp for $m = 2$ and $k = 1$, where the extremal $X^2 \subset S^3$ is the Clifford torus with curvature 1.

1.3.B. Euclidean secretly Gaussian **Inequality**. *If X^m is $\nexists Sc{>}0$, then the curvatures of immersions $f : X^m \to B^{m+k}(1)$ are bounded from below as follows:*

$$curv_f(X^m) \geq const \cdot \sqrt{\frac{m}{k}},$$

for some const < 10. (See [Gr2022] and references therein.)

[8] A few simple inequalities with no use of the scalar curvature are indicated in section 5.5.

Questions. Do $\nexists Sc{>}0$ manifolds satisfy the following inequality?

$$min.curv(X^m \hookrightarrow B^{m+k+1}(1)) \geq \max\left(\sqrt{\frac{m-1}{k}+1}, \sqrt{\frac{m-1}{m}+1}\right).$$

Do the curvatures of 2-tori in the unit ball satisfy $curv(\mathbb{T}^2 \hookrightarrow B^3(1)) \geq 3$?

Examples of $\nexists Sc{>}0$ Manifolds. Tori and product of tori with certain manifols *homeomorphic* (but *not diffeomorphic*) to spheres, $T^m \times \Sigma^n$, [9] admit no metrics with $Sc > 0$, see [Gr2021] and references therein.

The above can be improved for *enlargeable manifolds* X, e.g. for those which admit metrics with *non-positive sectional curvatures*, such as the m-tori for example.[10]

1.3.C. Enlargeable Codimension ≤ 2 Theorem . *The curvatures of compact enlargeable Riemannian m-manifolds X immersed to the unit ball $B^{m+k}(1)$ satisfy for $k = 1, 2$:*

$$[j/k] \qquad curv(X^m \hookrightarrow B^{m+k}(1)) \geq \frac{2j_\nu}{k\pi} - 1,$$

where $\nu = \frac{m+k}{2} - 1$ and j_ν is the first root of the Bessel function J_ν. (See [Gr2022] and references therein.)

One knows in his regard that $j_{-1/2} = \frac{\pi}{2}$, $j_0 = 2.4048...$, and if $\nu > 0$, then

$$\nu + \frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}} < j_\nu < \nu + \frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}} + \frac{3}{20}\frac{2^{\frac{2}{3}}a^2}{\nu^{\frac{1}{2}}}$$

where $a = \left(\frac{9\pi}{8}\right)^{\frac{2}{3}}(1+\varepsilon) \approx 2.32$ with $\varepsilon < 0.13\left(\frac{8}{8.847\pi}\right)^2$,

This implies, for instance, that

$$min.curv(\mathbb{T}^7 \hookrightarrow B^8(1)) \geq 3.$$

Codimension k Conjecture. *The inequality $curv(X^m \hookrightarrow B^{m+k}(1)) \geq \frac{2j_\nu}{k\pi} - 1$ holds for all compact enlargeable m-manifolds and all k.*

Remark. The above $[j/k]$-inequality can be equivalently formulated in terms of the *focal radius* of $X^m \hookrightarrow B^{m+k}$ (see next section), since

$$foc.rad(X \hookrightarrow Y) = \min\left(\frac{1}{curv(X \hookrightarrow Y)}, dist(X, \partial Y)\right)$$

for *Riemannian flat* manifolds Y.

If $k = 1, 2$ the focal radius version of the $[j/k]$-inequality generalizes to immersions of enlargeable manifolds to Riemnnian manifolds Y with *lower bounds on their scalar curvatures*, e.g. $Sc(Y) \geq j_\nu^2$. But no such generalization is possible for $k \geq 3$.

(Overoptimistic?) Conjecture. If the cohomology of a closed m-manifold X^m with coefficients in some field K contains l elements with non-zero product,

$$h_1, \smile ... \smile h_i \smile ... \smile h_l \neq 0, \ h_i \in H^*(X; K),$$

[9] Such Σ^n exists for all $n = 8l+1, 8l+2$, $l > 0$, see [Hit1973].

[10] A compact m-manifold X is *enlargeable* (see [G 2021]), if it admits a Riemannian metric g, a sequence of covering $\tilde{X}_i \to X$ and a sequence of λ_i-Lipschitz maps $(\tilde{X}_i, \tilde{g}_i) \to S^m(1)$ with *non-zero degrees*, such that $\lambda_i \to 0$ for $i \to \infty$.

e.g. $X^m = S^{m_1} \times ... \times S^{m_l}$, $m_1 + ... + m_l = m$,

Then the curvatures of immersion $f : X^m \to B^{m+k}(1)$ bounded from below as follows,

$$curv_f(X) \geq 0.1\frac{l^2}{mk}?$$

Clifford Tori Extremality Problem. *Does the m-torus admit an immersion to the unit $2m$-ball with curvature $< \sqrt{m}$?*

For all we know, all flat m-tori admit smooth isometric immersions to $B^{2m}(1)$ with curvatures < 10.

$\mathbf{m}^\beta$**-Problem,** *What is the minimal β, such that the tori of all dimensions m admit immersion to the unit $(m+1)$-balls,*

$$f : \mathbb{T}^m \hookrightarrow B^{m+1}(1),$$

with curvatures $curv_f(\mathbb{T}^m) \leq 100m^\beta$? (We shall see in section 4.1 that $\beta \leq \frac{3}{2}$)

Simply Connected Codim 1 Curvature Problem. *Do all compact smoothly imbedded* simply connected *hypersurfaces $X^m \subset \mathbb{R}^{m+1}$, e.g. products of spheres of dimensions ≥ 2, admit immersion to the unit ball,*

$$f : X^m \hookrightarrow B^{m+1}(1)$$

with curvature $curv_f(X) \leq 100$?

1.3.D. Rectangular Non-Expansion Theorem. *If a rectangular $2r_i$-solid admits an expanding map to a product of balls of radii R_ι,*

$$\bigtimes_{i=1}^{n}[-r_i, r_i] \to \bigtimes_{\iota=1}^{l} B^{m_\iota}(R_\iota), \;\; \sum_\iota m_\iota = n,$$

then

$[\sum(n_i/r_i)^2]$
$$\sum_{i=1}^{n}\frac{1}{r_i^2} \geq \frac{4}{\pi^2}\cdot\sum_{\iota=1}^{l}\frac{j^2_{\nu_\iota}}{R_\iota^2},$$

for $\nu_\iota = \frac{m_\iota}{2} - 1$.

$\square^m$-*Example.* If all $m_\iota = 1$ this reads

$$\sum_{i=1}^{n}\frac{1}{r_i^2} \geq \frac{4}{\pi^2}\cdot\sum_{\iota=1}^{n}\frac{\pi^2}{4R_\iota^2} = \sum_{\iota=1}^{n}\frac{1}{R_\iota^2}.$$

Proof of 1.3.D. If the distances d_i between the opposite faces of the solid $\square^n = \bigtimes_{i=1}^n[-r_i, r_i]$ with respect to a Riemannin metric g on $\square^n$ satisfy $d_i \geq 2r_i$ then, according to the *cube multi-width inequality* (see [WXY 2021])

$$\sum_1^n\frac{1}{r_i^2} \geq \frac{\inf_{x\in\square^n} Sc(g(x))}{\pi^2}\,\frac{n}{n-1},$$

where the $\rtimes$-stabilized form (see [Gr2022]) of this reads:

$$\sum_1^n\frac{1}{r_i^2} \geq \frac{\inf_{x\in\square^n} Sc^{\rtimes}(g(x))}{\pi^2}.$$

Since $Sc^{\rtimes}(B^m(r)) = 4\lambda_1(B^m(R)) = 4j_\nu^2/R^2,$, $\nu = \frac{m}{2} - 1$ (see[Gr2022]), since $Sc^{\rtimes}(A \times B) = Sc^{\rtimes}(A) \cdot Sc^{\rtimes}(B)$, and since the $\rtimes$-scalar curvature of the induced metric for equidimensional immersions $f : X \to (Y, g)$ satisfies $Sc^{\rtimes}(f^*(g)) \geq Sc^{\rtimes}(g)$ (see [Gr2022]), the proof follows.

1.3.F. Corollary: Expansion with Positive Codimension. Let

$$f : \square^{n-1} = \bigtimes_{i=1}^{n-k} [-r_i, r_i] \to Y^n = \bigtimes_{\iota=1}^{l} B^{m_\iota}(R_\iota), \ \sum_\iota m_\iota = n,$$

be an expanding immersion with curvature $\leq \frac{1}{\rho}$ and let $d = dist(f(\square^{n-k}, \partial Y))$. If $k = 1$ then,

$$[\textstyle\sum(n_i/r_i)^2] \qquad \sum_{i=1}^{n-1} \frac{1}{r_i^2} \geq \frac{1}{\pi^2} \cdot \sum_{\iota=1}^{l} \frac{j_{\nu_\iota}^2}{R_\iota^2} - \min\left(\frac{1}{4d^2}, \frac{1}{4\rho^2}\right),$$

and if $k = 2$ then

$$[\textstyle\sum(n_i/r_i)^2] \qquad \sum_{i=1}^{n-2} \frac{1}{r_i^2} \geq \frac{1}{\pi^2} \cdot \sum_{\iota=1}^{l} \frac{j_{\nu_\iota}^2}{R_\iota^2} - \min\left(\frac{1}{d^2}, \frac{1}{\rho^2}\right),$$

Proof. If $k = 1$, apply $[\sum(n_i/r_i)^2]$ to the normal exponential map

$$\square^{n-1} \times [-\rho/2, \rho/2] \to Y^n$$

and if $k = 2$ do this with the universal covering of the normal circle bundle $S^\perp(\rho/2) \to \square^{n-2}$ as in [Gr 2018].

Product of Balls Problem. Given positive numbers r_i, R_i and positive integers m_i, n_i, $i = 1, ...k$, such that $\sum_i m_i = \sum_i n_i$, evaluate, let it be only roughly, the maximal $\lambda > 0$, such that the product of m_i-dimensional r_i-balls $B^{m_i}(r_i)\mathbb{R}^{m_i}$ admit a λ-expanding map to the product of n_i-dimensional R_i-balls,

$$\bigtimes_{i=1}^{k} B^{m_i}(r_i) \to \bigtimes_{i=1}^{k} B^{n_i}(R_i).$$

Cube Extremality Problem. *Does, the unit n-cube $[-1, 1]^n$ admits an expanding map to the n-ball of radius $< \sqrt{n}$?*

1.3.1 The m-th Scalar curvature $Sc_{|m}$ and Focal Radius

Below we outline generalizations of the inequalities from the previous section to immersions to non-Euclidean manifolds Y.

Let $Sc_{|m}(Y)$ be the function on the tangent m-planes $T_y^m \subset T(Y)$ in a Riemannin manifold Y of dimension $\geq m$, which is the sum of the sectional curvatures κ of Y on the bivectors in T_y^m at y, that is the scalar curvature of submanifold $y \ni Y_y^m \subset Y$ tangent T_y^m, i.e. $T_y(Y_y^m) = T_y^m \subset T_y(Y)$ and having zero relative curvature in Y at y,

$$Sc_{|m}(Y, T_y^m) = Sc(T_y^m, y) = \sum_{i \neq j = 1, \ldots, n} \kappa(e_1 \wedge e_j)$$

for a frame of ortonormal vectors $e_i \in T_y^m$.

By the Gauss formula, the scalar curvature of $X^m \hookrightarrow Y$ satisfies:

$$Sc(X,x) = Sc_{|m}(Y, T_x(X)) + ||mean.curv(X,x)||^2 - ||\mathrm{II}(X,x)||^2$$

and if $Sc_{|m}(Y) \geq m(m-1)$, e.g. if $sect.curv(Y) \geq 1$, then, by an easy calculation,

$$[Sc_{|m}] \qquad curv(X^m \hookrightarrow Y) < \max\left(\sqrt{\frac{m-1}{k}}, \sqrt{\frac{m-1}{m}}\right) \Longrightarrow Sc(X) > 0,$$

which implies and generalize the "Gaussian obstruction" 1.3.A.

It is unclear if there is a similar generalization for enlargeable X but this is possible with the *focal radius of* X rather than with its curvature.

The focal radius of an immersed manifold $X \overset{f}{\hookrightarrow} Y$,

$$foc.rad(X) = foc.rad(X \hookrightarrow Y) = foc.rad_f(X)$$

is the supremum of those R, for which the differential of the *normal exponential map, denoted*

$$\exp^\perp : T^\perp(X) \to Y,$$

is *injective* along all normal segments of length $< R$, where, in the case of a non-complete Y or a presence of a boundary ∂Y, one has to say "*defined and injective...*".

1.3.G. ***Boundary of the Tube Formula.*** The focal radius of the boundary of the r-neighbourhood of $X \subset Y$ satisfies

$$foc.rad(\partial U_r(X)) = min(r, foc.rad(Y) - r).$$

If Y has constant sectional curvature, then the focal radii of submanifolds are intimately related to their curvatures in Y.

For instance,

$$foc.rad(X \hookrightarrow \mathbb{R}^N) = \frac{1}{curv(X \hookrightarrow \mathbb{R}^N)}.$$

and

$$foc.rad(X \hookrightarrow B^N(1)) = \min\left(\frac{1}{curv(X)}, dist(X, \partial Y)\right),$$

while the (available) relations between $curv(X)$ and $foc.rad(X)$ are limited for non-constant $sect, curv(Y)$. [11]

The inequality 1.3.C generalises in the focal form to immersions with codimensions $k = 1, 2$ of *enlargeable manifolds* X^m *to* Y *with* $Sc(Y) \geq 0$ as follows.

$$foc.rad(X^m \hookrightarrow Y) \leq \frac{k\pi}{2\sqrt{\lambda_1(X)}},$$

where λ_1 is the first Dirichlet eigenvalue of the Laplacian in X.

In fact, such an inequality holds for certain "topological focal radius".

For instance, let $X \to Y$ be a topological embedding.

[11] If $sect.curv(Y) \geq 1$, and $curv(X) \leq \alpha$, then $foc.rad(X)$ is bounded by the radii of circles in S^2 with curvatures α and if $sect.curv(Y) \leq \kappa$, then $foc.rad(X)$ is bounded from below by the radii of circles in surfaces with constant curvature κ.

$\bullet_1$ If $codim(X) = 1$ and the the boundary ∂Y contains two connected components separated by Y, where $Sc(Y) \geq 0$, then

$$dist_Y(X, \partial Y) \leq \frac{\pi}{2\sqrt{\lambda_1(X)}}.$$

$\bullet_2$ If $codim X = 2$, if Y is compact with a boundary, which contains a *non-zero* homology class $0 \neq s \in H_1(\partial Y)$, which *vanishes* in Y, then

$$dist_Y(X, \partial Y) \leq \frac{\pi}{\sqrt{\lambda_1(X)}}.$$

On Geometry of $[Sc_{|m}]$. One expects that positivity of $[Sc_{|m}](Y)$ for $m < n = dim(Y)$ has greater significance than positivity of Sc(Y)= $[Sc_{|m}](Y)$. Below is, albeit weak, a confirmation to this..

Let Y be a Riemannin manifold, the boundary ∂Y of which is divided into two disjoint parts, $\partial Y = \partial_- Y \sqcup \partial_+ Y$, where $\partial_\pm Y$ are unions of connected components of ∂Y.

Let

$$dist(\partial_- Y, \partial_+ Y) = 2r,$$

let the sectional curvature of Y be bounded from below,

$$\kappa(Y) \geq \kappa_-$$

and let

$$Sc_{|(n-1} \geq \sigma.$$

Then

Y contains a smooth hypersurface $X \subset Y$, which separates $\partial_- Y$ from $\partial_+ Y$ (recall that $\partial Y = \partial_- Y \sqcup \partial_+ Y$) *and such that the scalar curvature of the induced Riemannian metric in X satisfies:*

$[\sigma|\alpha]$ $$Sc(X) \geq \sigma - (n-1)\alpha_{\kappa_-}(r)^2,$$

where $\alpha_{\kappa_-}(r)$ denotes the curvature of the circle of radius r in the standart surface with constant curvature κ_-, e.g.

- $\alpha_1(r) = \frac{\cos r}{\sin r}$,
- $\alpha_0(r) = \frac{1}{r}$,
- $\alpha_{-1}(r) = \frac{e^r + e^{-r}}{e^r - e^{-r}}$.

Proof. Let $X_{[2r]} \subset Y$ be the $2r$-equidistance hypersurface to $\partial_- Y$ and $X_{[2r-r]} \subset Y$ be the r-equidistant to X_{2r} on the side of $\partial_- Y$. Then clearly

($\circ_r$) *the hypersurface $X_{[2r-r]}$ is $C^{1,1}$-smooth with the curvature, i.e. with the norm of the second fundamental form, bounded by $\alpha_{\kappa_-}(r)$.*

Hence, $X_{[2r-r]}$ can be approximated by C^∞-smooth hypersurfaces $X_\varepsilon \subset Y$ with curvatures bounded by $\alpha_{\kappa_-}(r) = \varepsilon$ for all $\varepsilon > 0$. QED.

Remark. If $n \leq 8$ then $\partial_- Y$ and $X_2 \subset Y$ can be separated by a smooth stable μ-bubble $X \subset Y$ such that the scalar curvature of a warped product metric $g^\rtimes = g^\rtimes(x,t) = dx^2 + \phi(x)^2 dt^2$ on $X \times \mathbb{T}^1$ is bounded from below in terms of $\sigma = \inf_y Sc(Y, y)$ and r as follows (see section 3.7 in [G(scalar) 2021]),

$$Sc(X) \geq \sigma - \frac{(n-1)\pi^2}{nr^2}.$$

Although this is not formally stronger than $[\sigma|\alpha]$, it is by far more general and informative. Probbaly, a version of this holds true for all n, but the present day techniques (due to Lohkamp and to Schoen-Yau) fail short of confirming this for $n \geq 9$.

Questions. (a) Does $(\circ_r)$ generalize to submanifolds $X \subset Y$ of codimensions $k > 1$, where Y is, in some way, "wide in k-directions"?

For instance, let Y be a Riemannin manifold homeomorphic to $X_0 \times B^k(1)$, where X_0 is a closed manifold of dimension $n-k$, let the sectional curvature of Y be bounded by $|\kappa(Y)| \leq 1$ and the injectivity radius by $inj.rad(Y) \geq 1$ (compare with [Gr2022]).

What else need you know about Y to effectively evaluate the minimal α, such that Y contains a submanifold $X \subset Y$ *homologous to* $X_0 = X_0 \times \{0\} \subset X_0 \times B^k(1) = X$, such that the curvature of X in Y is bounded by α?

What is the best bound on α in a presence of a *proper* (boundary-to-boundary) λ-*Lipschitz* map $X \to B^k(1)$?

The known (unless I am missing some) *quantitative transversality theorems* applied to maps $X \to B^k$ deliver submanifolds $X \subset Y$ with $\alpha \leq const_n$, but we need X with $\alpha \leq const_k$ for our purposes.

Alternatively, an inductive use of $(\circ_r)$ leads to a bound with

$$const \sim 100^{k(1+diam(Y))}$$

but this is not satisfactory either.

(b) How much (if at all) do (essential) global (geo)metric and/or topological properties of Riemannian n-manifolds Y with $Sc_{|m}(Y) \geq m(m-1)$ for $m \geq 3$ differ from those with $Sc(Y) \geq n(n-1)$?

For instance, does the product $\mathbb{T}^{n-2} \times S^2$, $n \geq 4$, admit a metric with $Sc_{|3} > 0$?

1.4 Remarks, Acknowledgements and the Plan of the Paper

The lower bounds on curvatures of tori (see section 1.3) in concert with the "natural symmetry" of Clifford's manifols may lead one to believe that such bounds persist in all codimensions. But when I mentioned this to Fedia Bogomolov, "everything is possible in large dimensions" – he responded.

Then my attempts to prove lower bounds on the curvatures of m-tori in n-dimensional balls for $n \sim 2m$ were arrested by what Gilles Pisier explained to me about norms of generic linear families of selfadjoint operators.

Also Gilles pointed out to me on the criticality of dimensions $N \sim m^2$ (example 3.1 in [FLM1977]) and the present state of art with Dvoretzky-Milman inequalities for the l_p-spaces was explained to me by Grigoris Paouris who also suggested to me the relevance [K1995] for evaluation of the Kolmogorov diameter D .

Then Bo'az Klartag and Noga Alon patiently explained me the essential properties on the spherical designs and construction of these based on binary codes, allowing sharp bound on D in moderately high dimensions. We present all this in section 2.

In section 3, we show how bounds on the Kolmogorov m-diameter of the space l_4^N translate to corresponding inequalities for curvatures $curv(X \hookrightarrow \mathbb{R}^{2N})$ for submanifolds X in the Clifford tori $\mathbb{T}^N \subset \mathbb{R}^{2N}$.

In section 4.1 we elaborate on the round torus construction from section 1 needed for immersions below $4m-2$.

In section 4.2. we exhibit codim 1 immersions with small curvatures as boundaries of "tubular neighbourhoods" of immersion with high codimension constructed in the previous sections and similarly construct expanding maps in the cases indicated in section 1.2.

In section 4.3 we describe a twisting procedure of immersed manifolds by regular homotopies with controlled curvature and in section 4.4. we outline a similar procedure based on *Poenaru-Elashberg's folding idea.*

In section 5.1 we compute the curvatures of generalized Veronese maps.

In behaviour 5.2 we evaluate the increase of min.curv under taking products of manifolds and under surgery.

In section 5.3 We give example of conservation and non/conservation of bounds on curvatures by connected sums of embedded manifolds.

In section 5.4 we say a few words on realization of homology classes in Riemannian manifolds by immersed and embedded submanifolds with small curvatures.

In section 5.5 we enlist what little is known on elementary (with no Dirac operators or geometric measure theory) lower bounds on curvatures of immersions.

2 Kolmogorov's $D = D(m,N,p)$, Hilbert's Theorem and Spherical Designs

K-Diameter $\sqrt[p]{\mathbf{D(m,N,p)}}$. Let $||y||_{L_p}$, $y = (y_1, ..., y_N) \in \mathbb{R}^N$ denote the normalized norm l_p,

$$||y||_{L_p} = \left(\frac{1}{N}\sum_{i=1}^{N}|y_i|^p\right)^{\frac{1}{p}}$$

Let $D(m,N,p)$ denotes the infimum of the numbers $D > 0$ such that $\mathbb{R}^N$ contains an m-dimensional linear subspace X, such that

$$||x||_{L_p}^p \le D||x||_{L_2}^p, \text{ for all } x \in X.$$

Observe that $D(1,N,p) = 1$, $D(m,m,p) = m^{\frac{p}{2}-1}$, that $D(m,N,p)$ is monotone increasing in m and decreasing in N and let

$$D(m,p) = D(m,\infty,p) = \lim_{N\to\infty} D(m,N,p).$$

2.1.A. Gamma Function Design Formula. If $p = 4,6,8...$, then a simple $O(m)$-averaging argument, shows that

$$[\Gamma/\Gamma] \qquad D(m,p) = \frac{\int_{S^{m-1}}|l(s)|^p ds}{\left(\int_{S^{m-1}}|l(s)|^2 ds\right)^{\frac{p}{2}}} = \frac{m^{\frac{p}{2}-1}\cdot 3\cdot 5\cdots(p-1)}{(m+2)\cdot(m+4)\cdots(m+p-2)},$$

where $l(s)$ is a non-zero linear function on on the sphere.

2.1.B. Hilbert Connection. In his proof of the Waring problem, Hilbert shows the existence of $M = \binom{m+p-1}{m-1} + 1$ *rational points* $s_i \in S^{m-1}$ and of positive *rational* weight $w_i > 0$, $\sum_1^M w_i = 1$, such that $\sum_i w_i l^d(s_i) = \int_{S^{m-1}} l^d(s)d$ *for all* linear functions on he sphere.

This, after partitioning each s_i into Δ atoms for Δ being the smallest common denominator $\mathcal{N}$ of w_i, becomes what is no-a-days called *spherical design* of cardinality $N = \mathcal{N}M$ of w_i, which yields (this is nearly obvious, see **2.1.C** below) the following.

$\mathbf{D(m,N)}$**-*Stabilization*:** $D(m,N,p) = D(m,\infty,p)$ for all sufficiently large $N \geq= N_{Hilb}(m,p)(\leq \mathcal{N}M)$, where – to be safe let it be rough– $N_{Hilb} \leq m^{m^p}$.

***Design Rationality*:** If $N \geq N_{Hilb}$ then the space l_p^N contains a *rational* linear subspace X of dimension m, such that

$$||x||_{L_p}^p = D(m,p)||x||_{L_2}^p \text{ for all } x \in X.$$

2.1.C. Spherical Designs and the Equality $D(m,N) = D(m,\infty)$

A *design of even degree* $p = 2,4,...$ *and cardinality* N on the sphere S^{m-1} is a map from a set Σ of cardinality N to the sphere, written as $\sigma \mapsto s(\sigma)$, such that the linear functions $l(s)$ on the sphere $S^{m-1} \subset \mathbb{R}^m$ satisfy

$$\frac{1}{N}\sum_{\sigma\in\Sigma} l^d(s(\sigma)) = \int_{S^{m-1}} l^d(s)ds,\ d = 2,...,p,$$

where ds is the $O(m)$ *invariant probability* measure on the sphere.

Hence, the linear map from the space $\mathbb{R}^{m\perp}(=\mathbb{R}^m)$ of linear functions on the sphere $S^{m-1} \subset \mathbb{R}^m$ to $\mathbb{R}^N = \mathbb{R}^\Sigma$ preserves both, the L_2 and the L_p-norms and, by the above $[\Gamma/\Gamma]$,

the existence a design of cardinality N *implies that* $D(m,N,p) = D(m,p)$.[12]

Non-rational designs, at least for $p = 4$, are known to exit for $N << N_{Hilb}$.

2.1.D $2m^2$**-Design Construction.** *If* $p = 4$*, and if* m *is a power of 2, then there exists a spherical designs of cardinality* $N = 2m^2 + 4m$. [13]

This, now for all m, shows that

(i) $$D(m,N,4) = \frac{3m}{m+2} \text{ for } N \geq 8(m^2+m).$$

$[\mathbb{R}^2$ in $l_4^3]$-*Example.* $D(2,N,4) = \frac{3}{2}$ for $N \geq 3$, with four (rational) planes $X \subset \mathbb{R}^3 = l_4^3$, where $||x|||_{L_4}^4 = \frac{3}{2}||x|||_{L_2}^4$: these are the normals to the vectors (1,1,1,), (1,1, -1), (1,-1,1), (1,-1,-1).

2.1.E. $D(m,N)$**-Inequalities.** If $N \lesssim m^2$, then upper bounds on $D^4(m,N,4)$ follow from the corresponding estimates in the randomization proofs of the Dvoretzky theorem for the l_p-spaces, where the following inequality follow from (the argument in) [PVZ2017] as it was spelled out in details in a mesage by Grigoris Paouris to me.

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[12]See [BB2009], [LW1993] for more about it.

[13]This was stated and proved in a written message by Bo'az Klartag to me. Also, Bo'az pointed out to me that the Kerdock code used in [K1995] yields designs for $m = 4^k$ and $N = \frac{m(m+2)}{2}$.

(ii) $D(m,N,4) \leq 3 + const_{\textbf{(ii)}} \frac{m^2}{N}$ for $N \geq m^2$;[14]

(iii) $D(m,N,4) \leq const_{\textbf{(iii)}} \frac{m^2}{N}$ for $2m \leq N \leq m^2$.[15]

2.1.F. $D(m,N)$ **Concentration Property**. The existence of m-subspaces $X \in l_4^N$ in [FLM1977] and [PVZ2017], such that

$$[D] \qquad \|x\|^4_{L_4} \leq D\|x\|^4_{L_2}, \ x \in X,$$

is derived from a *lower bound the measure* of those m-subspaces $X \subset \mathbb{R}^N$, where this inequality fails for some $x \in X$.

In particular, the argument used in [FLM1977] implies that the measure μ_D of those $X \subset \mathbb{R}^N$ with respect to the $O(N)$-invariant probability measure in the Grassmanian $Gr_m(\mathbb{R}^N$ where $\|x\|^4_{L_4} \geq D\|x\|^4_{L_2}$, for some $x \in X$ satisfies:

If ,

$$D > \frac{3m}{m+2}$$

then

$$\mu_D \to 0, \ \textit{for } N \to \infty.$$

Nash Connection. Besides applications to lower bounds on curvatures of immersions (see next section), Hilbert's argument, combined with a Nash-like twist, leads to C^2-smooth isometric Riemannian immersions with (large) *prescribed* curvatures and also to a solution of the *differential geometric Warning problem*:

construction of *isometric C^1-immersions of manifolds with symmetric differential forms of degrees $d > 2$*, (see 2.4 (B)(4) on p. 205 in [Gr1986] and [Gr2017]).

3 Equivariant Immersions $\mathbb{R}^m \to S^{2N-1}$ and Euclidean $\mathcal{D}(m,N)$-Theorem for $N \geq 4n$

3.A. Curvatures of the Clifford Tori. Let

$$\mathbb{T}^N \subset S^{2N-1} \subset B^{2N}(1) \subset (B^2(1))^N \subset \mathbb{R}^{2N}$$

be the Clifford torus and observe that the second quadratic form of this torus in the the ambient Euclidean space $\mathbb{R}^{2N} \supset S^{2N-1} \supset \mathbb{T}^N$, regarded as a quadratic form with values in the normal bundle, is

$$\mathrm{II}{=}\sqrt{N}\textstyle\sum_{i=1}^N \nu_i dt_i^2,$$

where t_i are the cyclic coordinates on the torus and $\{\nu_i \in T^\perp(\mathbb{T}^N \subset \mathbb{R}^{2N})\}$ is the corresponding orthonormal frame of *normal* vectors to $\mathbb{T}^N$.

This, in terms of the orthonormal *tangent* frame $\{e_i = \frac{\partial}{\partial t_i} \in T(\mathbb{T}^N)\}$, means that

$$\mathrm{II}\colon e_i \otimes e_i \mapsto \sqrt{N}\nu_i \text{ and } \mathrm{II}\colon e_i \otimes e_j \mapsto 0 \text{ for } i \neq j.$$

[14]This follows from **(i)** for $N \geq 8(m^2+m)$ and, if $const_1$ is large, also for (some) $N \leq 8(m^2+m)$. Besides, the inequality $D^4(m,m^2,4) \leq const$ follows from (the proof of) example 3.1 in [FLM1977].

[15]Since $D(m,N,4) \leq D(m,m,4) = m$ for all m and N, the significance of this inequality for $N \sim m$ depends on the value of $const_2$.

Thus, the curvature of $\mathbb{T}^N$ in B^N along a unit tangent vector $\bar{x} \in T(\mathbb{T}^N)$,

$$\bar{x} = \sum_i x_i e_i, \text{ where } \sum_i x_i^2 = 1,$$

is

$$curv(\mathbb{T}^N, \bar{x}) = ||\mathrm{II}(\bar{x} \otimes \bar{x})|| = ||\mathrm{II}(\sum_i x_i e_i \otimes \sum_i x_i e_i)|| =$$

$$||\mathrm{II}(\sum_{ij} x_i x_j (e_i \otimes e_i)|| = \sqrt{N} || \sum_i x_i^2 \nu_i|| = \sqrt{N}\sqrt{\sum_i x_i^4} = \sqrt{N}\frac{\sqrt{\sum_i x_i^4}}{||\bar{x}||^2} =$$

where $||\bar{x}||^2 = |\bar{x}|_{l_2}^2 = \sum_{i=1}^N x_i^2$.

Hence,

$$(\bigstar) \qquad curv(\mathbb{T}^N, \bar{x}) = \left(\sqrt[4]{N}\frac{||\bar{x}||_{l_4}}{||\bar{x}||_{l_2}}\right)^2 = \left(\frac{||\bar{x}||_{L_4}}{||\bar{x}||_{L_2}}\right)^2,$$

where, recall, the L_p-norms refer to the finite probability spaces with N equal atoms,

$$||\bar{x}||_{L_p} = \frac{||\bar{x}||_{l_p}}{\sqrt[p]{N}}.$$

3.B. Proof of the Euclidean $\mathcal{D}(\mathbf{m},\mathbf{N})$-Theorem 1.1.A for $N \geq 2m$**.** The above ($\bigstar$) implies the existence of an equivariant isometric immersion from the Euclidean m-space to the Clifford N-torus,

$$f^{\odot} : \mathbb{R}^m \to \mathbb{T}^N \subset S^{2N} \subset \mathbb{R}^{2N}$$

with the relative curvature $curv_{\mathbf{E}}^{\mathbf{e}}(f^{\odot})$ (for the Euclidean metrics $\mathbf{e}$ in $\mathbb{R}^m$ and $\mathbf{E}$ in $\mathbb{R}^{2N}$) equal to $\sqrt{D(m,N)} = \sqrt{D(m,N,4)}$.

Hence,

$$\mathcal{D}(\mathbf{m},\mathbf{N}) \leq \sqrt{D(m,M)}$$

for all m and $N \geq 2M$; thus the above $D(m,N)$-inequalities **(i)**,**(ii)**,**(iii)** yield the corresponding $\mathcal{D}(\mathbf{m},\mathbf{N})$ inequalities in 1.1.

In addition to that, if the l_4^N-space contains a rational m-subspace X with $\frac{||x||_{L_4}^4}{||x||_{L_2}^4} = D$, then $\mathbb{T}^N$ contains an m-subtorus with the ambient Euclidean curvature $\sqrt{D}$.

3.C. Proof of $[\mathbf{N} >>]$ **from 1.1.E.** Embed $X^m \to \mathbb{R}^{2m} \subset \mathbb{R}^N$, $N > 2m$, and apply orthogonal transformations $o \in O(M)$ to X. Since X is compact (non-compact manifolds are irrelevant here) the $D(m,N)$-concentration 2.1.F implies that there exist an $o_\varepsilon \in O(n)$, such that all tangent vectors $\tau \in o(T(X)) \subset \mathbb{R}^N$ satisfy

$$||\tau||_{L_4}^4 \leq \left(\frac{3m}{m+2} + \varepsilon_N\right)||\tau||_{L_2}^4, \text{ where } N_\varepsilon \to \infty \text{ for } N \to \infty .$$

Thus, arguing as earlier, the λ-scaled manifold X imbeds to the Clifford torus $\mathbb{T}^N \subset S^{2N-1} \subset \mathbb{R}^{2N}$ with

$$curv(X \hookrightarrow \mathbb{R}^{2N}) \leq \left(\sqrt{\frac{3m}{m+2} + \varepsilon_N} + \epsilon_\lambda\right), \text{ where } \epsilon_\lambda \to 0 \text{ for } \lambda \to \infty$$

and the proof follows.

3.D. δ-Approximation in Non-Euclidean Riemannian Manifolds. The derivation of the δ-approximation from expanding Euclidean maps in section 1.1 easily generalizes, albeit with limitations, to Riemannian manifolds as follows.

Theorem. Let Y be a complete Riemannin manifold[16] with the sectional curvature $|sect, curv(Y)| \leq \kappa^2$ and let $f : X \to Y$ be a continuous map.

If the induced bundle $f^*(T(Y)) \to X$ contains a subbundle isomorphic to $X \times \mathbb{R}^N$, (i.e. a trivial one) and if X admits an immersion to $\mathbb{R}^N$, e.g. $2m - 1 \leq N \leq dimY - dim(Y) - 1$, then, for all positive $\delta \leq \frac{1}{2\kappa}$, the map f can be δ-approximated by immersions $f_\delta : X \to Y$, such that

$$curv_{f_\delta}(X) \leq \frac{1+2\kappa}{\delta}\sqrt{\underline{\mathcal{D}}(m,N)},$$

where

$$\underline{\mathcal{D}}(m,N) \leq \frac{3m}{m+2} + const\frac{m}{\sqrt{N}} \text{ for } N \geq 2m$$

and

$$\underline{\mathcal{D}}(m,N) \leq \frac{6m^{\frac{3}{2}}}{N-m} \text{ for } N \leq 2m.$$

Proof. Proceed as in the proof of 1.1.B, where instead of adding $\delta \cdot f^{\odot} \circ \psi_{\delta^{-1}\lambda}$ to f'_ε we the compose exponential map with a (fiberwise injective) bundle homomorphism from the trivial bundle $X \times \mathbb{R}^N$ to X over the smooth map f'_ε, (this map ε-approximates f).

3.1 Subtori in Non-Equilateral Clifford Tori

All invariant N-tori in the sphere $S^{2N-1} \subset \mathbb{R}^{2n}$ are (equal, up to isometries of S^{2N-1}, to) the orbits of the product action of N-copies of the standard action of $\mathbb{T}^1$ in the plane. where these orbits are equal to the non-equilateral Clifford tori

$$\mathbb{T}^N(\bar{r}) \bigtimes_{i=1}^{N} S^1(r_i), \text{ for } \bar{r} = (r_1, ..., r_N), \text{ where } ||\underline{r}||^2 = \textstyle\sum_i r_i^2 = 1$$

Then, similarly to the above ($\bigstar$), the values of the curvature operator of this torus at the unit tangent vectors $\bar{x} = (x_1, ...x_N) \in T(\mathbb{T}^N(\bar{r}))$ are

$$(\bigstar\bar{\mathcal{R}}) \qquad curv(\mathbb{T}^N_{\bar{r}}, \bar{x}) = \left\|\sum_i \frac{x_i^2}{r_i}\nu_i\right\| = \sqrt{\sum_i \frac{x_i^4}{r_i^2}}$$

where, if all $r_i = \frac{1}{\sqrt{N}}$, this reduces to ($\bigstar$) for

$$\sqrt{\sum_i \frac{x_i^4}{r_i^2}} = \sqrt{\frac{\sum_i |x_i|^4}{N}}$$

and where we denote

$$||x||_{L_4(\bar{r})} = \sqrt[4]{\sum_{i=1}^{N} \frac{x_i^4}{r_i^2}}$$

[16]One may allow a boundary, but this is a minor problem.

3.1.A. ***Conclusion.*** There is a one-to-one correspondence between

$$\textit{equivariant } \mathbb{R}^m \subset S^{2N-1} \textit{ with } curv(\mathbb{R}^m) < \alpha$$

and pairs $(\bar{r}, X)$*, where* $\bar{r} = (r_1, ..., r_N)$ *is a unit vector with positive entries,*

$$\sum_{i=1}^{N} r_i^2 = 1,\ \ r_i > 0,$$

and subspaces $X \subset Y = \mathbb{R}^N = l_2^N$ *is a such that all* $x \in X$ *satisfy*

$$||x||_{L_4(\bar{r})} < \sqrt{\alpha} \cdot ||y||_{L_2},$$

where, recall, the L_2-norm of $y \in Y$, including $y \in X \subset Y$, is

$$||y||_{L_2} = \sqrt{\frac{\sum_{i=1}^{N} y_i^2}{N}} = \frac{||y||}{\sqrt{N}}.$$

Conceivably, m-torical orbits not contained in $\mathbb{T}^N_{Cl}$, e.g. those *maximizing the m-volumes* of the respective m-tori actions, may have slightly smaller curvatures than Kolmogorov's $D(m, N)$, that is, as we know, is equal to the infimum of the curvatures of m-subtori in $\mathbb{T}^M_{Cl}$.

This can be stated with the $\bar{r}$-counterpart of Kolmogorov's $D(m, N)$, denoted $\Diamond(m, N)(\leq D(m, N))$ that is the infimum of the suprema of the ratios of the two norms:

$$\Diamond(m, N) = \inf_{Y, \bar{r}} \sup_{0 \neq y \in Y} \frac{||y||_{L_4(\bar{r})}}{||y||_{L_2}},$$

where the infimum is taken over all m-dimensional linear subspaces $Y \subset \mathbb{R}^N$ and all positive unit vectors $\bar{r}$.[17]

Question. Is, ever, $\Diamond(m, N) < D(m, N)$?

The space $\mathcal{I}_\alpha = \mathcal{I}(\mathbf{m}, \mathbf{N}, \alpha)$ *of isometric equivariant immersions* $\mathbb{R}^m \hookrightarrow S^{2N-1}$ *with curvatures* $\leq \alpha$ is a semi algebraic subset in the (Euclidean) space $J_N(m, N)$ of N-jets at $0 \in \mathbb{R}^m$ of smooth maps $\mathbb{R}^m \to \mathbb{R}^N$[18], which is invariant under the action of the orthogonal group $O(2N)$, and where the $O(2N)$-orbit of an $I \in \mathcal{I}$ in S^{2N-1} is equal to

$W_I \backslash O(2N)/\mathbb{T}^N$, where W_I is the subgroup of the Weyl group of $O(2N)$, which preserves I, (this is empty for generic I).[19]

There can be something geometrically interesting in the $O(N)$-topology of $\mathcal{I}_\alpha$ depending on α, but all one can say off hand is the *Petrovsky*-(Thom-Milnor) *bound on the homology of* $\mathcal{I}_\alpha$ by the algebraic degree of this set.

---

[17] Grigoris Paouris has sent to me a message with an evaluation of $\Diamond_{\bar{r}} = \inf_Y \sup_{0 \neq y \in Y} \frac{||y||_{L_4(\bar{r})}}{||y||_{L_2}}$, for several classes of $\bar{r}$.

[18] The space $J_k(m, N)$ is isomorphic to the space of polynomial maps $\mathbb{R}^m \to \mathbb{R}^N$ of degrees $\leq k$

[19] The corresponding space $\mathcal{X}(m, N, \sqrt{\alpha})$ of m-subspaces X in L_4^N with $\frac{||x||^4_{L_4}}{||x||^4_{L_2}} = \sqrt{\alpha}$, which, albeit being also semi algebraic, has more combinatorial flavour than $\mathcal{I}$.

4 Normal Immersions in Small Codimensions

4.1 Proof of Euclidean $\mathcal{D}(m,N)$-Theorem for $N \le 2m$

$\rtimes$-Construction. Let $\phi_1 : X_1 = X_1^{m_1} \hookrightarrow \mathbb{R}^{m_1+n_1}$, be an immersion with a *trivial* normal normal bundle, where this "triviality" is implemented by a smooth map

$$\Phi_1 : X^1 \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_1+n_1}$$

and let $\phi_2 : X_2 = X^{m_2} \to \mathbb{R}^{n_1}$ be another immersion. If ϕ_2 lands in the r-ball in $\mathbb{R}^{n_1}$ for some $r_1 > 0$,

$$\phi_2(X_2) \subset B_0^{n_1}(r) \subset \mathbb{R}^{n_1}$$

and

$$curv_{\phi_1}(X_1) \le \alpha_1 < \frac{1}{r},$$

then the composed map $(x_1, x_2) \mapsto \Phi_1(x_1, \phi_2(x_2)$ is an *immersion,* say

$$\phi_1 \rtimes \phi_2 : X_1 \times X_2 \to \mathbb{R}^{m_1+n_1}.$$

Recall that the *normal connection* $\nabla^\perp$ in the (trivial) normal bundle

$$X_1 \times \mathbb{R}^{n_1} = T^\perp(X_1) = T(\mathbb{R}^{m_1+n_1}) \ominus T(X^1) \to X_1$$

is defined by the field $\tau^\perp$of tangent m_1-planes in $X_1 \times \mathbb{R}^{n_1}$, which are normal to the Euclidean fibers $x_1 \times \mathbb{R}^{n_1}$ with respect to the (flat) Riemannin metric induced by the map $\Phi_1 : X^1 \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_1+n_1}$.

Flat Split Bundles and $\nabla^\perp$-Trivial Immersions The connection $\nabla^\perp$ is called *flat split* if the map Φ_1 is $\nabla^\perp$*-parallel* that is the field $\nabla^\perp$ is normal to the fibers $x_1 \times \mathbb{R}^{n_1}$ with respect the product metric in $X^1 \times \mathbb{R}^{n_1}$ and the immersion ϕ_1 is called $\nabla^\perp$*-trivial* in this case.

4.1.A. List of $\nabla^\perp$-Trivial Examples. (a) Immersions $\mathbb{R}^1 \to \mathbb{R}^n$ are $\nabla^\perp$-trivial.

(b) Codimension 1 immersion of orientable manifolds, $X^m \to \mathbb{R}^{m+1}$, are $\nabla^\perp$-trivial.

(c) Equivariant immersions of tori, $\mathbb{T}^m \to \mathbb{R}^n$, are $\nabla^\perp$-trivial.

(d) Direct products of $\nabla^\perp$-trivial-immersions $\phi_i : X_i \to \mathbb{R}^{n_i}$

$$\bigtimes_i \phi_i : \bigtimes_i X_i \to \mathbb{R}^{\Sigma_i n_i}$$

are $\nabla^\perp$-trivial.

(e) The above "semidirect products" $\phi_1 \rtimes \phi_2 : X_1 \times X_2 \to \mathbb{R}^{m_1+n_1}$ of $\nabla^\perp$-trivial $\phi_1 : X_1 \to \mathbb{R}^{m_1+n_1}$ and $\phi_2 : X_2 \to \mathbb{R}^{n_1}$ are $\nabla^\perp$-trivial.

4.1.B. (Obvious) **$\rtimes$-Normality Lemma.** Let $\phi_1 : X_1 \to \mathbb{R}^{n_1}$ and $\phi_2 : X_2 \to \mathbb{R}^{n_1}$ be $\nabla^\perp$-trivial immersions. Then:

$\bullet_{norm}$ If $\phi_1 : X_1$ and ϕ_2 are normal (see 1.1) then $\phi_1 \rtimes \phi_2$ is also normal.

$\bullet_{curv}$ If $\phi_2(X_2) \subset B^{n_1}(r) \subset \mathbb{R}^{n_1}$, then

$$foc.rad_{\phi_1} \rtimes \phi_2(X_1 \times X_2) \ge \min(foc.rad_{\phi_2}(X_2), foc.rad_{\phi_1}(X_1) - r)$$

and in the normal case the relative curvature of $\phi_1 \rtimes \phi_2$ (as well as the curvature $curv(X) = foc.rad(X)^{-1}$ itself), satisfies the corresponding inequality.

$$curv(\phi_1 \rtimes \phi_2) \le (\min(curv(\phi_2)^{-1}, curv(\phi_1)^{-1} - r))^{-1}.$$

4.1.C. Torus-by-Torus Construction. Let

$$[-1,1]\times\mathbb{T}^1 \to [-2,2]^2 \supset B^2(2)$$

be the map obtained by rotation of the segment $[0,2$ around the origin in the plane (which is an immersion away from the "interior" boundary circle) and let

$$f_1 = f_0^{\times k} : [-1,1]^k \times \mathbb{T}^k = ([-1,1]^k \times \mathbb{T}^1)^k =\to ([-2,2]^2)^k = [-2,2]^{2k},$$

$$f_2 : [-1,1]^k{\times}\mathbb{T}^{3k} = [-1,1]^k{\times}\mathbb{T}^k{\times}\mathbb{T}^{2k} \to [-2,2]^{2k}{\times}\mathbb{T}^{2k} = ([-2,2]^k{\times}\mathbb{T}^k)^2 \to [-4,4]^{4k}$$

....

$$f_i : [-1,1]^k \times \mathbb{T}^{k2^i-k} \to [-2^i,2^i]^{k2^i}.$$

It follows by the construction, that this map is normal and that the normal exponential map of the central torus

$$\mathbb{T}^{2^i-1} = 0\times\mathbb{T}^{2^i-1}$$

(immersed actually embedded) to the cube $[-2^i,2^i]^{k2^i}$ is injective in the interior of $[-1,1]^k{\times}\mathbb{T}^{k2^i-k}$. Hence, the curvature of this torus and the (relative) curvature of the immersion f_i are bounded by 1 and the corresponding scaled map f : $\mathbb{T}^{k2^i-k} \to B^{k2^k}$ satisfies

$$curv_f(\mathbb{T}^{k2^i-k}) = curv^{\mathbb{T}^{k2^i-k}}(f) \le 2^i\cdot\sqrt{k2^i},$$

or, in terms of $m = k2^i - k$,

$$curv_f(\mathbb{T}^m) \le \left(\frac{m}{k}+1\right)\sqrt{m+k},$$

which implies for all m and $k\le m$:

$$curv_f(\mathbb{T}^m) = curv^{\mathbb{T}^m}(f) < 6\frac{m^{\frac{3}{2}}}{k}.$$

The proof of theorem 1.1.B is concluded.

4.2 Proofs of the Codim 1 and the Rolled Band Theorems

.

Let $f : X^m \to Y$ be an immersion with $foc.rad_f(X) = R$ and $S^\perp(r)(X) \to X$ be the bundle of normal r-spheres $S_x^{N-m-1}(r) \subset T_x^\perp(X) = T_{f(x)}(Y) \ominus T_x(X) = \mathbb{R}^{N-m}$.

If $r < R$ then the normal exponential map $E : S^\perp(r)(X) \to Y$ is an immersion, where $foc.rad_E(S^\perp(r)(X)) = \min(r, R-r)$.

For instance, if $X \to B^N(1)$ is an immersion with trivial normal bundle and $curv_F(X) \le$, then the immersion

$$E_f : \left(1+\frac{1}{2c}\right)^{-1} E : X\times S^{N-m-1} = S^\perp\left(\frac{1}{2c}\right)(X) \to B^N(1)$$

has

$$curv_{E_f}\left(X \times S^{N-m-1} \hookrightarrow B^{N-m-1}\right) \le 2c\left(1+\frac{1}{2c}\right) = (2c+1).$$

4.2.A. Codim1 Conclusion. This, applied to immersions of tori $\mathbb{T}^{l-1} \to B^N(1)$ with large N curvature $\mathbb{T}^{l-1} = \sqrt{\frac{3(l-1)}{l+1}}$, yields codimension codimension one immersions with small curvature as stated in 1.1.G.

4.2.B. Generalization from l-Tori to l-Polyhedra. Given a compact polyhedral (or cellular) space P of dimension l, there exists a compact N-manifold X, for all $N \ge 2l-1$, such that:

$\bullet_P$ there is a continuous map $K \to X$, which is a homotopy equivalence in dimensions $< N/2$, i.e. this map induces isomorphisms of the homotopy groups, $\pi_i(P) \to \pi(X)$ for $i < N/2$;

$\bullet_{200}$ if $N \ge 200l^2$ then, for all $\varepsilon > 0$, X admits an immersion to$B^{N+1}(1)$ with

$$curv(X \hookrightarrow B^{N+1}(1) \le 1 + \sqrt{\frac{3l}{l+2}} + \varepsilon.$$

In fact, the boundary of the regular neighbourhood of P embedded to $\mathbb{R}^{N+1}$ can be taken for X.

Embedding Remark. This, X, by its very construction, *embeds* to $\mathbb{R}^{N+1}$, but one can show (section 5.3) that there is *no universal bound on the curvature* of embeddings of X to the unit ball in $\mathbb{R}^{N+1}$.

For instance if P is a connected sum of different lens spaces, e.g.

$$P_k = \#_{i=1}^k S^3/\mathbb{Z}_{p_i},$$

where $p_1 < ... < p_i < ... < p_k$ are prime numbers, then the curvatures of all smooth embeddings $F : X \to B^{N+1}(1)$ satisfy:

$$curv_F(X) \ge \log\log(k)/N^N.$$

Question. What, roughly, is the minimum of the curvatures of embeddings $\mathbb{T}^l \times S^N \to B^{N+l+1}(1)$?

4.2.C. The proof of the "rolled band theorem proceeds similarly to the above.

Let $f : \mathbb{R}^m \to B^{m+M}(1)$ be an immersion with curvature bounded by $\mathcal{D} = \mathcal{D}(m, m+M)$ as in 1.1. let

$$e = e_f : \mathbb{R}^m \times B^M(r) \to \mathbb{R}^m \to B^{m+M}(1+r),\ r < \frac{1}{\mathcal{D}},$$

be the normal exponential map for $\mathbb{R}^m$ immersed to $\mathbb{R}^{m+M} \supset B^{m+M}$ and let

$$E_\lambda : \mathbb{R}^m \times B^M(r) \to \mathbb{R}^m \to B^{m+M}(1) \text{ for } (x,b) \mapsto (1+r)^{-1}e(\lambda x, b).$$

If λ is sufficiently large, then the map E_λ is expanding in the $\mathbb{R}^m$ directions, i.e. it expands $\mathbb{R}^m \times b$ for all $b \in B$ and since it is isometric in the B^M-directions it is expanding on $\mathbb{R}^m \times B^M(r)$... except for one problem:

the normal M-ball bundle $B^\perp(r) \to \mathbb{R}^m$ of the immersed $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m+M}$ is trivial, it is indeed, isomorphic to the product $\mathbb{R}^m \times B^M(r)$ but the map $(x,b) \mapsto \lambda(x,b)$

is not necessarily expanding with respect to the (Euclidean) metric induced by the exponential map. (Look at the planar map $(x, y) \mapsto (0, 10x + y)$

Fortunately, the normal bundles of our immersions constructed in sections and 3. are *flat split*, (see 4.1) the map E_λ is expanding and it can be taken for the required F_r in.

4.2.D. Expanding Maps F_r for all m and M. The above argument delivers expanding maps $F_r : \mathbb{R}^m \times B^M(r) \to B^{M+m}(1)$ provided $r \leq (1+\Delta)^{-1}$, where Δ is taken according to the $\mathcal{D}(m, N)$ inequalities (see section 1.1 and 3).

$$\Delta = \sqrt{\frac{3m}{m+2}} + C_o \frac{m}{\sqrt{M}}, \text{ for } M \geq m,$$

and

$$\Delta = 6\frac{m^{\frac{3}{2}}}{M} \text{ for } M < m.$$

.

4.3 Proof of the Regular Homotopy/Approximation Theorem.

Step 1. *Slicing.* Given an immersed manifold

$$X = X^m \overset{\phi}{\hookrightarrow} \mathbb{R}^n,\ n > m,$$

, and (small) positive numbers $\varepsilon, \delta > 0$ there exists an immersion

$$X \overset{\varphi}{\hookrightarrow} \mathbb{R}^n$$

regularly homotopic tp ϕ, such that

$\bullet_{curv_\varphi}$ $curv_\varphi(X) \leq \varepsilon$,

$\bullet_\delta$ the first coordinate function $y_1(x) = y_1(\varphi(x))$ of $y = \phi(x) \in \mathbb{R}^n = \{y_1, ...y_n\}$ is proper Morse, where there are no critical points of y_1 on the δi levels of y_1 for integer $i = ... - 2, -1, 0, 1, 2...$, i.e. the hyperplanes where $y_1 = \delta i$ in $\mathbb{R}^n$ are transversal to $\varpi(X) \subset \mathbb{R}^n$ and

$\bullet_\varepsilon$ the curvatures of these δi levels are bounded by ε.

Proof. If X is compact, then $\bullet_{curv_\varphi}$ achieved achieved by scaling: $x \mapsto \lambda\phi(x)$ for a large λ and then one gets $\bullet_\delta$ by a preliminary generic rotation of $\phi(X) in \mathbb{R}^n$, where then the critical values of $y_1(x)$ moved to the centers of the segments $[\delta i, \delta(i+1)]$, let $\frac{1}{\delta} = o(\lambda)$and conclude the proof with the following obvious (but essential)

4.3.A. Levels Curvature Sublemma. Let $y(x)$ be a Morse function on a compact Riemannian manifold X and x_0 be a critical point, where $y(x_0) = 0$. Then the curvatures of the δ-levels $f^{-1}(\delta) \subset X$ satisfy

$$curv(f^{-1}(\delta) = o\left(\frac{1}{\delta}\right).$$

Step 2. *Zigzag Folding and Compression.* Reflect the X-bands $y_1^{-1}[\delta i, \delta(i+1) \subset X$ in the hyperplanes $y_1 = \delta i$, $i \in \mathbb{Z}$, and thus "compress" $\varphi(X)$ to a zigzag map ζ from X to the Euclidean δ-band between a pair of such hyperplane, say between $y_1 = 0$ and $y_1 = \delta$.

Step 3. *Twisted Regularization with Controlled Curvature.* There exists a smooth 10δ-approximation of ζ by a smooth immersion $\zeta_\circ : X \to \mathbb{R}^n$, such that

$\bullet_\epsilon$ the immersion $\zeta_\circ$ is equal to ζ outside the ϵ-neighbourhood of the*corners of* ζ, that is the subset $y_1^{-1}(\delta\mathbb{Z}) \subset X$, where $\epsilon > 0$ en is a given number which may be taken much smaller than δ;

$\bullet_{reg}$ the immersion $\zeta_\circ$ is regularly homotopic to φ,

$\bullet_{curv/\delta}$ the curvature $\zeta_\circ$is bounded by $\frac{1}{\delta}$

Proof. To see how it works, let $\theta_\circ$ and $\theta_{\mathscr{P}}$ be two immersions of the circle to the plane, each having a single corner point, both with the same corner angle. If we align these corners properly and attach the immersions one to another at the corner points, we obtain a composed smooth immersion θ_* where, if $\theta_{\mathscr{P}}$ is $\mathscr{P}$-shaped, this f_* is regularly homotopic to $f_\circ$.

Now, in he case of a corner along a hypersurface $X_i = \varphi^{-1}\delta)$ attach the product $X_i \times \mathscr{P}$ to $\zeta(X)$ along this corner and by doing it to all X_i we obtain a smooth immersion regularly homotopic toφ where the conditions $\bullet_\epsilon$ and $\bullet_{curv/\delta}$ are easily achievable 10δ close to ζ. Details are left to the reader.

Step 4. *Rolling Bands into Balls.* The band $\mathbb{R}^{n-1} \times [-10\delta, 11\delta] \supset \zeta_\circ(X)$ is mapped to $B^n(1)$ by "rolled band" immersion $F_r : \mathbb{R}^n \times [-r, r] \to B^n(r)$ for r from 1.2.D, where F_r is restricted to the sub-band $\mathbb{R}^n \times [-r/2, r/2]\mathbb{R}^n \times [-r, r]$ and where we then let $\delta = \frac{r}{42}$.

In order estimate the curvature of the composed map $\Phi = F_r \circ \zeta_\circ$,

$$X \overset{\zeta_\circ}{\to} \mathbb{R}^n \times [-r/2, r/2] \overset{F_r}{\to} B^n(1),$$

by $curv_{\zeta_\circ}(X) \le c = \frac{1}{\delta}$ we recall the construction of the underlying normal immersion

$$f = F_r|_{\mathbb{R}^{n-1}\times\{0\}} : \mathbb{R}^{n-1} \to B^n(1 - r),$$

where $curv(f) \le 6(1 - r)^{-1} n^{\frac{3}{2}}$ and where also (the differential of) this map has *controllably bounded anisotropy*,

$$\frac{||d\tau_1||}{||d\tau_2||} \le 2n$$

for all unit tangent vectors $\tau_1, \tau_2 \in T(X)$. It follows that the curvature $curv_\Phi(X)$ is bounded essentially in the same way as that of F_r,

$$curv_\Phi(X) \le 420 n^{\frac{3}{2}},$$

and the corresponding approximation inequality follows as in the proof in the genera case of the δ-approximation theorem. (This δ and that in $[-10\delta, 11\delta]$, albeit similar, are not the same.)

4.3.B. Immersions to non-Euclidean Y. The above argument, unlike the proof of the the δ-approximation theorem as explained in 3.D doesn't generalize to immersions from X to general Riemannian manifolds Y.

Yet, a combination of the above "twisted regularization" on the top of a routine induction by skeleta delivers the following.

4.3.C. Rough Exponential Bound on Curvature. Let Y be a complete Riemannin manifold with $|sect.curv| \le \kappa^2$ and let $f : X = X^m \hookrightarrow Y$ be a smooth immersion.

If $dim(Y) > m$ then, for all positive $\delta \leq \frac{1}{\kappa}$, the map f can be δ-approximated by immersions $f_\delta : X \to Y$, which are regularly homotopic to f and such that

$$curv_{f_\delta}(X) \leq \frac{(1+\kappa)100^m}{\delta}.$$

4.4 Unfolding Folds and other Singularities.

Below is another proof of the regular homotopy/approximation theorem for *orientable hypersurfaces*, which leads to a better, possibly sharp in some cases, bounds on the curvature.

Unfolding Lemma. Let $X = X^m$ be an orientable manifold and $f : X \to \mathbb{R}^{m+1}$ be an immersion. Then, for all $\varepsilon > 0$, there is an immersion,

$$\zeta_\circ : X \to \mathbb{R}^m \times [-1,1],$$

which is reguraly homotopic to f and such that

$$curv_{\zeta_\circ}(X) \leq 1+\varepsilon.$$

Proof. Apply Poenaru's h-principle for pleated maps (see (C) on p.56 in [Gr1986]), and obtain a smooth map $f_1 : X \to \mathbb{R}^{m+1}$ regularly homotopic to f, such that the only singularity of the normal projection $\zeta : X \to \mathbb{R}^m \subset \mathbb{R}^{m+1}$ is a folding along a smooth hypersurface $\Sigma = \Sigma^{m-1} \subset X$.

Make the curvature of the immersion $\zeta : \Sigma \hookrightarrow \mathbb{R}^m$ as small as you wish by λ-scaling as we did earlier and thus also separate different part of Σ far one from another, such that, on the balls of large radii $R \sim \lambda$ in X, the scaled map is ε-close to the standard fold $(x_1, ..., x_m) \mapsto (x_1, ..., x_m^2)$.

"Unfold" $\zeta \rightsquigarrow \zeta_\circ = (\lambda\zeta, y) \in \mathbb{R}^{m+1}$, where $y : X \to \mathbb{R}$ is a smooth function on X, which, in the obvious normal coordinates, depends only on the last coordinate $x = x_m$, where it is ε-close to a lift $\eta_\circ : \mathbb{R} \to \mathbb{R}_+ \times [-1,1]$ of the standard fold $\mathbb{R} \to \mathbb{R}_+$, $x \mapsto y = x^2$, where $\eta_\circ(x) = (x, y(x))$ and where

the x-segment $[-1,1]$ is sent by $\eta_\circ$ to the semicircle in the half plane $\{x,y\}_{y\geq 0}$ and $\eta_\circ(x) = -1$ for $x < -1$ and $\eta_\circ(x) = 1$ for $x > 1$.

Conclude the proof by rolling the band $\mathbb{R}^m \times [-1,1]$ into the ball as in the above step 4.

Remarks. (a) Our unfolding with controlled curvature quantifies a single step in *removal of the singularities* argument (see [GE1971] and section 2.1 in [Gr1986].)

To do the same for all step and thus unfold more general Thom-Boardman singularities with controlled curvature start by observing that our image curve $\eta_\circ(\mathbb{R}) \subset \mathbb{R}_+ \times [-1,1]$, (which is is only C^1-smooth), is equal to the boundary of the 1-neighbourhood of the ray $[1,\infty) \subset \mathbb{R} \times [-1,]1]$.

Then, to unfold $\Sigma^{1,...,1}$, of depth k, where $1, ..., 1 = \underbrace{1, ..., 1}_{k}$, the natural model to use is the boundary of the 1-neighbourhood of the positive quadrant $\mathbb{R}^k_+ \subset \mathbb{R}^k \times [-1,1]$, which has $curv \leq 1$ as well. But I haven't checked if this actually works.[20]

[20]Beware of non-coorientable folds, such as of the Möbius strip along he central line.

(b) It could be interesting to quantify the approximation procedure of *smooth maps by immersion in Sobolev spaces* from [GE 1971'] and also a similar approximation in [Be1991].

(c) It is unclear how to "controllably unfold" in $\mathbb{R}^{m+l}$ more general singularities of smooth maps $X^m \to \mathbb{R}^m \subset \mathbb{R}^{m+l}$.

This leaves the following question open.

Do smooth immersions $f : X^m \to \mathbb{R}^{m+l}$ are regularly homotopic to immersions $f_\circ$, the curvatures of which are bounded up to a multiplicative constant by the minimal relative curvatures of $\nabla^\perp$-trivial immersions of flat tori $\mathbb{T}^m \to \mathbb{R}^{m+l}$.

For instance it remains problematic if

all m-manifolds X admit immersions $f : X \to B^{2m}(1)$ with curvatures $curv_f(X) \le cost\sqrt{m}$, say for $const = 100$.

5 Miscellaneous

5.1 Veronese Maps.

Besides invariant tori, there are other submanifolds in the unit sphere S^{N-1}, which have small curvatures and which are transitively acted upon by subgroups in the orthogonal group $O(N)$.

The generalized Veronese maps are a *minimal equivariant isometric* immersions of spheres to spheres, with respect to certain homomorphisms (representations) between the orthogonal groups $O(m+1) \to O(m+1)$,

$$ver = ver_s = ver_s^m : S^m(R_s) \to S^m = S^{m_s} = S^{m_s}(1),$$

where

$$m_s = (2s+m-1)\frac{s+m-2)!}{s!(m-1!} < 2^{s+m} \text{ and } R_s = R_s(m) = \sqrt{\frac{s(s+m-1)}{m}},$$

for example,

$$m_2 = \tfrac{m(m+3)}{2} - 1,\ R_2(m) = \sqrt{\tfrac{2(m+1)}{m}} \text{ and } R_2(1) = 2,$$

(see [DW1971]If $s = 2$ these, called *classical Veronese maps*, are defined by taking squares of linear functions (forms) $l = l(x) = \sum_i l_i x_i$ om $\mathbb{R}^{m+1}$,

$$Ver : \mathbb{R}^{m+1} \to \mathbb{R}^{M_m},\ M_m = \frac{(m+1)(m+2)}{2},$$

where tis $\mathbb{R}^{M_m}$ is represented by the space $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^{m+1})$ of quadratic functions (forms) om $\mathbb{R}^{m+1}$,

$$Q = \sum_{i=1,j=1}^{m+1,m+1} q_{ij} x_i x_j.$$

The Veronese map, which is (obviously) equivariant for the natural action of the orthogonal group group $O(n+1)$ on $\mathcal{Q}$, where, observe, this action fixes the line $\mathcal{Q}_\circ$ spanned by the form $Q_\circ = \sum_i x^2$ as well as the complementary subspace $\mathcal{Q}_\diamond$ of the *traceless forms* Q, where the action of $O(n+1)$ is irreducible and, thus, it has a *unique, up to scaling* Euclidean/Hilbertian structure.

Then the normal projection[21] defines an equivariant map to the sphere in $\mathcal{Q}_\diamond$

$$ver : S^n \to S^{M_m-2}(r) \subset \mathcal{Q}_\diamond,$$

where the radius of this sphere, a priori, depends on the normalization of the $O(m+1)$-invariant metric in $\mathcal{Q}_\diamond$.

Since we want the map to be isometric, then we either take $r = \frac{1}{R_2(m)} = \sqrt{\frac{m}{2(m+1)}}$ and keep $S^m = S^m(1)$ or If we let $r = 1$ and $S^m = S^m(R_2(m))$ for $R_2(m) = \sqrt{\frac{2(m+1)}{m}}$.

Also observe that the Veronese maps, which are not embeddings themselves, factor via embeddings of projective spaces to spheres

$$S^m \to \mathbb{R}P^m \subset S^{M_m-2} \subset \mathbb{R}^{M_m-1} = \mathcal{Q}_\diamond,\ M_m = \frac{(m+1)(m+2)}{2}.$$

Curvature of Veronese. Let is show that

$$curv_{ver}\left(S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)\right) = \sqrt{\frac{R_2(1)}{R_2(m)} - 1} = \sqrt{\frac{m-1}{m+1}}.$$

Indeed, the Veronese map sends equatorial circles from $S^m(R_2(m)$ to planar circles of radii $R_2(m)/R_2(1)$, the curvatures of which im the ball B^{M_m-1} is $\sqrt{R_2(1)/R_2(m)} = \sqrt{\frac{2m}{m+1}}$, and the curvatures of these in the sphere,

$$curv(S^1 \subset S^{M_m-2}(1)) = \sqrt{curv(S^1 \subset B^{M_m-1}(1))^2 - 1},$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \hookrightarrow S^{M_m-2}(1)$itself. QED.

It may be hard to prove (conjecture in section 1) that Veronese manifolds have the smallest possible curvatures among non-spherical m-manifold in the unit ball: *if a smooth compact m-manifold X admits a smooth immersion to the unit ball $B^N = B^N(1)$ with curvature $curv(X \hookrightarrow B^N) < \sqrt{\frac{2m}{m+1}}$, then X is diffeomorphic to S^m.*

It is more realistic to show that the Veronese have smallest curvatures among submanifolds $X \subset B^N$ *invariant under subgroups in $O(N)$, which transitively act on X.*

Remark. Manifolds X^m immersed to S^{m+1} with curvatures < 1 are diffeomorphic to S^n, see 5.5, but, apart from Veronese's, we can't rule out such X in S^N for $N \geq m+2$ [22] and, even less so, non-spherical X immersible with curvatures $< \sqrt{2}$ to $B^N(1)$, even for $N = m+1$.

It seems hard to decide this way or another, but it may be realistic to try to prove *sphericity of simply connected* manifolds immersed with curvatures < 1 to $S^N(1)$ for all N.

The curvatures of Veronese maps can be also evaluated with the *Gauss formula,* (teorema egregium), which also gives the following formula for curvatures of all ver_s:

[21]The splitting $\mathcal{Q} = \mathcal{Q}_\circ \oplus \mathcal{Q}_\diamond$ is necessarily normal for all $O(m+1)$-invariant Euclidean metrics in $\mathcal{Q}$.

[22]Hermitian Veronese maps from the complex projective spaces $\mathbb{C}P^m$ to the spaces $\mathcal{H}_n$ of Hermitian forms on $\mathbb{C}^{m+1}$ are among the prime suspects in this regard.

From Veronese to Tori. The restriction of the map $ver_s : S^{2m-1}(R_s) \to S^{N_s}$ to the Clifford torus $\mathbb{T}^m \subset S^{2m-1}(R_s)$ obviously satisfies

$$curv_{ver_s}(\mathbb{T}^m) \leq A_{2m-1,s} + \frac{\sqrt{m}}{R_s} = \sqrt{3 - \frac{5}{2}m + \varepsilon(m,s)}$$

for

$$\varepsilon(m,s) = \frac{2}{4m^2} - \frac{4m-2}{s(s+2m-2)} + \frac{5(2m-1)}{2ms(s+2m-2)} - \frac{2m-1}{(ms(s+2m-2))^2}.$$

This, for $s >> m^2$, makes $\varepsilon(m,s) = O\frac{1}{m^2}$
Since $N_s < 2^{s+2m}$,
starting from $N = 2^{10m^3}$

$$curv_{ver_s}(\mathbb{T}^m) < \sqrt{3 - \frac{5}{2}m}.$$

where it should be noted that
the Veronese maps restricted to the Clifford tori are $\mathbb{T}^m$*-equivariant*
and that
this bound is *weaker than the optimal one* $\frac{||y||^2_{l_4}}{||y||^2} \geq \sqrt{3 - \frac{3}{m+2}} + \varepsilon$ from the previous section.

Remarks. (a) It is not hard to go to the (ultra)limit for $s \to \infty$ and thus obtain an
equivariant isometric immersion ver_∞ of the Euclidean space $\mathbb{R}^m$ to the unit sphere in the Hilbert space, such that

$$curv_{ver_\infty}(\mathbb{R}^m \hookrightarrow S^\infty) = \sqrt{\frac{(m-1)(2m+1)}{(m+1)^2}} = \sqrt{2 - \frac{5}{m+1} + \frac{2}{(m+1)^2}},$$

where equivariance is understood with respect to a certain unitary representation of the isometry group of $\mathbb{R}^m$.

Probably, one can show that this ver_∞ realizes the *minimum* of the curvatures among all equivariant maps $\mathbb{R}^m \to S^\infty$.

(b) Instead of $vers_s$, one could achieve (essentially) the same result with a use of compositions of the classical Veronese maps, $ver : S^{m_i} \to S^{m_{i+1}}$, ${}_{i+1} = \frac{(m_i+1)(m_i+2)}{2} - 2$,

$$S^{m_1} \hookrightarrow S^{m_2} \hookrightarrow ... \hookrightarrow S^{m_i},$$

starting with $m_1 = 2m-1$ and going up to $i = m$. (Actually, $i \sim \log m$ will do.)

5.2 Product Manifolds, Connected Sums and Related Constructions

Let $f_i : X_i^{m_i} \to \mathbb{B}^{m_i+1}(1)$, $i = 1, ..., l$, be immersions with focal radii r and let $f_0 : X_0^{m_0} \to B^l(1)$ be an immersion with $foc.rad_f(X_0^{m_0}) = r_0$,
Then the $\rtimes$-construction (see 4.1) delivers an immersion

$$f : X = \bigtimes_0^l X_i \to B^N(1),\ N = l + \sum_1^l m_i,$$

such that

$$foc.rad_{f_\times}(X_\times) \geq \max_{0<\lambda\leq 1} \frac{\min(r-\lambda, \lambda r_0)}{\sqrt{l}+\lambda r_0.}$$

Similarly, if $X_0^{m_0}$ admits a $\nabla^\perp$-trivial (see 4.1) immersion to $B^M(1)$ with focal radius r_0, then X admits an immersion to $B^{M+k}(1)$ for all $k \geq 1 - M + \sum_0^l m_i$, such that

$$foc.rad_{f_\times}(X_\times) \geq \max_{0<\lambda\leq 1} \frac{\min(r_0-\lambda, \lambda r/\sqrt{l})}{\sqrt{l}+\lambda r/\sqrt{l}}$$

5.2.A. Example: Product of Spheres. Let

$$X = X^m = \bigtimes_i S_i^m, \ \sum_i m_i = m,$$

and let $\mu = \min_i m_i$. Then there exists an immersion $f : X \to B^m + 1(1)$, such that

$$curv_f(X) \leq const_\mu m^{\frac{\mu+2}{\mu+1}}$$

Proof. Adopt the torus-by-torus construction **4.1.C** to product of spheres, where instead of squaring maps at each step, use (Cartesian) product of at least μ of maps, where then the above inequality for $foc.rad$ translated to curvature apply.

Embedding Remark. Observe that the resulting maps $X^m \to B^{m+1}(1)$ are *embeddings.*

5.2.B. Connected Sums. If m-manifolds X_i, $i = 1, 2, ..., l$, admit immersions to the unit ball $B^n = B^n(1)$, $n > m$, with the curvatures bounded by a constant C, then the connected sum $X_1\#...\#X_l$ can immersed to B^n with curvature bounded by $5C$.

Proof. Make *geometric connected sums* of all $X_i \hookrightarrow B^n$ with the unit equatorial sphere $S^m \subset S^n = \partial B^n$, where this is done with each X_i individually with a copy of $S^m \subset B^n$ by connecting X_i with $S_1^m = S^m$ with a tube with curvature $< 5C$. Then the connected sum between X_i is implemented by making similar tubes between S_i^m.

Example. Since there are 2-Tori in the unit 3-ball with $curv = 3$, the minimal possible curvatures of orientable surfaces X satisfy

$$min.curv(X_{ori}^2 \hookrightarrow B^3(1)) < 15,$$

while nonorientable ones have

$$min.curv(X^2 \hookrightarrow B^3(1)) \leq 5min.curv(\mathbb{R}P^2 \hookrightarrow B^3(1)) < 50,$$

the Boy surface seem to have curvature about 10, Probably, all surfaces have $min.curv < 10$, but it is unclear, not even for the 2-torus, what actually minimal curvatures of surfaces in $B^3(1)$ are.

Attaching k-Handles for $k \geq 2$. To attach a handle to a sphere $S^{k-1} \subset X$ with a controlled the curvature, with a controllable increase of the curvature, one needs a regular δ-neighbourhood of this sphere in X with δ controllably bounded from below: this which would allow attaching a k handle with the curvature increase roughly by $1/\delta$.

For instance, if $k = 2$ an $S^1 \subset X$ is the shortest non-contractible curve in X, then it does admits such a neighbourhood in X with δ controllably bounded from below by the curvature of X; thus attaching with certain normal frames 2-handles to it is possible with curvature increase by a definite multiplicative constant.

In general one can show the following.

5.2.C. Handles Stretch Proposition. (Compare with 4.3.C.) Let an immersed manifold $X_\diamond^m \overset{\phi}{\hookrightarrow} B^N(1)$ be obtained from $X^m \overset{f}{\hookrightarrow} B^n(1)$ by attaching l-handles for $l \leq k$ where, all steps surgery keep in the *class of immersed manifolds.*

Then ϕ is regularly homotopic to an immersion $\phi_1 : X_\diamond \hookrightarrow B^n(1)$, such that

$$curv_{\phi_q}(X_\diamond) \leq C^{2k} curv_f(X)$$

for $C \leq 10\ 000$.

Sketch of the Proof. Regularly homotop f in $B^n(1)$ to an immersion f_1 with $curv_{f_1}(X) \leq 100^{2k} curv_f(X)$ and such that that the f_1-induced Riemannian metric in a (small) neighbourhood U of the $2k$-skeleton of a smooth triangulation of X is by an arbitrarily large (independently of U) factor λ greater than the f-induced metric.

Assume without loss of generality that all spheres S^i, at which the surgery performed are located and in U don't intersect there (this is possible for $m \geq 2k$, which we may assume with no problem) and choose λ so large that the union of these spheres has a nice thick regular neighbourhood, where the surgery can be made with at most 100^{2k} increase in the curvature.

Remark. It is not hard to visualise an actual proof along these lines but I don't see how to write it down in a readable form.

5.3 Embeddings with Small Curvatures

Connected Sums of Embedded Manifolds. If $X = X^m$ admits an embedding (i.e. a immersion with no self-intersection) to $B^{m+1}(1)$ with curvature $\leq c$, then the connected sums of $2l$-copies of X embed to $B^{m+1}(1)$ with curvatures $< 100c$.

Proof. Let $X_1 \subset B^{m+1}(1)$ be obtained from X by attaching a single 1-handle $S^{m-1} \times [0,1]$, such that $curv X_1 \subset B^{m+1}(1) < 10c$.

Let $\tilde{X}_l$ be the natural cyclic covering of X_1 of order l and let $\bar{X}_l$ be obtained by cutting $\tilde{X}_l$ along the sphere $S^{m-1} \subset \tilde{X}_l$ from the handle.

Observe that this $\bar{X}_l$ is a manifold with two spherical boundary components and that it (almost) naturally embeds to ${}^{m+1}(1)$ with curvature $< 10c$.

Let $\bar{X}'_l \subset B^{m+1}(1) \setminus \bar{X}_l$ be obtained by a slight normal displacement of $\bar{X}_l$ and let us attach $\bar{X}'_l$ to $\bar{X}_l$ along a pair of nearby $(m-1)$- spheres and also fill in the remaining two boundary spheres with m-balls. Clearly, the resulting manifold, call it X_{2l}, is diffeomorphic to the connected sum of $2l$ copies of X and it is not hard to arrange an embedding of X_{2l} to the unit ball with curvature < 100.

Exercises. (a) Let $X = X^m$ be a connected sum of an arbitrary number of manifolds diffeomorphic to product of spheres. Show that X embeds to the unit $(m+1)$-ball with curvature$< 500 \cdot 2^{\frac{m}{2}} m^{\frac{3}{2}}$.

Hint. Embed mutually non-diffeomorphic products of spheres into 2^m disjoint r-balls in $B^{m+1}(1)$ of radii $r = 2^{-\frac{m+2}{2}}$.

(b) Let $X = X^m$ be disconnected closed manifold, which contains l *mutually* *non-diffeomorphic* components. Show that

$$curv_f(X \hookrightarrow B^{m+1}) \ge const_m l,\ const_n \ge \frac{1}{(10m)^m},$$

for all embeddings $f : X \hookrightarrow B^{m+1}(1)$.

(c) Construct closed m-dimensional manifolds X_i, $i = 1, 2, ...$ for all $m \ge 6$, such tat all of them embeds to $B^7(1)$ and such that embedding of connected sums of l among these manifolds have curvatures $\ge constl$.

Question. Can one have these X_i embeddable to $\mathbb{R}^{m+1}$ with curvatures < 1 000 000?

5.4 Cycles with Small Curvature

Our equidimensional expanding maps are effective in delivering immersed submanifolds with controllably bounded curvatures, because these maps themeselves, besides being expanding, have controllably bounded second derivatives.

In general, it is hard to

construct a immersed m-dimensional submanifolds $X \hookrightarrow Y$ with *small curvature* and with *non-zero* homology classes $[X] \in H_m(Y)$.

Apparently, all known results of this kind badly depend on the dimension and/or codimension of X, see [CDMW2016]

A happy exception is the codimension one case, $m = n - 1$, where there is no topological obstructions for the existence of X and where an equidistant smoothing delivers hypersurfaces with controllably small curvatures as follows.

Let Y be a *proper Riemannian band* of dimension n, that is a Riemannin manifold, the boundary ∂Y of which is divided into two disjoint parts, $\partial Y = \partial_- Y \sqcup \partial_+ Y$, where $\partial_\pm Y$ are unions of connected components of ∂Y, and denote by d the *width* of Y,

$$d = width(Y) =_{def} dist(\partial_- Y, \partial_+ Y).$$

Let us d_1-equidistantly push $\partial_- Y$ inside Y for $d_1 < d$ and then d_2-equidistantly move the resulting hypersurface, denoted ∂_{-d_1}, back toward $\partial_- Y$ with $d_2 < d_1$.

That is, ∂_{-d_1} is equal to the (topological) boundary of the d_1-neighbourhood $U_{d_1}(\partial_- Y) \subset Y$ and the result of the second move, call it $X_\circ = \partial_{-d_1|+d_2} \subset U_{d_1}(\partial_- Y)$, is the boundary of $U_{d_2}(\partial_{-d_1}) \subset U_{d_1}(\partial_- Y)$.

Let us evaluate the curvature of $X_\circ$ in terms of the sectional curvatures of Y, where we observe the following.

1. If Y has constant sectional curvature $\pm\kappa^2$, then $X_\circ$ is $C^{1,1}$-smooth and

$$foc.rad(X_\circ) \ge (\min(d_2, d_1 - d - 2));$$

accordingly $curv(X) \le \alpha^\pm_\kappa(\min(d_2, d_1 - d - 2))$ for the function $\alpha^\pm$ from 1.B.

2. If.more generally, the sectional curvatures of Y is pinched between two values, that are the curvatures of two standard surfaces $S_\pm$ with constant curvatures,

$$sect.curv(S_-) \le sect.curv(Y) \le sect.curv(S_+),$$

then the curvature of $X_\circ$ is bounded by the maximum the two numbers:

$\bullet_1$ the first number is the curvature of the circle of the radius d_2 in S_-;

$\bullet_2$ the second number is the curvature of the circle $S^1(r) \subset S_+$, such that the curvature of the concentric circle $S^1(r+d_2)$ is equal to the curvature of the d_1-circle in S_-;

It follows, for instance, that

$(\circ_d)$ if

$$-1 \leq sect.curv(Y) \leq 1$$

and $d = width(Y) \leq 1$, then

Y contains a smooth hypersurface, which separates $\partial_- Y$ from $\partial_+ Y$ and such that

$$curv(X) \leq \frac{4}{d}.$$

Corollary. Let Y be a complete Riemannian n-manifold with $|sectcurv(Y)| \leq \kappa^2$ and with $inj.rad(Y) \geq r$.

Then

$(\circ_{\kappa,r})$ *all integer $(n-1)$-dimensional homology classes $h \in H_{n-1}(Y)$ are realizable by smoothly immersed oriented hypersurfaces $X \hookrightarrow Y$ with $curv(X) \leq 10\kappa + \frac{10}{r}$.*[23]

Indeed, given a homology class $h \in H_1(Y)$, apply $(\circ_d)$ to the infinite cyclic covering of Y, which is defined by this class.

Questions. (a) Do $(\circ\circ_d)$ and $(\circ_r)$ meaningfully generalize to submanifolds $X \subset Y$ of codimensions $k > 1$, where Y is, in some way, "wide in k-directions"?

For instance, Let Y be a Riemannin manifold homeomorphic to $X_0 \times B^k(1)$, where X_0 is a closed connected orientable manifold of dimension $n-k$, let the sectional curvature of Y be bounded by $|\kappa(Y)| \leq 1$ and the injectivity radius by $inj.rad(Y) \geq 1$.

What else need you know about Y to effectively bound the minimal possible curvature of a submanifold $X \subset Y$ *homologous to* $X_0 = X_0 \times \{0\} \subset X_0 \times B^k(1) = X$?

What is the best bound on this curvature in a presence of a *proper* (boundary-to-boundary) λ-*Lipschitz* map $X \to B^k(1)$?

Are, similarly to $(\circ\circ_\kappa, r)$, non-zero multiples of the homology classes $h \in H_m(Y)$, for all $m \leq dim(Y)$, realizable by immersed m-dimensional submanifolds $X \hookrightarrow Y$ with $curv(X) \leq 100m^{100}\left(\kappa + \frac{1}{r}\right)$?

D. **From Focal Radius to Expansion.** Let us turn to the

opposite problem: In what cases does the the r-neighbourhood $U_r(X) \subset X$ of an embedded manifold $X \subset Y$ with *"large" universal covering*, e. g. for X homeomorphic to $\mathbb{T}^m$, and with large $foc.rad(\mathbb{X})$ receive an expanding map from a "*large manifold*" e.g. from $B^m(R) \times B^{n-m}\left(\frac{r}{100}\right)$ with large R?

Here the answer is positive for $m = n-1$ and $m = n-2$:

if X receives expanding maps from the balls $B^m(R)$ for all R (as e.g. the m-torus does), then, in the case $m = n-1$, the neighbourhood $U_r(X)$ receives expanding maps from $B^m(R) \times B^1\left(\frac{1}{\sqrt{2}}r - \varepsilon\right)$ for all $R \to \infty$ and positive $\varepsilon \to 0$.

And if $m = n-2$, then $U_r(X)$ receives such maps from $B^{m+1}(R) \times B^1\left(\frac{r}{2\sqrt{2}} - \varepsilon\right)$.

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[23]If Y is, Riemannian flat, then the term $10/r$ is unneeded and if Y is almost flat one can do without it for multiples of h and I am not certain about examples where the term $10/r$ is truly needed.

Proof. The required map for $m = n-1$ and coorientable $X \subset Y$ is obtained with the obvious splitting $U_r(X) = X \times B^1(r)$ and the case $m = n-2$ follows by applying this to the hypersurface $Z = \partial U_{r/2}(X) \subset U_r(X)$, where, clearly, $foc.rad(Z) = \frac{1}{2} foc.rad(X) \geq \frac{r}{2}$, and where the case of a non-trivial normal bundle of $X \subset Y$ needs a little thinking about.

But when it comes to $m \leq n-3$ nothing of the kind seems to be true, where the apparent difficulty stems from the following phenomenon.

If $m, k \geq 2$, then the topologically trivial sphere bundle $V = \mathbb{R}^m \times S^k \to \mathbb{R}^m$ admits an orthogonal connection ∇ with an arbitrary small curvature such that all smooth sections $\phi : \mathbb{R}^n \to V$ satisfy.

$$\sup_{x \in \mathbb{R}^m} \|\nabla \phi(x)\| = \infty.$$

Despite this, our $U_r(X)$, still looks large for all m and large $r = foc.rad(X)$, but I don't know, how to make precise sense of largeness for these U_r.

Here is a specific question.

Let us regard $U = B^k(r) \times B^m(R)$ as (the total space of) a $B^k(r)$-bundle over the ball $B^m(R)$, let ∇ be a Euclidean connection in this bundle and g_∇ the corresponding Riemannin metric on U, that is the sum of the differential quadratic form induced by the map $U = B^k(t) \times B^m(R) \to B^m(R)$ with the Euclidean metrics in the fibers $B^k_x(r) \subset U$, $x \in B^m(R)$ extended to $T(U)$ by zero on the ∇-horizontal vectors.

For which r, R and $\underline{R}$ the manifolds (U, g_∇) admit *no* expanding maps $(U, g_\nabla) \to B^{m+k}(\underline{R})$ for all connections ∇?

Conversely, from what kind of manifolds do (U, g_∇) receive expanding maps.?

5.5 Elementary Lower Bounds on Curvature and upper Bounds on Expansion

5.5.A. Small Curvature Observation. *Immersed compact manifolds* $X^m \hookrightarrow S^N(1)$, $m \geq 2m$ *with* $curv(X^m) \leq \delta$ *as well as* $X^m \hookrightarrow B^N(1)$ *with* $curv(X^m) \leq 1 + \delta$ *for a small* $\delta > 0$ *keep close to an equatorial m-sphere in* $S^m \subset S^{N-1} = \partial B^N(1)$*; thus, they are diffeomorphic to* S^m. In fact, it is not hard to show, that

$\delta = 0.01$, *is small enough for this purpose,*

The expectation of a significantly greater δ for $N = m+1$, is confirmed only in the spherical case by *Jian Ge Rigidity theorem.*

5.5.B. *If* $min.curv(X^m \hookrightarrow S^{m+1}) \leq 1$, *then* X^m *is homeomorphic to* S^m.

In fact, if $m = 2$ this follows from Gauss' theorema egregium + Gauss-Bonnet, while the proof of two slightly different generalizations/modifications of this theorem are given in

See [Ge 2021] and in section 3.7.3 in [Gr2021].

5.5.C. *Conjecture*. The best candidates for the smallest curvature immersions of *non-spherical* manifolds to spheres and to balls are, as we mentioned in section 1, (isometric) *Veronese embeddings* of the real projective spaces,

$$\mathbb{R}P^m \hookrightarrow B^{\frac{m(m+3)}{2}}(1),$$

which, actually, land in the boundary of this ball, where

$$curv(\mathbb{R}P^m \hookrightarrow S^{\frac{m(m+3)}{2}-1}(1)) = \sqrt{\frac{m-1}{m+1}} < 1,$$

(section 5.1). For instance the real projective plane embeds to the unit 4-sphere with curvature $\sqrt{\frac{1}{3}}$. and to the unit 5-ball $B^5(1)$ with curvature $2\sqrt{\frac{1}{3}}$.

****

Let $f : X \to Y$ be an expanding map between compact manifolds with boundaries

The simplest invariant which is monotone increasing under f is *the inradius* of X,

$$inrad(Y) \geq inrad(X),$$

where $inrad(X) = \sup_{x\in X} dist(x, \partial X0.$

In fact,

$$dist(x, \partial X) \leq dist(f(x), \partial Y) \text{ for all } x \in X,$$

since (some connected component of) the f pullback of a curve from $f(x)$ to ∂Y in Y connects x with partial X.

This also shows that if

$$dist(x_1, x_2) \leq dist(x_1, \partial X),$$

then

$$dist(f(x_1), f(x_2)) \geq dist(x_1), x_2),$$

that is f is distance increasing on all balls in the interior of X.

Example of a Corollary. Let $X \subset \mathbb{R}^n$ be the convex hull of two balls of radii r_1 and $r_2 \leq r_1$ in $\mathbb{R}^n$, such that distance d between their centers is $\geq r_1$ and let $f : X \to \mathbb{R}^n$ be an expanding map. Then

$$diam(f(X)) \geq 2r_1 + \frac{r_1(d+r_2)}{r_1+d+r_2} + \frac{r_2 r_1}{r_1+d+r_2} = \frac{r_1(d+2r_2)}{r_1+d+r_2}$$

In fact, the image of f contains the union of an r_1-ball $B_1 = B(r_1) \subset R^n$ and an r-ball $B_2 = B(r)$ for $r = \frac{r_1(d+2r_2)}{r_1+d+r_2}$, with the center of B_2 in the boundary ∂B_1.

Question. Do expanding self-maps $X \to X$ of compact manifolds with boundaries send $\partial X \to \partial X$?

5.6 More Questions

A. How large can be the ratio

$$min.curv(X^m \hookrightarrow B^M(1))/min.curv(X^m \hookrightarrow B^{M+1}(1))$$

provided X immerses to the Euclidean space $\mathbb{R}^M$, e.g. for $M \geq 2m-1$?

Is this ratio bounded by a universal constant, say by $const \leq 100$?

B. What is the *homological Morse spectrum* of the function $\mathcal{M} : f \mapsto curv_f(X)$ on the space of immersions $f : X \to Y$?

(An $r \in \mathbb{R}_+$ is in the Morse spectrum of a function $\mathcal{M} : \mathcal{F} \to \mathbb{R}_+$ if there exists a homology class $h \in H_*(\mathcal{F}; A)$ with some coefficient group A, such that the r-sublevel of $\mathcal{M}$ contains h, i.e. h is contained in the image of the inclusion homomorphism

$$H_*(\mathcal{M}^{-1}[0,r]; A) \to H_*(X; A)$$

while the lower sublevels $\mathcal{M}^{-1}[0,p] \subset \mathcal{M}^{-1}[0,r]$, $p < r$, don't contain h. See [Gr1988], [Gr2017] for more about it.)

C. Are there meaningful lower bounds on the averages of powers of the curvatures of immersions,

$$\frac{1}{vol_X} \int_X (curv_x(X \hookrightarrow Y)^p dx \text{ for } p \geq 1?$$

(Some inequalities may follow from the bounds on the Yamabe invariant for 4-manifolds, see [LB2021] and references therein).

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