Immersions, Curvatures and Spherical Designs

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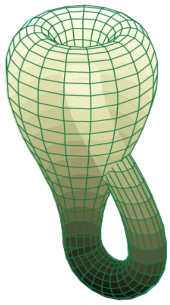

Immersions. A C^1-map $f : X \to Y$ between smooth manifolds is an *immersion* if the differential of f nowhere vanishes,

$$df(\tau) = 0 \implies \tau = 0.$$

The curvature of an immersed X in a Riemannian Y is *the supremum of Y-curvatures* , of geodesics $\gamma \subset X$, for the induced Riemannin metric in X,

$$curv(f(X)) = curv_f(X) = curv(X \overset{f}{\hookrightarrow} Y) = curv(X \hookrightarrow Y),$$

Minimal Curvature Problems. What is the infimum of curvatures of immersions $f : X \to Y$,

$$min.curv(X \hookrightarrow Y),$$

e.g. where Y is a unit ball $B^N = B^N(1) \subset \mathbb{R}^N$ and X is a product of spheres $X = S^{m_1} \times .... \times S^{m_l}$.

Conjectures.

A. *All $X^m \subset \mathbb{R}^n$:*

$$min.curv(X^m \hookrightarrow B^n) < 100.$$

B. *For all C, almost all X^m:*

$$min.curv(X^m \hookrightarrow B^n) > C.$$

C. *???*

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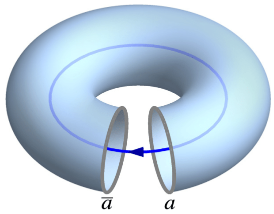
Standard torus:

$$min.curv(\mathbb{T}^2 \hookrightarrow B^3) \overset{=?}{\leq} 3$$

Clifford Torus.

$$(\mathbb{T}^1 \subset B^2(1)^m \subset B^{2m}(\sqrt{m}) \implies$$



$$min.curv\left(\mathbb{T}^m \hookrightarrow B^{2m}(1)\right) \overset{=?}{\leq} \sqrt{m}$$

$$min.curv(\mathbb{T}^3 \hookrightarrow B^4) \leq 2\sqrt{2} + 1 < 4$$

$$min.curv(\mathbb{T}^7 \hookrightarrow B^8) \leq 8 + 2\sqrt{2} + 1 < 12$$

....

$$min.curv(,,,,) < m^{\frac{3}{2}}, \ m = 2^k - 1.$$

$$min.curv(X^m = (S^k)^{k+2} \hookrightarrow B^{m+1}) \overset{=?}{\leq} 2\sqrt{k+1} + 1 = \sqrt[4]{m+1} + 1.$$

$$min.curv(X^2 \hookrightarrow B^3) < 50.$$

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Conjecture. The inequality $curv(X^m \hookrightarrow B^N(1)) < \sqrt{\frac{2m}{m+1}}$ must imply that X is diffeomorphic to the sphere S^m.

Veronese embeddings of the real projective spaces,

$$\mathbb{R}P^m \hookrightarrow B^{\frac{m(m+3)}{2}}(1),$$

(lands in the boundary of this ball):

$$curv(\mathbb{R}P^m \hookrightarrow B^{\frac{m(m+3)}{2}}(1)(1)) = \sqrt{\frac{2m}{m+1}}$$

.

For instance the real projective plane embeds to the unit 4-sphere with curvature $\sqrt{\frac{1}{3}}$. and to the unit 5-ball $B^5(1)$ with curvature $2\sqrt{\frac{1}{3}}$.

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Theorem 1.

$$min.curv(\mathbb{T}^{n-k} \hookrightarrow B^n) \geq \frac{2n}{k\pi} \cdot \sqrt{\frac{\lambda_1(B^n))}{n(n-1)}} - 1, \ k = 1, 2, ???.$$

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(Cecchini-Zeidler, Wang-Xie-Yu)

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$$min.curv(\mathbb{T}^2 \hookrightarrow B^3) \geq \frac{6}{\pi} \cdot \sqrt{\frac{\pi^2}{6}} - 1 = \sqrt{6} - 1 \approx 1.449. \ (\lambda_1(B^3) = \pi^2)$$

$$3 \geq min.curv(\mathbb{T}^2 \hookrightarrow B^3) \geq 1.449.$$

$\lambda_1(B^1) = \frac{\pi}{2}$, $\lambda_1(B^2) > 5.78$, $\lambda_1(B^3) = \pi^2$

$$2 + n + 2^{\frac{2}{3}} a \nu^{\frac{1}{3}} < \sqrt{\lambda_1(B^N)} < 2 + n + a 2^{\frac{2}{3}} \nu^{\frac{1}{3}} + \frac{3}{20} \frac{2^{\frac{5}{3}} a^2}{\nu^{\frac{1}{2}}}$$

where $\nu = \frac{n}{2} - 1$, $a = \left(\frac{9\pi}{8}\right)^{\frac{2}{3}} (1 + \varepsilon) \approx 2.32$ with $\varepsilon < 0.13 \left(\frac{8}{8.847\pi}\right)^2$,

$$min.curv(\mathbb{T}^7 \hookrightarrow B^8(1)) > 3.$$

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(Provisional) **Theorem 2.** All $X^m \hookrightarrow \mathbb{R}^n$:

$$min.curv(X^m \hookrightarrow B^n) < 500 m^{\frac{3}{2}} (/(n-m)?).$$

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$$\lim_{N \to \infty} min.curv(X^m, B^N) \leq \sqrt{\frac{3m}{m+2}}$$

$$min.curv(\mathbb{T}^m, B^N) \leq \sqrt{\frac{3m}{m+2}} \text{ for } N \geq m^{m^4}$$

.

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Codim =11 Example. Let

$$X = S^k \times \underbrace{S^1 \times ... \times S^1}_{l-1}.$$

If $k \geq l^{l^4}$, then there exists an immersion

$$F : X \hookrightarrow B^{k+l}(1)$$

with

$$curv_F(X) \leq 1 + 2\sqrt{\frac{3l-3}{l+1}} < 4.5.$$

$$min.curv(X, B^N) = 1 + 2\sqrt{\frac{3l-3}{l+1}}?$$

(Beweis fiir die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches Problem)Hilbert(1909).

(Averaging set. A generalization of mean values and spherical designs, Seymour-Zaslavsky(1984).)

Given finitely many continuous functions f_i, $i \in I$, on the unit interval, there exists a *finite* subset $S \subset [0,1]$, such that

$$\frac{1}{card(S)} \sum_{s \in S} f_i(s) = \int_0^1 f_i(t)dt \rightsquigarrow$$

$$\rightsquigarrow \sqrt{\frac{3m}{m+2}}.$$

δ-Approximation. Let $X = X^m$ be a smooth manifold and $f : X \to \mathbb{R}^N$ a continuous map.

$\bullet^{\geq}$ If $N \geq 2m - 1$ then f can be δ-approximated by smooth immersions

$$f_\delta : X \hookrightarrow \mathbb{R}^N, \delta > 0,$$

regularly homotopic to f and with curvatures

$$curv_{f_\delta}(X) \leq \frac{1}{\delta}\left(\sqrt{\frac{6m-2}{2m+1}} + C_o \frac{m}{\sqrt{N}}\right) + o\left(\frac{1}{\delta}\right), \ \delta \to 0,$$

where "δ-approximated" means that

$$dist_{\mathbb{R}^N}(f_\delta(x), f_0(x)) \leq \delta, \ x \in X.$$

$\bullet^{\leq}$ If X admits an immersion to $\mathbb{R}^n$, $n < N$, and $N \leq 2m$, then f can be δ-approximated by smooth immersions

$$f_\delta : X \hookrightarrow \mathbb{R}^N, \delta > 0,$$

with curvatures

$$curv_{f_\delta}(X) \le \frac{1}{\delta}\frac{6n^{\frac{3}{2}}}{N-n} + o\left(\frac{1}{\delta}\right).$$

Proof. Let $\phi : X \to \mathbb{R}^n$ be a smooth immersion [1] and observe the following.

(**II**) Besides, the above (I) our argument doesn't apply to immersions to $\mathbb{R}^n$ withut passing to $\mathbb{R}^{n+1}$ but this is taken care of by the following (see section???).

1.1.E. Regular Homotopy/Approximation Theorem. Let $f : X = X^m \to \mathbb{R}^n$ be an immersion. If $n > m$, then f can be δ-approximated by immersions $f_\delta X \hookrightarrow \mathbb{R}^n$ which are regularly homotopic to f and such that

$$curv_{f_\delta}(X) \le \frac{500}{\delta} m^{\frac{3}{2}} + o\left(\frac{1}{\delta}\right).$$

1.H. Remarks/Questions. We don't know how close this inequality to the minimal values of the curvatures of codim1 immersions of products of spheres is.

(a) For instance let P^{l-1} be an $(l-1)$-dimensional manifold diffeomorphic to a product of spheres where some of these have dimensions ≥ 2. Then, if $k >> l$, there exist immersions

$$F_\varepsilon : S^k \times P^{l-1} \hookrightarrow B^{k+l}(1)$$

with

$$curv_{F_\varepsilon}(S^k \times P^{l-1}) \le 1 + 2\sqrt{\frac{3l-3}{l+1}} + \varepsilon$$

for all $\varepsilon > 0$.

But this is *unclear* for $\varepsilon = 0$, even for the product $S^1 \times S^k$, which embeds to the ball $B^{k+2}(1)$ with curvature 3 for all k and where we *don't know* if there are immersions of $S^1 \times S^{k+2}$ (or other closed non-spherical manifolds of dimension $k+1$) to the unit ball $B^{k+2}(1)$ with curvatures < 3.

(b) It is not impossible according to what we know, that m-dimensional products of spheres of dimensions ≥ 2 admit immersions to $B^{m+1}(1)$ with curvature $<$100.

But the best we can do (see section ???) are immersions with curvatures $\lesssim m^{\frac{4}{3}}$.

[1] All X^m immerse to $\mathbb{R}^{2m-1}$, if $m \ge 2$, by the Whitney theorem.