Isoperimetric Inequalities in Geometry, Analysis, Algebra and Probability Theory, Fall 2023

In The Process of rewriting and editing

Misha Gromov

December 14, 2023

Abstract

These notes are to accompany my lectures at the Courant Institute in the Fall 2023. Besides presenting basic theorems, we try to show several different proofs of most of them.

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1 Magnificent Seven

1.1 Classical Euclidean and non-Euclidean Isoperimetry

The volumes of all bounded domains $X \subset \mathbb{R}^n$ are bounded by "areas" their boundaries

$$[Is]_X \quad Vol_n(X) \leq C_n Vol_{n-1}(\partial X)^{n/n-1},$$
where the constant $C_n$ is such that the unit balls $B^n = B^n(1) \subset \mathbb{R}^n$ satisfy the equality
$$Vol_n(\partial B^n) = C_n Vol_{n-1}(S^{n-1})^{n/n-1},$$
where $S^{n-1} = \partial B^n$ is the unit sphere. For instance $C_2 = \frac{1}{4\sqrt{\pi}} = \frac{\pi}{2\sqrt{\pi}}$ and $C_3 = \frac{1}{6\sqrt{\pi}} = \frac{(4/3)\pi}{(4\pi)^{3/2}}$.

Furthermore, the equality $Vol_n(\partial X) \leq C_n Vol_{n-1}(\partial X)^{n/n-1}$ implies that $X$ is a round ball.

Thus

\text{Among all domains with a given volume, balls and only balls have minimal surface area.}

(This is obvious for $n = 2$ by calculus of variations: extremal $Y = \partial X$ are closed curves with constant curvature, hence, circles, where justification of regularity of extremal $Y$ is easy for $\dim(Y) = 1$.)

\subsection{1.1.1 Sharp Isoperimetric Inequalities in Spheres, Balls, Hyperbolic Spaces and Gaussian Spaces}

Besides the above, the sharp isoperimetric inequalities are known in all simply connected fully homogeneous Riemannian manifolds $X$, where "fully homogeneous" signify that all isometries between subsets $X_1 \leftrightarrow X_2$ between subsets $X_1, X_2 \subset X$ extends to isometries $X \leftrightarrow X$ and where the extremal hypersurfaces, i.e. hypersurfaces of given "areas" enclosing maximal volumes are metric spheres.

(The only fully homogeneous Riemannian, as well as finite dimensional geodesic, spaces besides the Euclidean ones are spheres, real projective spaces, and hyperbolic spaces, where the real projective spaces $S^n/\pm 1$ are non-simply connected.

According to [Vil 2023], the extremal hypersurfaces in $S^n/\pm 1$, are $S^k \times S^l/\pm 1$ for $O(k+1) \times O(l+1)$-equivariant (Clifford) embeddings. $S^k \times S^l \hookrightarrow S^n$, $k + l = n - 1$.\footnote{I have not studied yet the proof of this theorem.}

Also one knows (we prove this in section???) that the extremal hypersurfaces $S$ in the (say open) balls $B \subset X$ in the above (simply connected fully homogeneous Riemannian manifolds) $X$ are totally umbilical.

(These, in the present case, are intersections of fully homogeneous, hence complete, hypersurfaces $Y \subset X$ with $B$.)

For instance, hypersurfaces, which divide Euclidean $n$-balls $B$ in equal halves are intersection of $B$ with hyperplanes.

(This brings to one’s mind Bourgain’s Difficult Slicing Problem. \footnote{https://www.weizmann.ac.il/math/klartag/sites/math.klartag/files/uploads/bourgain_slicing_problem.pdf} Let $X \subset \mathbb{R}^n$ be convex body of unit volume. Does there exist a hyperplane $H \subset \mathbb{R}^n$, such that $vol_{n-1}(H \cap K) \geq \delta$ for some universal $\delta > 0$, say for $\delta = 0.1$?)

\subsection{1.1.?? Metric&Measure, $\mu$-Isoperimetry&$\mu$-Extremality}

Let us extend the range of isoperimetric phenomena to Riemannian manifolds $X$, e.g,
\[ X = \mathbb{R}^n \] with smooth non-Riemannian measures \( \mu(x) = \phi(x)dx \) on them, e.g. to measures \( \mu \) on \( \mathbb{R}^n \) with finite mass \( M = \mu(\mathbb{R}^n) \).

Here a cooriented hypersurface \( Y \subset \mathbb{R}^n \), which divide the space into halves with given masses, say \( M_- \) and \( M_+ \), \( M_1 + M_+ = M \), is called \( \mu \)-extremal if it minimizes the integral \( \int_Y \phi(y)dy \) and thus solves the \( \mu \)-isoperimetry problem.

In general, the solution to such a problem seems fairly complicated but for the Gaussian \( \mu \) it comes up with an unexpectedly neat solution.

1.1. Tsirelson-Sudakov-Borell Theorem. If

\[ \mu(x) = e^{-|x|^2} dx, \]

then the \( \mu \)-extremal hypersurfaces are affine hyperplanes.

This follows from the isoperimetric inequality for \( S^N \) for \( N \to \infty \), since the Gauss measure on \( \mathbb{R}^n \) is equal to the limit of the push-forwards of the normalized spherical measures on the spheres \( S^{N+n-1}(R) \), \( R = \sqrt{N+n} \), under the normal projections \( \mathbb{R}^{N+n} \to \mathbb{R}^n \) (see section ???).

1.2 Sobolev and Gagliardo–Nirenberg

Smooth functions \( f \) with compact support in \( \mathbb{R}^n \) satisfy

\[
\left( \int |f(x)|^{n-1} dx \right)^{n^{-1/2}} \leq C_n \int \|df(x)\|dx
\]

with the above constant \( C_n \).

In fact, as we shall see presently, if all compact domain \( V \) in a Riemannian \( n \)-manifold \( X \) satisfy

\[
\text{vol}(V) \leq C \text{vol}_{n-1} (\partial V)^{n-1} \Leftrightarrow
\]

for some constant \( C \), then the inequality

\[
\left( \int |f(x)|^{n-1} dx \right)^{n^{-1/2}} \leq C_n \int \|df(x)\|dx
\]

holds for the functions \( f \) with compact supports in \( X \) (Maz 1960).

1.3 Minkowski Concavity

Given subsets \( X, Y \subset L \) in a linear \( n \)-space \( L(= \mathbb{R}^n) \), the Minkowski sum is

\[ X + Y = \{x, y\} \subset L, x \in X, y \in Y, \]

that is the image of the product \( X \times Y \subset L \times L \) under the addition map

\[ L \times L \xrightarrow{\oplus} L, (l_1, l_2) \mapsto l_1 + l_2 \]

\textbf{Minkowski} \( \frac{1}{n} \)-\textbf{Inequality.} The volume of the Minkowski sum of arbitrary open subsets in \( \mathbb{R}^n \) satisfies

\[
\left[\frac{1}{n}\right]_M (\text{vol}(X+Y))^{1/n} \geq (\text{vol}(X))^{1/n} + (\text{vol}(Y))^{1/n}.
\]

\[
\frac{1}{n} \text{This, applied to powers of } |f| \text{ yields all Sobolev Inequalities } \sqrt{\int |f|^p} \leq \text{cons} \sqrt{\int \|df\|^q} \text{ for } p,q \geq 1 \text{ and } p \leq nq/(n-q).
\]

6
Equivalently,
\[
\left( \frac{\text{vol} \left( \frac{X+Y}{2} \right)}{2} \right)^{1/n} \geq \frac{1}{2} \left( \text{vol}(X)^{1/n} + \text{vol}(Y)^{1/n} \right),
\]
where \( X+Y \) geometrically is the set of the centers of the segments \([x, y] \subset \mathbb{R}^n \) for \( x \in X \) and \( y \in Y \).

If \( Y \) is an infinitesimal ball \( Y = B^n(o(1)) \) then \( \frac{1}{n} \) implies the isoperimetric inequality for \( X \).

1.4 Almgren’s Sharp Filling Inequality

Let \( Y \subset \mathbb{R}^N \) be a piecewise smooth \((n-1)\)-cycle e.g. a smoothly embedded sphere \( S^{n-1}(1) \subset \mathbb{R}^N \). Then \( Y \) bounds a piece-wise smooth \( n \)-chain \( X \subset \mathbb{R}^N \),
\[
\partial X = Y,
\]
which satisfy the above \([Is]_X\)
\[
\text{Vol}_n(\partial X) \leq C_n \text{Vol}_{n-1}(\partial X)^{n/n-1},
\]
where the equality holds only for flat round spheres \( S^n(R) \subset \mathbb{R}^n \subset \mathbb{R}^N \).

Moreover, if \( Y = f(S^{n-1}(1)) \), then the map \( f : S^{n-1}(1) \to \mathbb{R}^N \) extends to a smooth map \( F : B^n(1) \to \mathbb{R}^N \), such that
\[
\text{Vol}_n(F : B^n(1)) \leq C_n(\text{vol}_{n-1} f(S^{n-1}))^{n/n-1},
\]
where this \( F \) can be chosen to be a smooth embedding for \( 2N > 3n \).

1.5 Loomis-Whitney Inequality and Subadditivity of Entropy

Let \( X \subset \mathbb{R}^n \) be a measurable subset and let \( X_i \subset \mathbb{R}^{n-1} = \mathbb{R}^n_i = \{x_i = 1, \ldots, n\} \), be the normal projections of \( X \) to the hyperplanes \( \mathbb{R}^{n-1}_i = \{(x_1, \ldots, x_n) \} \subset \mathbb{R}^n \).

\[
\text{vol}_n(X) \leq \prod_{i=1}^{n} \text{vol}_{n-1}(X_i)^{1/n-1}.
\]

This, almost obviously, implies a non-sharp isoperimetric inequality, namely
\[
\text{vol}(X) \leq \frac{1}{(2n)^{n/n-1}} \text{vol}_{n-1}^{n/n-1}(\partial X)
\]
with equality for cubes \( X = [0,1]^n \).

In turn, \([\text{Loo-Whi}]\) almost obviously, follows from the Shannon \((\text{Boltzmann?})\) inequality:

\[\text{vol}(X) \leq \frac{1}{(2n)^{n/n-1}} \text{vol}_{n-1}^{n/n-1}(\partial X)\]

This makes sense for subsets in Riemannian manifolds, while the additive \([\cdot]^{1/n}_M\) generalises to subsets in Lie Groups.
Strong Subadditivity of Entropy.

Let \( \mu = \mu_{123} \) be a probability measure on \( \mathbb{R}^3 \) let \( \mu_1, \mu_2, \mu_3 \) be the push-forwards of \( \mu \) to the coordinate lines (coordinate marginals of \( \mu \)) and \( \mu_{12}, \mu_{13}, \mu_{23} \) be the push-forwards of \( \mu \) to the coordinate planes. (Thus, \( \mu_1 \) is the marginal of \( \mu_{12} \) as well as of \( \mu_{13} \), etc.)

Then Boltzmann’s entropies of these satisfy

\[
\text{ent}(\mu_{123}) \leq \text{ent}(\mu_{12}) + \text{ent}(\mu_{13}) - \text{ent}(\mu_1),
\]

where the entropy of a measure \( \mu = \mu(x) = \phi(x)dx \) for a positive function \( \phi(x) \geq 0 \), such that \( \int_X \phi(x)dx = 1 \) and such that \( \log \phi \) is summable on the support \( S = S(\phi), \left\{ \int_S |\log \phi(x)|dx < \infty \right\} \), are evaluated by the Boltzmann-Gibbs formula:

\[
\text{ent}(\mu) = -\int \phi(x) \log \phi(x)dx.
\]

1.6 Poincaré Inequality on the unit n-Sphere \( X = S^n(1) \).

If

\[
\int_X f(x)dx = 0,
\]

then

\[
\int_X f^2(x)dx \leq \frac{1}{n^2} \int_X \|df\|^2 dx.
\]

Equivalently, all smooth maps \( F : X \to \mathbb{R}^N \) satisfy

\[
\frac{n^2}{2\text{vol}(X)} \int_{X \times X} \| F(x_1) - F(x_2) \|^2 dx_1 dx_2 \leq \frac{1}{\text{Vol}(S^{n-1})} \int_{UT(X)} \| \partial_{\tau_x} F(x) \|^2 d\tau_x dx,
\]

where \( UT(X) \) is the unit tangent bundle of \( X \) and \( \partial_{\tau_x} = dF(\tau_x) \), \( \tau_x \in S^{n-1} = UT_x(X) \), is the derivative of \( F \) by the vector \( \tau_x \).

---

\(^5\)This can be taken for the definition of entropy for all measure spaces \( (X, dx) \), e.g. for \( X = \mathbb{R}^N \) (Boltzmann’s N-partical gas) and/or for Shannon’s finite or countable sets \( X = \{ x_i \} \) with atoms \( x_i \) of equal weights, where \( \text{ent}(\mu) = -\sum_{i} \log \mu(x_i) \), compare https://www.crmarsh.com/pdf/Charles_Marsh_Continuous_Entropy.pdf.
1.7 Selberg Theorem, Selberg $\frac{1}{2}$-Conjecture and Ramanujan Graphs.

Preparation to Selberg. There is a strictly decreasing sequence of subgroups of finite index in the free group on two generators,

\[ F_2 = \Gamma_1 \supset \Gamma_2 \supset ... \supset \Gamma_l \supset ..., \]

with the following property.

Let \( X \) be a compact connected Riemannian manifold, such that the fundamental group \( \pi_1(X) \) admits a homomorphism onto \( F_2 \), e.g. \( X \) is a smooth bounded planar domain with at least two holes, such as a small neighbourhood of the figure \( \infty \).

Then there exist compact Riemannian manifolds \( X_l \) and \( l \)-sheeted coverings \( X_l \to X \), where \( l \to \infty \) and such that all smooth functions \( f(x_l) \) on \( X_l \) with \( \int_{X_l} f(x_l) dx_l = 0 \) satisfy

\[ \int_{X_l} \|df(x_l)\|^2 dx_l \geq \text{const} \cdot \int_{X_l} f^2(x_l) dx_l, \]

where the constant \( \text{const} = \text{const}(X) > 0 \) doesn't depend on \( l \).

Equivalently maps \( F : X_l \to \mathbb{R}^N \) satisfy

\[ \frac{1}{\text{vol}(X_l)^2} \int_{X_l \times X_l} \|F(x_l) - F(x'_l)\|^2 dx_l dx'_l \leq \frac{2}{\text{const}} \]

In truth, the above are "coarse corollaries" of a particular instance of a precise form of such an inequality for a specific family of complete non-compact Riemann surfaces \( X_l \) with constant curvatures \(-1\) and with finite volumes proved by Selberg in 1965.

These Selberg’s \( X_l \) are the quotients,

\[ X_l = H^2/\Gamma_l, \]

where \( H^2 \) is the hyperbolic plane and \( \Gamma_l \) are subgroups in the group of \( (2 \times 2) \)-matrices \((a_{ij})\) with integer entries and determinants one,

\[ \Gamma_l \subset SL(2, \mathbb{Z}), \]

where \( SL(2, \mathbb{Z}) \) naturally acts on \( H^2 \) by isometries and where \( \Gamma_l \) consists of matrices congruent to upper triangular ones \( \mod l \), i.e. where the entry \( a_{12} = 0 \mod l \).\(^6\)

---

\(^6\)These are locally isometric maps with \( l \) pullbacks of all \( x \in X \). In particular \( \text{vol}(X_l) = d \cdot \text{vol}(X) \).

\(^7\)One’s experience in the classical PDE – (Hersch \( S^2\)-eigenvalue theorem, Rayleigh–Faber–Krahn Inequality...) points to the opposite:

\[ \frac{\int_{X_l \times X_l} \|F(x_l) - F(x'_l)\|^2 dx_l dx'_l}{\int_{X_l} \|df(x_l)\|^2 dx_l} \to 0 \]

for \( l \)-sheeted coverings of compact manifolds: the ground frequency of an oscillating membrane \( X \) must tend to zero for \( \text{size}(X) \to \infty \). This is true in the "real world", and, probably, true in mathematics under some reasonable assumptions on \( X \), but... this is not so in general.

\(^8\)Strictly speaking, these \( X_l \) are not quite coverings of \( X_1 = H^2/SL_2(\mathbb{Z}) \), since torsion elements in \( SL(2, \mathbb{Z}) \) do not act freely on the hyperbolic plane \( H^2 \), but this needs only a minor adjustment of our terminology.
Selberg proved that the above constant in his case, call it $\lambda = \text{const}(X_{Selb})$ is bounded from below by $3/16$ and conjectured that $\lambda \geq 1/4$; The best current bound is $\lambda \geq \frac{1}{4} \left( \frac{2}{\pi^2} \right)^2$.

Remark: Mathematics behind $\lambda$ is fundamentally different from what is seen in the other six famous "isoperimetric theorems".

The later essentially depend on similarities between the geometries of the spaces these theorems apply to with the geometries of the corresponding Euclidean models, where everything boils down to the inequality $|f(1) - f(0)| \leq \int_0^1 |f(t)| dt$ with occasional use of the $O(n)$-symmetries.

But the geometry of the spaces $X_l$ for $l \to \infty$ is maximally non-Euclidean,\(^\text{10}\) which, in fact, follow from the inequality

$$\liminf_{l \to \infty} \lambda(X_l) > 0.$$

## 2 Methods of Proofs

### 2.1 Coarea Inequality, Volumes of Cones, Divergence and Green’s Formula

#### 2.1.A. Coarea Equality

Let $X$ be a Riemannian manifold, e.g. the Euclidean $N$-space $X = \mathbb{R}^N$ let $X_0 \subset X$ be a subset, e.g. the origin $\{0\} \subset \mathbb{R}^N$ and let

$$d_0(x) = \text{dist}(x, X_0)$$

be the distance function to $X_0$ (e.g. $d_0(x) = \|x\|$ for $X_0 = \{0\} \subset \mathbb{R}^N$).

Let $V \subset X \setminus X_0$ be a measurable, e.g an open subset in the complement of $X_0$. Then the $(N-1)$-volumes, $N = \text{dim}(X)$, of the intersections $V_r = V \cap d_0^{-1}(r) \subset V$ of $V$ with the $r$-levels of the function $d_0$ satisfy:

$$\int_0^\infty \text{vol}_{N-1}(V_r) dr = \text{vol}_N(V).$$

#### 2.1.B. Coarea Inequality

Let $V \subset X$ be a smooth $n$-dimensional submanifold. Then the intersections $V_r = V \cap d_0^{-1}(r) \subset V$, which are smooth $(n-1)$-submanifolds for almost all $r$ by the Sard theorem, satisfy:

$$\int_0^\infty \text{vol}_{n-1}(V_r) dr \leq \text{vol}_n(V).$$

#### 2.1.C. Cone Inequality

Let $\text{Cone}_{x_0}(Y) \subset \mathbb{R}^N$ be the cone over a submanifold $Y = Y^{n-1} \subset \mathbb{R}^N$ from a point $x_0 \in \mathbb{R}^n$.

Then the $n$-volume of this cone is

$$\text{vol}_n(\text{Cone}_{x_0}(Y)) = \frac{1}{n} \int_Y \|x_0 - y\| \sin \angle(x_0 - y, T_y(Y)) dy \leq \frac{1}{n} \int_Y \|x_0 - y\| dy$$


\(^{11}\)See [https://www3.nd.edu/~lnicolae/Coarea.pdf]

\(^{12}\)This is seen clearly for intersections $V_r$ of a curve or surface $V \subset \mathbb{R}^3$ with (a family of) parallel planes $H_r \subset \mathbb{R}^3$. In fact, this inequality applies to Lipschitz maps $d : X \to \mathbb{R}$ between general metric spaces and measures $ds$ in $X$, see [https://en.wikipedia.org/wiki/Eilenberg%27s_inequality].
Let $Y$ mean of the the unit ball $2$. 2.2 Volumes of Radial Projections and Isoperimetry in $\mathbb{R}^n$.

Often amounts to particular specifications of this "indirectly".

Bounds on $\mathbb{R}^n$ often amounts to particular specifications of this "indirectly".

Therefore, the volume of the cone from some point $x_0$ is bounded by the volume and the diameter of $Y$ as follows

$$[\text{Cone}] \quad \text{vol}_n(\text{Cone}_o(Y)) \leq \frac{N}{n\sqrt{2(N+1)}} \text{diam}(Y) \text{vol}_{n-1}(Y),$$

since

there exists a ball $B_o(r) \subset \mathbb{R}^n$, $o \in \mathbb{R}^N$, of radius $\frac{\text{diam}(Y)}{\sqrt{2(N+1)}}$, which contains $Y$ by Jung’s theorem.

If $N = n$ and $Y \subset \mathbb{R}^n$ is a closed naturally (say, inward) cooriented hypersurface, then the angle between the vector $x - y \in \mathbb{R}^n$ and he tangent space $T_y(Y)$ comes with a $\tau$-sign and the volume of the domain $X \subset \mathbb{R}^n$ bounded by $Y$ is equal to the absolute value of the "signed volume" of the cone, i.e.

$$[\text{Cone}], \quad |\text{vol}(X)| = \frac{1}{n} \int_Y \| x_0 - y \| \sin \theta(x_0 - y, T_y(Y)) dy$$

This yields a non-sharp isoperimetric inequality for $n = 2$ by Jung’s theorem,

$$\text{area}(X) < \frac{1}{4\sqrt{3}} \text{length}(Y)^2.$$ (Jung’s theorem for closed curves gives you $\text{area}(X) \leq \frac{1}{2} \text{length}(Y)^2$.)

But since the diameters of (connected) hypersurfaces for $n \geq 3$ are not controlled by their $(n-1)$-volumes for $n \geq 3$, this only indirectly leads to non-trivial bounds on $\text{vol}(X)$ by $\text{vol}_{n-1}(\partial X)$ and actual proofs of isoperimetric inequalities often amounts to particular specifications of this "indirectly".

2.1.G Gauss-Green Formula. If $\tau$ is a vector field in $\mathbb{R}^n$ with divergence $n$, then

$$[\text{div} = n] \quad \text{vol}(X) = \frac{1}{n} \int_Y (\tau(y), \nu(y)) dy \leq \frac{1}{n} \int_Y \| \tau \| dy,$$

where $\nu$ is the unit normal vector field on the boundary $Y = \partial X$ and which reduces to the above $[\text{Cone}], \nu$ applied to $\text{grad} \| x_0 - x \|^2$, that is the gradient of the squared distance function $x \mapsto \| x_0 - x \|^2$.

2.2 Volumes of Radial Projections and Isoperimetry in Balls

Let $Y$ be a smooth. (or piecewise smooth) $(n-1)$-dimensional submanifold in the unit ball $Y = Y^{n-1} \subset B^N(1) \subset \mathbb{R}^N$.

Since $N > n - 1$, the integral $\int \frac{1}{21^n} dx$ converges at zero in $\mathbb{R}^N$ and the mean of the $\text{dist}^{n-1}$-function in the ball $B^N(r) \subset \mathbb{R}^N$ satisfies,

$$\frac{1}{\text{vol}_N(B^N(1))} \int_{B^N(1)} \| x - x_0 \|^{-(n-1)} dx \leq \text{const}_N^{14}$$

13 The sharp inequality for $n = 2$ is $\text{area}(X) \leq \frac{1}{2} \text{length}(Y)^2$.
14 Since the measures of balls concentrate near their boundaries, this $\text{const}_N \rightarrow 1$ for $N \rightarrow \infty$. 

11
for all $x_0 \in \mathbb{R}^N$.

It follows, that for all submanifolds $Y$ in $\mathbb{R}^N$, (not necessarily contained in particular balls) and all $r$-balls $B^N(r) \subset \mathbb{R}^N$, there exist $x \in B^N(r)$, such that

$$\int_Y \|y-x\|^{-(n-1)}dy \leq \frac{\text{const}_N \text{vol}_{n-1}(Y)}{r^N}.$$ 

Therefore, for all domains $V \subset \mathbb{R}^N$, which contain an $r$-ball $B^N(r)$,

$$B^N(r) \subset V \subset \mathbb{R}^N,$$

the cylinder of the radial projection, call it $\psi = \psi_x$, $x \in B^N(r)$, of a submanifold $Y = Y^{n-1} \subset B^N(r)$ from some point $x \in B^n(r)$ to the boundary $\partial V$ of $V$ satisfies:

$$\text{vol}_n(\text{cyl}_\psi) \leq \text{vol}_{n-1}(Y) \frac{\text{const}_N}{n} \left( \frac{\text{diam}(V)}{r} \right)^n.$$ 

2.1.D. Corollary: Smaller Half Inequality. Let $V \subset \mathbb{R}^N$ be a connected domain and let a hypersurface $Y = Y^{n-1} \subset V \subset \mathbb{R}^N$ divide $V$ in two subdomains, $V_1, V_2 \subset V$ with common boundary $V_1 \cap V_2 = Y$ in $V$. Then

$$\min(\text{vol}_N(V_1), \text{vol}_N(V_2)) \leq \text{vol}_{n-1}(Y) \frac{\text{const}_N}{n} \left( \frac{\text{diam}(V)}{\text{inrad}(V)} \right)^n.$$ 

2.1.D. Exercises. (a) Let $V \subset \mathbb{R}^N$ be a convex domain and $Y = Y^{n-1} \subset V$. Show that there exists a point $x \in V$.

(b) Let $V$ satisfy ???

2.2.1 "Involutive" Proof of Isoperimetry in Convex Sets and Cheeger Constant in Manifolds with $Ric \geq -\text{const}$

Cheeger Constant for Convex Sets. Let $Y, W_1, W_2 \subset \mathbb{R}^n$ be closed subsets of finite volumes, such that all straight segments $[w_{i,2}] \subset \mathbb{R}^n$, $w_i \in W_i$, $i=1,2$ intersect $Y$.

Then – this 99% obvious – one of the subsets, say $W_1$ contains a point $w_1$ and the second subset contains a subset $W'_2 \subset W_2$, such that

- $\text{vol}_n(W'_2) \geq \frac{1}{2} \text{vol}(W_2)$;
- each segment $[w_1, w'_2], w'_2 \in W'_2$, intersect $Y$ at a point $y \in Y$, such that $\text{dist}(w_1, y) \geq \text{dist}(y, w'_2)$.

Conclusion. Let $Y$ a smooth hypersurface divide a convex domain $W \subset \mathbb{R}^n$ into two parts $W_1$ and $W_2$. Then

$$\min(\text{vol}(W_1), \text{vol}(W_2)) \leq 2^n \text{vol}_{n-1}(Y) \times \text{diam}(W).$$

Sharp Needle Remark. The sharp inequality

$$\min(\text{vol}(W_1), \text{vol}(W_2)) \leq \text{vol}_{n-1}(Y) \times \text{diam}(W)$$

(this is sharp for all partitions of infinitesimally thin cylinders, $W = W_0 \times [0, d]$, where $W_0$ is infinitesimally small) is proven in section ??? by the (1960) Payne-Weinberger [PW1960] needle decomposition argument prove

2.1E. Cones, Mapping Cylinders and Volumes /Measures of non-Injective Maps ???

2.1.E. Geodesic Cones: Riemannian and non-Riemannian ???
2.3 Parallel Displacement of Volume and Isoperimetry

Given a bounded domain $V \subset \mathbb{R}^n$ with a smooth boundary, (or any Borel subset for this matter) it is geometrically obvious and is justified below that $V$ can’t be almost invariant under parallel translations by vectors $x \in \mathbb{R}^n$ with norm $\|x\| \leq d$ for $d \gg \text{vol}(V)^{1/n}$.

For instance if $d \geq R$, where $R = R_V$ is the radius of the ball $B^n(R = R_V) \subset \mathbb{R}^n$ such that $\text{vol}(B^n(R)) = 2\text{vol}(V)$, that is

$$R = \frac{(2\text{vol}(V))^{1/n}}{\text{vol}(B^n(1))^{1/n}},$$

then

$\star 1/2$ at least half of the volume of $V$ is transported out of $V$ by an $x$ with $\|x\| \leq R_V$.

This means that

$$\text{vol}(V \cap V + x) \leq \frac{1}{2} \text{vol}(V),$$

where $V + x = \{v + x\}$ is the $x$-translate of $V$.

Then clearly, since $\|x\| \leq R$,

$$\text{vol}(V_x) \leq R \cdot \text{vol}_{n-1}(\partial V) = \frac{(2\text{vol}(V))^{1/n} \cdot \text{vol}_{n-1}(\partial V)}{\text{vol}(B^n(1))^{1/n}}$$

and

$$\text{vol}(V) \leq 2\text{vol}(V_x) \leq \frac{2(2\text{vol}(V))^{1/n} \cdot \text{vol}_{n-1}(\partial V)}{\text{vol}(B^n(1))^{1/n}},$$

which can be rewritten as an isoperimetric inequality with a non-sharp constant, $[\text{Iso}_2]$.

$$\text{vol}(V) \leq \frac{2^{n+1/n-1} \cdot \text{vol}_{n-1}(\partial V)^{n/n-1}}{\text{vol}(B^n(1))^{1/n-1} \cdot \text{vol}_{n-1}(\partial V)}.$$

Proof of $\star 1/2$. Let $D \subset V \times B^n(2R) \subset \mathbb{R}^{2n}$ be the subset of the pairs $(v,x)$, such that $v + x \in V$ and let us evaluate the $2n$-volume of $D$ in two ways.

$$\text{vol}_{2n}(D) = \int_V \text{vol}_n(V \cap (B_v(2R)) \text{dv} \leq \text{vol}_n(V)^2,$$

where $B_v(2R) \subset \mathbb{R}^n$ is the $2R$-ball with center $v$. Thus,

$$\text{vol}_{2n}(D) = \int_{B^n(2R)} \text{vol}_n(V \cap (V + x) \text{dx} \leq \text{vol}(V)^2 \leq \frac{1}{2} \text{vol}(V) \times \text{vol}(B^n(R)),$$

since $\text{vol}(B^n(R)) = 2\text{vol}(V)$.

Therefore, there exits an $x \in B^n(R)$, such that

$$\text{vol}_n(V \cap (V + x) \leq \frac{1}{2} \text{vol}(V).$$

15 Compare with $???$
by the mean value theorem. QED. Reference to Minkowski (for small $R$) and to Saloff-Coste.

Similar proof for subdomains in a convex sets.

Euclidean Geometry and Descendants: Semi-Algebraic Integral Formulas, Calculus of variations, Rearrangements

Non-Euclidean Symmetry: Amenability, T-property and ...

Probability

Linear Algebra and Algebraic Geometry

2.4 Sharp Santalo’s Argument for $n = 2$ and non-Sharp for $n \geq 3$

Let $X \subset \mathbb{R}^2$ be a bounded planer domain with smooth boundary $Y = \partial X$ and proceed with the proof of the isoperimetric inequality

$$\text{area}(X) \leq \frac{\text{length}(Y)^2}{4\pi}$$

as follows.

A. Let $\phi(x,y)$ be the norm of the differential of the radial projection from $Y$ to the unit circle $S^1(1) \subset \mathbb{R}^2$.

Let $V \subset X \times Y$ be the set of pairs $(x,y)$, such that the segment $[x,y] \subset \mathbb{R}^2$ is contained in $X$ and let

$$V_x = V \cap \{x\} \times Y \subset V \quad \text{and} \quad V_y = V \cap X \times \{y\} \subset V.$$ 

Let $X'$ be the disk $B^2 = B^2(R(a))$ with

$$\text{area}(X' = X'(a)) = a = \text{area}(X).$$

Then, clearly,

\[ \int_{V_x} \phi(x,y) dy = 2\pi \quad \text{for all} \quad x \in X \]

and

\[ \int_{V_y} \phi(x,y) dx \leq \int_X \phi(x', y') dx' = c(v) \quad \text{for all} \quad y \in Y, \]

since the levels of the $x$-function $\phi_y(x) = \phi(x,y)$ are $r$-circles tangent to $Y = \partial X$ at $y$ and $\phi$ is monotone decreasing in $r$.

It follows that on the one hand

\[ \int_V \phi(x,y) dx dy = \int_X dx \int_{V_x} \phi(x,y) dy \geq 2\pi \cdot \text{area}(X); \]

and on the other hand

\[ \int_V \phi(x,y) dx dy = \int_V dy \int_{V_y} \phi(x,y) dx \leq \text{length}(Y)c(a) \]

Thus

$$\text{length}(Y)c(a) \geq 2\pi \cdot \text{area}(X)$$
and since this becomes the equality for \(X'\),

\[
c(v) = 2\pi \cdot \frac{\text{area}(X')}{\text{length}(Y')} = \sqrt{\pi a},
\]

this implies that

\[
\frac{\text{area}(X)}{\text{length}(X)^2} \leq \frac{\text{area}(X')}{{\text{length}(X')}^2}(= 1/4\pi).
\]

QED.

**Commentary.** Santalo’s argument is, logically, the most elementary among known proofs of the sharp 2d-isoperimetric inequality; besides, this proof gives an exemplary form of the deviation of the ratio \(\frac{\text{area}(X)}{\text{length}(\partial X)^2}\) from that for the ball \(X' = B^2 \subset \mathbb{R}^2\).

???: [area], [length]-Divergence and Greens Formula. There is a (unique) vector field \(\nu = \nu_y = \nu_y(x)\), for all \(y \in Y = \partial X\), normal to the levels of the function \(\phi_y(x) = \phi(x, y)\), such that

\[
\text{div}(\nu_y)(x) = \text{div}_X(\nu_y) = \phi_y(x) \quad \text{and} \quad \|\nu_y(x)\| = 1/2 \quad \text{for all } y \text{ and } x
\]

and the Green’s formula delivers an alternative proof of the key inequality [area] in the length form

\[
\int_{Y_y} \phi(x, y) \, dx \leq \text{length}(Y)/2.
\]

If \(\text{sect.curv} \neq 0\). The above generalizes to surfaces with non-zero, e.g constant < 0, curvatures, as follows

**Non-Sharp Santalo for all dimensions \(n\).** Let \(X \subset \mathbb{R}^n\) be a compact domain with a smooth boundary \(Y\) and let

\[
\psi(x, y) = \text{dist}(x, y)^{(n-1)}
\]

(instead of the above \(\phi(x, y)\)). Then, clearly,

\[
\int_Y \psi(x, y) \, dy \geq a_n = \text{vol}_{n-1}(S^{n-1}(1)), \quad x \in X,
\]

and

\[
\int_X \psi(x, y) \, dx \leq b_n \cdot r = \int_{B^n(r)} \|y\|^{-(n-1)} \, dx, \quad y \in Y = \partial X.
\]

where \(B^n = B^n_0(r(v)) \subset \mathbb{R}^n\) is the ball with volume \(v = \text{vol}_n(X)\).

Evaluate the integral of \(\psi\) over \(X \times Y\) (instead of \(V \subset X \times Y\)). as earlier,

\[
a_n \cdot \text{vol}_n(X) \leq \int_{X \times Y} \psi(x, y) \, dx \, dy \leq b_n \cdot r \cdot \text{vol}_n-1(Y).
\]

Write \(\text{vol}_n(X) = \text{vol}_n(B^n(r)) = \beta_n r^n\) and conclude to the inequality

\[
\text{vol}_n(X)/r = \text{vol}_n(X)^{n-1/n} \leq C_n \text{vol}_n-1(Y), \quad \text{for } C_n = a_n^{-1} b_n / \beta_n^{1/n}.
\]

???: On Divergence and Green’s Formula
2.5 Santalo Formula and Croke Inequality

A point $x$ in a Riemannian manifold $X$, e.g. $x \in X = \mathbb{R}^n$, is ray-surrounded by a subset $Y \subset \mathbb{R}^n$ if all geodesics in $X$ issuing from $x$ intersect $Y$.

Let $\overrightarrow{[Y]} \subset X$ be the set of all $x$ surrounded by $Y$.

Observe that $\overrightarrow{\partial V} \supset V$ for all bounded domains $V \subset \mathbb{R}^n$.

**Theorem.** Let $X$ be a complete simply connected Riemannian $n$-manifold with $\text{sect.conv}(X) \leq 0$, e.g. $X = \mathbb{R}^n$, and let $Y = Y^{n-1} \subset X$ be a smooth submanifold. Then

$$\left(\text{vol}\left(\overrightarrow{[Y]}\right)\right)^{n-1} \leq \text{const}_n(\text{vol}_{n-1}(Y))^n$$

for where

$$\text{const}_n = \frac{\text{vol}_{n-2}(S^{n-2})^{n-2} \cdot \left(\int_0^{\pi/2} \cos(t)^{n-2} \sin(t)^{n-2} dt\right)^{n-2}}{\text{vol}_{n-1}(S^{n-1})^{n-1}}$$

**Proof.**

**Sharp 4d-Isoperimetric Corollary.** Bounded domains $V$ in complete simply connected Riemannian 4-manifolds $X$ with $\text{sect.conv}(X) \leq 0$ satisfy:

$$\text{vol}_4(V) \leq \left(\text{const}_4(\text{vol}_{n-1}(\partial V))\right)^{4/3}$$

for

$$\text{const}_4 = \frac{\text{vol}(B^4)^3}{\text{vol}(S^3)^4} = \frac{\text{vol}_2(S^2)^2 \cdot \left(\int_0^{\pi/2} \cos(t)^2 \sin(t)^2 dt\right)^2}{\text{vol}_{n-1}(S^{n-1})^3}.$$ 

******************************************************************

Bound on $\text{vol}_{m_1+m_2+1}$ of the set

$$\overrightarrow{[Y_1,Y_2]} \subset UT(\mathbb{R}^N) = \mathbb{R}^N \times S^{N-1}(1)$$

of the unit tangent vectors to the segments $[y_1,y_2] \subset \mathbb{R}^N$ joining points $y_1 \in Y_1$, and $y_2 \in Y_2$, $Y_1 = Y_1^{m_1}$, $Y_2 = Y_2^{m_2}$ by

$$\text{const}_N \text{vol}_{m_1}(Y_1) \cdot \text{vol}_{m_2}(Y_2).$$

Measure of flags of affine $(k,k-1)$-subspaces $A^k_1$, $A_k - 1$) in $\mathbb{R}^{n+k}$, such that both half-spaces $A_k \subset A^k$ intersect $Y = Y^{n-1} \subset \mathbb{R}^{n+k}$,

(Alternatively, where $A^{k-1}$ is linked with $Y$.)

2.6 Steiner Symmetrizations and Isoperimetry in Balls with Constant Curvature

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A. Mirror Symmetry by Parallel Rearrangement

Let $V \subset \mathbb{R}^n$ be a bounded Borel (e.g. closed or open) subset and let $H \subset \mathbb{R}^n$ be a hyperplane. Then there exists a unique subset

$$V_H = \text{symm}_H(V) \subset \mathbb{R}^n,$$

such that

- the intersections of $V_H$ with all lines $L \subset \mathbb{R}^n$ normal to $H$ are closed segments, the Lebesgue measures (lengths for segments) of which are equal to those of the intersections $L \cap V$.

- $V_H$ is symmetric under the reflection in $H$, that is all segments $L \cap V_H$ intersect $H$ at their middle points.

B. Central Symmetrization.

If we symmetrize $V$ successively in the mutually orthogonal linear hyperplanes, then the resulting subset, call it $V_{\text{uni}}$, will be centrally symmetric with respect to the origin $0 \in \mathbb{R}^n$ and this central symmetry will be preserved by all further $H$-symmetrizations $\text{symm}_H$.

C. Trivial Observation and Useful Corollary.

Let $V \subset \mathbb{R}^n$ be a bounded centrally symmetric domain. Then $H$-symmetrizations increase the volumes of the intersections of symmetrized sets $V_H = \text{symm}_H(V)$ with balls,

$$\text{vol}(V \cap B^n_0(r)) \geq \text{vol}(V \cap B^n_0(r)), \ r \geq 0,$$

for all (linear) hyperplanes $H \subset \mathbb{R}^n$.

Moreover, if $\varepsilon$-much of additional measure of $V$ could have been put to $B^n_0(r)$, that is if

$$\text{min}(\text{vol}(V \setminus B^n_0(r)), \text{vol}(B^n_0(r)), \text{vol}(V)) \geq \varepsilon > 0,$$

then there exists an $H$ such that the above inequality is controllably strict;

$$\text{vol}(V_1 \cap B^n_0(r)) \geq \text{vol}(V \cap B^n_0(r)), \ r \geq \delta > 0,$$

where this strictly positive $\delta$ depends (only) on $\varepsilon > 0$ as well as on $R$ and $\text{vol}(V)$.

Corollary. If $\text{Vol}(V) = \text{Vol}(B^n_0(R))$, then there exists a sequence of hyperplanes $H_i$, such that the sequence $V_i$ of $H_i$-successive symmetrizations of $V$,

$$V_i = \text{symm}_{H_i}(V_{i-1})$$

volume-wise converges to $B^n_0(R)$, that is

$$\text{vol}(B^n_0(R)) \setminus V_i \to 0 \text{ for } i \to \infty.$$

Now comes the basic step in Steiner’s (attributed to Steiner?) symmetrization proof of the isoperimetric inequality.

B. Area Decrease of the Boundary.

The boundaries of symmetrized of polyhedral domains$^{18}$ satisfy

$$\text{vol}_{n-1}(\partial(V_H)) \leq \text{vol}_{n-1}(\partial(V)).$$

In fact this inequality trivially reduces to the special case, where $V$ is a trapezoid in the plane.

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$^{17}$https://www.math.utah.edu/~treiberg/Steiner/SteinerSlides.pdf

$^{18}$???
A. Symmetric Trapezoid Lemma. Among all trapezoid \( T \subset \mathbb{R}^2 \) with a given height and the lengths of the two parallel bases, an isosceles \( T \) has the minimal sum of the lengths of its two side legs because the differential of the distance function \( \phi_0 : x \mapsto \text{dist}(x, x_0) \) depends only on the direction (but not the length length) of the segment \([x, x_0] \), namely

\[
d\phi_0(\tau) = \frac{\langle \tau, x-x_0 \rangle}{\|x-x_0\|},
\]

the horizontal derivative of the sum of the lengths of the legs of \( T \) vanishes if and only if \( T \) is isosceles.

In fact, the derivative of the sum of distances \( \text{dist}(x, x_1) + \text{dist}(x, x_2) \) in an arbitrary Riemannian manifold \( X \), where \( x \) runs along a geodesic \( L \subset X \) vanishes iff the angles \( \alpha_1 \) and \( \alpha_2 \) of minimizing segments \([x, x_1] \) and \([x, x_2] \) with \( L \ni x \) at \( x \) are \( \pi \)-complementary i.e. \( \alpha_1 + \alpha_2 = \pi \).

However simple, this is sole of Steiner's symmetrization, which, together with the cone inequality 2.1.C yield the following.

B. Isoperimetric Conclusion. All bounded polyhedral domains in \( \mathbb{R}^n \) satisfy

\[
\text{vol}(V)^{n-1}/\text{vol}(\partial V)^n < \text{vol}(B^n(1))^{n-1}/\text{vol}(S^{n-1}(1))^n.
\]

Complexity Remark. The statement and the above proof of this inequality for \( n = 3 \) is limited to the high school level of geometry and if you comfortable with calculus and rudimentary differential geometry, this trivially generalizes to domains \( V \) with almost everywhere smooth boundaries in all complete simply connected spaces with constant sectional curvatures (i.e. inspheres and hyperbolic spaces).

Strangely however, this (19th century) argument was considered incomplete and difficult by the early 20th century mathematicians, e.g. Blaske (above is from his book "Kreis und Kugel", 1916) and Loomis and Whitney 1949.

Correction Term in Symmetrization and Isoperimetric Stability of Balls.

2.7 Formal Schwarz Symmetrization

The Geometric Schwarz Symmetrization transforms domains \( V \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \) to \( V_\circ \subset \mathbb{R}^n \), such that


V_0 are invariant under the orthogonal group O(n - 1) of rotation around the 0 × R axes

and

the intersections of V_0 with the hyperplanes parallel to ℝ^{n-1} ⊂ ℝ^n, say H_t = ℝ^{n-1} × \{t\} are balls, say B^{n-1}(R_t) = H_t ∩ V_0 with the (n - 1)-volumes equal to these of the intersections V ∩ H_t.

vol_{n-1}(B^{n-1}(R_t)) = vol_{n-1}(V ∩ H_t).

The isoperimetric inequality in H_t, along with the trapezoid lemma ???? and an obvious integral (same Schwartz?) inequality show that

vol_{n-1}(∂V_0) ≤ vol_{n-1}(∂V).

This, technically speaking, seems as a trivial generalization of the area decrease property for the Steiner symmetrization, but it has a much wider range of application when combined with the 1-dimensional calculus of variation.

In fact, a single Schwarz symmetrization reduces the general isoperimetric problem for V ⊂ ℝ^n to that for O(n - 1)-invariant domains, that are obtained by rotating a domain V_● in the plane P = ℝ^2 ⊃ ℝ^2 ⊃ ℝ^n which contains the 0 × ℝ axes.

Thus, the (first) variation extremality condition for the isoperimetric problem, that is the constancy of the mean curvature of the hypersurface ∂V_{extr}, translates to a certain second order ordinary differential equation for the boundary of V_● ⊂ ℝ^n.

Then (by the same argument Newton shown that the elliptic orbits are the only solutions of the second law with the inverse quadratic attraction):

among all O(n - 1)-invariant closed Euclidean hypersurfaces S with given (n - 1)-volumes, spheres maximize the volumes of domains V ⊂ ℝ^n bounded by S.

This implies, by Schwartz, similar extremality of spheres among all domains S in the Euclidean space ℝ^n.

Isoperimetry in Euclidean and Non-Euclidean Balls. The Schwartz symmetrization equally applies to intersections of V with concentric spheres H_t = S^{n-1}(t) ⊂ ℝ^n, thus deriving the sharp isoperimetric inequality in ℝ^n from those in the (n - 1)-spheres (rather than in ℝ^{n-1}).

In fact, the spherical Schwartz symmetrization effortlessly extends to all complete simply connected spaces X_κ with constant sectional curvatures κ, i.e. spheres and hyperbolic spaces; this yields

the isoperimetric extremality of balls in all X_κ.

Moreover, this applies to bounded subdomains V inside (possibly concave and unbounded) domains D = D_λ ⊂ X_κ bounded by complete connected umbilic hypersurfaces Y = Y_λ ⊂ X_κ.  

\footnote{These Y ⊂ X_κ are
\bullet all κ-spheres and (totally geodesic) hyperplanes (which are spherical for λ = κ > 0);
\bullet κ < 0 horospheres and equidistants to hyperplanes.

In all cases, the isometry group of X_κ, which preserves Y, is (bi)transitive on Y, where the induced Riemannian metric g_λ on Y = Y_λ has constant curvature λ ≥ κ. The only bounded domains D_λ with ∂D_λ = Y_λ are balls bounded by spheres, (where λ > 0) and where unbounded D include complements to balls, for instance.}
among all bounded domains \( V \subset D \), the ones with minimal \( \text{vol}_{n-1}(\partial V) \) (here \( \partial V \) denotes the topological boundary of \( V \) in the ball) are those, where their boundaries \( \partial V \subset B^n \) are umbilic hypersurfaces normal to \( Y = \partial D \).

Remark. If \( D = D_\lambda \) is concave, then the boundaries of the extremal \( V \) have sectional curvatures \( \mu \geq \lambda \) and if \( D_\lambda \) is the complement to a ball or to a horoball (when \( \kappa < 0 \)), then the extremal \( V \subset D \) are balls, the boundaries of which don’t have to intersect \( \partial D_\lambda \).

Formal Schwartz. The above symmetrization argument essentially depends on the trapezoid and Schwartz inequalities but it doesn’t truly need any symmetry as it applies, for instance, to domains \( V \) in cylindrically split Riemannian manifolds,

\[ V \subset X = X_0 \times \mathbb{R} , \]

where it leads to a lower bound on \( \text{vol}_{n-1}(\partial V) \) in terms of \( \text{vol}_n(V) \) provided the "corresponding" bounds hold for domains

\[ W_t = V \cap X_t \subset X_t = X \times \{t\} = X_0, \quad t \in \mathbb{R} \]

and where "corresponding" means "some bounds" on \( \text{vol}_{n-1}(W_t) \) in terms of \( \text{vol}_{n-2}(\partial W_t) \).

For instance, if all \( W \subset X_0 \) satisfy the \((n-1)\)-dimensional Euclidean isoperimetric inequality,

\[ \text{vol}_{n-1}(W)/\text{vol}_{n-2}(\partial W)^{n-1/n-2} \leq \text{vol}_{n-1}(B^n)/\text{vol}_{n-2}(S^{n-2})^{n-1/n-2} , \]

then all \( V \subset X_0 \times \mathbb{R} \) satisfy the \( n \)-dimensional Euclidean isoperimetric inequality,

\[ \text{vol}_n(V)/\text{vol}_{n-1}(\partial V)^{n/n-1} \leq \text{vol}_n(B^n)/\text{vol}_{n-1}(S^{n-1})^{n/n-1} . \]

(Here \( B^n = B^n(1) \subset \mathbb{R}^n \) is the unit ball and \( S^{n-1} = \partial B^n \) is the unit sphere.)

Furthermore, a simple elaboration of this argument shows that if \( X = X_1 \times X_2 \), where \( X_i = X_i^n, \ i = 1,2, \) are Riemannian manifolds, such that

- all \( V_i \subset X_i, \ i = 1,2, \) satisfy the \( n_i \)-dimensional Euclidean isoperimetric inequalities, then
- all \( V \subset X \) satisfy the \((n_1 + n_2)\)-dimensional Euclidean isoperimetric inequality.\(^{21}\)

2.8 "Isoperimetric" Proof of Sobolev’s bounds on the integrals \( \int \| f \|^p \) by \( \int \| df \|^q \)

Let \( X \) be a Riemannian \( n \)-manifold without a boundary and \( f(x) \geq 0 \) be a Lipschitz function on \( X \) with a compact support. Let \( B(t) = f^{-1}(t) \subset X \) be the \( t \)-levels of \( f \) and let \( A(t) = f^{-1}(t, \infty) \subset X \) be the compact domains in \( X \) bounded by \( B(t) \).

2.6.A. Maz’ya-Cheeger Conditional Inequality. If

\[ \nu \leq c(\nu) \] for some \( \nu \geq 1 \) and all \( t \geq 0 \),

(e.g. \( \nu = n/n - 1 \) as in the Euclidean isoperimetric inequality) then

\[ \| f \|_{L_p} = \sqrt[p]{\int f(x)^p dx} \leq \frac{p}{\nu} \| f \|_{L_q} \leq \frac{p}{\nu} \sqrt[q]{\int \| df(x) \|^q dx} \]

for all \( p \geq 1 \) and

\[ \frac{1}{p} = \frac{1}{q} - \frac{1}{\nu} \]

Proof for \( q = 1 \) and \( p = \nu \). Here this inequality reads:

\[ \| f \|_{L_\nu} = \sqrt{\int f(x)^\nu dx} \leq c \| df \|_{L_1} \leq c \int \| df(x) \| dx, \]

where it is immediate if \( f \) is the characteristic function \( \chi = \chi_A \) of a smooth domain \( A \subset X \), since \( \| f \|_{L_\nu} = vol_\nu(V) \) for all \( p \) and \( \| df \|_{L_1} = vol_{\nu-1}\partial C \), where, either the integral \( \int_X \chi(x) dx \) is understood in the distribution sense, or, more geometrically, as the limit of smooth or Lipschitz approximations of \( \chi \), of \( f \), e.g.

by the trapezoidal functions

\[ \chi_{A\nu}(x) = \max\left( 0, 1 - \frac{1}{\varepsilon} \text{dist}(x, A) \right), \varepsilon \to 0. \]

Thus, \( \| f \|_{L_\nu} \) is the same as \( \| f \|_{L_\nu} \) for \( q = 1 \).

Next, if \( f = \sum_i \alpha_i \chi_{A_i} \), then

\[ \| f \|_{L_\nu} \leq \sum_i |\alpha_i| \| \chi_{A_i} \|_{L_\nu} \]

by convexity of the norm \( \| \cdot \|_{L_\nu} \), and if \( \alpha_i \geq 0 \), then

\[ \| df \|_{L_1} = \sum_i |\alpha_i| \| \chi_{A_i} \|_{L_1}. \]

It follows that \( \| f \|_{L_\nu} \) does imply \( \| f \|_{L_\nu} \) for such \( f = \sum_i \alpha_i \chi_{A_i} \).

Then we approximate our general positive Lipschitz function \( f(x) \) by such sums, where one takes \( A(t_1) = f^{-1}(t_1, \infty) \), (or smooth approximations to \( A(t_1) \), see remark ??? below) (see Lemma ??? below), and thus conclude he proof for \( q = 1 \).

The proof for \( q \geq 1 \). The above applied to \( f^\frac{\nu}{p} \) and the Gölde (\( r, q \))-inequality, where \( r(\frac{p}{\nu} - 1) = p \), \( q = \left( 1 - \frac{1}{\nu} \right)^{-1} \) yield:

\[ \| f \|_{L_p}^\frac{\nu}{p} \leq \| df \|_{L_1} \leq c \| df \|_{L_q} \leq \frac{c}{\nu^\frac{p-1}{\nu}} \frac{\nu}{p} \| df \|_{L_q}, \]

thus

\[ \| f \|_{L_p} \leq \| f \|_{L_q} \]

One may be justifiably worried with possible intersections of boundaries \( \partial A_i \) of different subsets \( A_i \), but this introduces no correction terms at least for smooth \( \partial A_i \); besides our \( \partial A_0, \partial A_i = B(t_i) = f^{-1}(t_i) \) do not intersect anyway.
QED.

*On Sharp Sobolev.* If $q = 1$ the inequality $\| \ldots \|_\nu = \| \ldots \|_{\nu, 1} \| \nu$ is sharp but it is only exceptionally sharp with our constant $pc/\nu$ for $q > 1$ and $p > \nu$, since the extremal functions $f_{\text{extr}}$ are not like $\chi_X$ in general, but they are associated to smooth solutions of certain Bessel-like (ordinary) differential equations, (see ???, below), from which sharp constants can be derived.

**2.6.B. $L_2$-Example.** If $\nu = 1$ and $p = q = 2$, then the above inequality for $c = 1$, which says that

$$\| f \|_{L_2} \leq 2 \| df \|_{L_2},$$

is sharp for the cylinder $X = \mathbb{R} \times S^1$ with the the (hyperbolic) metric $dx^2 = dt^2 + e^{-2t} ds^2$ (compare with ???)..

**2.6.C. Obvious Approximation Lemma.** Let $\varphi(x)$ be a smooth function on a Riemannian manifold, e.g. on the Euclidean $n$-space, with compact support $X$. Then there exist decreasing families of smooth bounded domains

$$X \supset X_1 \supset X_2 \subset \ldots \supset X_k$$

, such that (multiples of) the sums of trapezoidal functions

$$f_{k, \varepsilon}(x) = \frac{\sup_{\varepsilon} \varphi(x)}{k} \sum_{i=1}^{k} \chi_{X_i, \varepsilon}(x)$$

uniformly, hence $L_p$ for all $p$ converge to $f$ for $k \to \infty$ and $\varepsilon \to 0$ and the differentials of these sums $L_1$ converge to $df$,

$$df_{k, \varepsilon}(x) \to_{L_1} df$$

• The set $X_i$ is contained in the interior of $X_{i+1}$ for all $i$,

$$X_i \Subset X_{i+1}, i = -2, -1, 0, 1, 2, \ldots$$

**2.6.D. Remark: Sard Theorem and Lipschitz $\varphi$.** ???

**2.6.D. Sharp Sobolev.**

**2.8.1 Cavalieri Principle, Coarea Formula, Pushforward Measures and Formal Symmetrization**

Let $\varphi : x \mapsto t \in \mathbb{R}$ be a smooth function on $\mathbb{R}^n$, let

$$A_\varphi(t) = \{ x \in \mathbb{R}^n \}_{\varphi(x) \leq t}$$

be the sublevels of $\varphi$ and

$$B_\varphi(t) = \varphi^{-1}(t) \subset \partial A_\varphi(t)$$

be the levels of $\varphi$.

Then

**A**

$$\int_{\mathbb{R}} \text{vol}(A_\varphi(t)) dt = \int_{\mathbb{R}^n} \varphi(x) dx$$

and

**B**

$$\int_{\mathbb{R}} \text{vol}_{n-1}(B_\varphi(t)) dt = \int_{\mathbb{R}^n} |d\varphi(x)| dx.$$
To see this, check A and B for the function which is equal to \( x \mapsto \|x\| \) in an annulus \( r \leq \|x\| \leq R \) and which is zero outside this annulus and then extend this to all \( \phi \) by linearity of integrals and differentials.\(^{23}\)

More generally, and equally obviously, all continuous (or just Borel) functions \( \psi(x) \) with compact supports on an arbitrary Riemannian manifold \( X = (X,g) \),\(^{24}\) e.g. on \( X = (\mathbb{R}^n; g_{\text{Eucl}} = \sum_i dx_i^2) \)

\[
\int_X \psi(x) \|d\phi(x)\| \, dx = \int_{t=\infty}^{t=\infty} dt \int_{B_\sigma(t)} \psi \, db_t
\]

for all smooth functions \( \phi = \psi(x) \) on \( X \).

This, in terms of the \( \phi \)-pushforward of the Riemannian measure \( dx \), reads:

\[
\phi_*(dx) = \mu(t) \, dt \text{ for } \mu(t) = \int_{B(t)} \|d\phi\|^{-1} \, db_t
\]

Thus, for instance

\[
\int_X \phi(x)^p = \int_{-\infty}^{\infty} t^p \mu(t) \, dt
\]

and

\[
\int_X \|d\phi(x)\| \, dx = \int_{-\infty}^{\infty} b(t) \, dt \text{ for } b(t) = \text{vol}_{n-1}(B_t)
\]

Let us assume (to avoid irrelevant terminological complications) that the support \( S \) of the function \( b(t) \) is a union of disjoint intervals and replace the \( t \)-parameter in \( S \) by \( s \), such that \( ds = b^{-1}(t) \, dt \).

**Formal Symmetrization.** From now on, we think of \( S = S(X) = S_d(X) \) as an oriented 1-dimensional Riemannian manifold with the metric \( ds^2 \), where the function \( t(s) \) is viewed as a kind of symmetrization of \( \phi(x) \), now denoted \( \phi(s) \).

Observe that \( ds^2 \) depends only on the partition of \( X \) to the levels of the function \( \phi(x) \) (but not on the values of \( \phi \) on these levels) and that there is a natural map, say \( \sigma : X \to S \), such that, for all segments \( S_0 \subset S \).

\[
\text{vol}_{n}(\sigma^{-1}(S_0)) = \int_{S_0} b(s) \, ds,
\]

where \( b(s) = b(\phi(s)) = \text{vol}_{n-1}(\sigma^{-1}(s)) \).

Also observe that

\[
\int_X \phi^p(x) \, dx = \int_S \phi^p(s) b(s) \, ds,
\]

\[
\int_X \|d\phi(x)\| \, dx = \int_S \|d\phi(s)\| b(s) \, ds
\]

and

\[
\int_X \|d\phi(x)\|^p \, dx \geq \int_S \|d\phi(s)\|^p b(s) \, ds \text{ for } p \geq 1,
\]

where the later follows from \([\text{Symm}]_d\) and convexity of the function \( z \mapsto z^p \), \( p \geq 1 \).

\(^{23}\) Alternatively, refer to Cavalieri/Fubini for A and to the coarea formula for B.

\(^{24}\) The Riemannian metric \( g \) on \( X \) is assumed continuous, although bounded Borel measurable will do here.
Symmetrization and Comparison Inequalities. Let \( Y = (Y,h) \) be a Riemannian \( m \)-manifold, e.g. \( m = n - 1 \), or \( m = 1 \). with \( \text{vol}_m(Y) = 1 \) and let \( X = Y \times \mathbb{S}^1 \) with the (warped product) metric \( g = ds^2 + b(s)^2 h \), i.e. such that \( \text{vol}_m(Y, b(s)^2 h) = b(s) = \text{vol}_{n-1}(\sigma^{-1}(s)) \).

Then \( S_{b(s)}(X) = Y \times \mathbb{S}^1 \) and if \( Y \) is a homogeneous manifold, e.g. the sphere, this is commonly called a "symmetrization of" \( X \).

### 2.9 Metric Measure Spaces, Moment Maps, Maxwell Law, and Gaussian Isoperimetry

All of the above symmetrizations equally applies to Riemannian manifolds with (more or less) arbitrary measures \( \mu \) on them, where the Riemannian metric is used for evaluation of the norms of differentials \( \|df\| \) while integrals are taken with respect to \( \mu \) rather then with the Riemannian measure.

Thus symmetrization keeps us within the same category of metric measure spaces.

But if you want you can return, at least for smooth measures \( \mu(x)p(x)dx \), to the pure metric Riemannian category by passing from \( (X,g,\mu) \) to the warped product of \( X \) with the circle \( X \times \mathbb{S}^1 \) with the metric \( dx^2 + p(x)^{2/n}ds^2 \).

### 2.10 Cabrè’s ABP-Proof of the Classical Isoperimetric Inequality

25 Logic of the Proof. Assume without loss of generality that the boundary of a smooth bounded domain \( X \subset \mathbb{R}^n \) has the same \((n-1)\)-volume as the unit sphere,

\[ \text{vol}_{n-1}(\partial X) = \text{vol}_{n-1}(S^{n-1}), \]

and let \( f(x) \) be a smooth function, such that

\[ \Delta(f) = a \text{ and } df(\nu) = 1. \]

Then the proof would trivially follow from the inequality

\[ a^{-1}\text{vol}_{n-1}(\partial X) = a^n\text{vol}(X) \geq n^nz\text{vol}(B^n(1)) = n^nz\text{vol}_{n-1}(S^{n-1}) = n^nz\text{vol}_{n-1}(\partial X) \]

where \( z \) is proven below by constructing a map from a part of \( X \) onto \( B^n(1) \) with Jacobian \( \leq a^n/n^z \).

Let \( X \subset L = \mathbb{R}^n \) be a bounded domain with a smooth boundary \( \partial X \) and let \( f : X \rightarrow \mathbb{R} \) be a smooth function.

Let \( L' = \mathbb{R}^n \) be the linear dual of \( L \), let us identify the tangent spaces \( T_x(L) \), \( x \in L \supset X \) with \( L \) and let \( df : X \rightarrow L' \) be the map, which thus corresponds to the differential \( df : x \mapsto T_x(L) \).

25https://pdfs.semanticscholar.org/0b0f/91abb26f8ae7c6d304f0881f646d28cabf7a.pdf
**ABP**\(^{26}\) Lemma. Let \( X \subset \mathbb{R}^n \) be the subset, where \( f \) is **locally convex**, i.e. where the Hessian of \( f \), (full second differential) \( d^2 f/\partial x_i \partial x_j \) of \( f \) is **semipositive definite**.

If the normal derivative of \( f \) on he boundary of \( X \) is **bounded from below** by a positive constant \( R \),
\[
    df(\nu) \geq R,
\]
where \( \nu : \partial X \to T(X) \) is the **outward looking** unit normal vector field, then the image \( \overline{d}(X) \subset L' = \mathbb{R}^n \) contain the \( R \)-ball in \( L' \),
\[
    \overline{\partial f}(X) \supset B^n(R) \subset L' = \mathbb{R}^n.
\]

Let \( S' = S^{n-1}(1) \subset L' = \mathbb{R}^n \) be the unit sphere in the dual space to \( L \) and let \( R(s') \) be the infimum
and let \( T_\nu(\partial X) \subset T(\partial X) \) be the set of the **supporting hyperplanes** to \( X \) that are the hyperplanes tangent to \( \partial X \) at the intersection points of \( \partial X \) with the boundary of the convex hull \( \text{conv}(X) = \text{conv}(\partial X) \).

Let \( S' = S^{n-1}(1) \subset L' = \mathbb{R}^n \) be the unit sphere in the dual space to \( L \) and \( T_\nu(\partial X) \subset T(\partial X) \) be the set of the vectors \( s' \in S' \).

Let \( R_f(s') \) be the maximum of the values of the differential \( df \) at the outward looking unit normal vector field to the boundary of \( X \) at the points, where the supporting hyperplane is tangent to \( \partial X \), i.e. at \( x \in \partial X \cap \nu s' \). \(^{27}\)

Let \( R_f \geq 0 \) and let \( U'(R_f) \subset L' \) be the set of the vectors \( r(s') s' \in L' \) for all \( s' \in S' \) and \( r(s') \leq R_f \). (If \( R \) is constant this is the \( R \)-ball.)

**ABP Corollary.** The Laplacian of \( f \) satisfies
\[
    \int_X |\Delta(f)|^n dx \geq \int_{\partial X} |\Delta(f)|^n dx \geq n^n \text{vol}(B^n(R)).
\]

**Proof.** The arithmetic/geometric inequality applied to the (real positive) eigenvalues of the Hessian of \( f \) at he points in \( X_\nu \) shows that the Jacobian of he map \( df : X_\nu \to B^n(R) \) satisfies
\[
    \text{Jac}(df) \leq (\Delta(f))^n / n^n,
\]
while the integral of this Jacobian over \( X_\nu \) is bounded from below by the integral over our \( R \)-ball of the multiplicity of the map \( df \),
\[
    \int_{X_\nu} |\text{Jac}(f)|^n dx \geq \int_{B^n(R)} \text{card}(df)^{-1}(l') dl', l' \in B^n(R) \subset L',
\]
where \( \text{card}(df)^{-1}(l') \geq 1 \) on \( B^n(R) \) by the ABP-Lemma.

**Linear PDE-Recollection.** (Neumann Boundary problem.) Let \( a(x) \) and \( b(y) \) be smooth functions in \( x \in X \) and \( y \in \partial X \). If
\[
    [GF] \quad \int_X a(x) dx = \int_Y b(y) dy
\]
\(^{26}\)Aleksandrov–Bakelman–Pucci.
\(^{27}\)A supporting hyperplane to an \( X \) can be tangent to \( \partial X \) at several points \( x \).
(this is Green’s formula), then there exists a smooth function \( f(x) \), such that the Laplacian of \( f \) and the normal derivatives of \( f \) satisfy
\[
\Delta f(x) = a(x) \quad \text{and} \quad df(\nu(y)) = b(y).
\]

**Proof of the Isoperimetric Inequality.** Let us assume without loss of generality that the boundary of \( X \) has the same \((n-1)\)-volume as the unit sphere,
\[
\text{vol}_{n-1}(\partial X) = \text{vol}_{n-1}(S^{n-1}),
\]
let \( b = 1 \) and let \( a > 0 \) be a constant, such that
\[
[a/1] \quad a \cdot \text{vol}(X) = \text{vol}_{n-1}(\partial X).
\]

Let \( f(x) \) be a smooth function, such that
\[
\Delta f(x) = a \quad \text{and} \quad df(\nu) = 1.
\]

Then the above ABP inequality reads
\[
a^{n-1}\text{vol}_{n-1}(\partial X) = a^n\text{vol}(X) \geq n^n\text{vol}(B^n(1)) = n^{n-1}\text{vol}_{n-1}(S^{n-1}) = n^{n-1}\text{vol}_{n-1}(\partial X)
\]
Hence \( a \geq n \) and \([a/1]\) shows that
\[
\text{vol}(X) \leq \frac{1}{n}\text{vol}_{n-1}(\partial X).
\]

QED.

*Question.* Is there a *natural* Borel (measurable) correspondence
\[
X \times S^{n-1} \leftrightarrow \partial X \times B^n
\]
which would geometrically implement the inequality
\[
\text{vol}_{2n-1}(X \times S^{n-1}) \leq \text{vol}_{2n-1}(\partial X \times B^n)?
\]

Or, maybe a *natural* family of similar correspondences between powers of these sets
\[
(X \times S^{n-1})^N \leftrightarrow (\partial X \times B^n)^N
\]
\(N = 1, 2, \ldots\), which implements the inequality
\[
\lim_{N \to \infty} \sqrt[N]{\text{vol}_{N(2n-1)}(X \times S^{n-1})^N} \leq \lim_{N \to \infty} \sqrt[N]{\text{vol}_{N(2n-1)}(\partial X \times B^n)^N}
\]

**2.11 Dimension 2: Steiner, Santalo, Cabre, Wirtinger...**

**3 Laplace Operators on Riemannian manifolds \( X \) and Eigen Values \( \lambda_1(X) \) and \( \lambda_2(X) \).**

Let \( X \) be a smooth Riemannian \( n \)-manifold, e.g. a domain in \( \mathbb{R}^n \) or a smooth closed hypersurface, such as the \( n \)-sphere \( S^n \subset \mathbb{R}^{n+1} \).

Recall that the Laplace operator on \( X \) is defined as
\[
\Delta f(x) = \text{div grad} f(x) = \sum_{i=1}^{n} \partial_i^2 f(x)
\]
where $\partial_i f_x(X)$ are orthonormal tangent vectors at $x$ and $\partial^2 f$ is the second derivatives along the geodesic issuing from $x$ in the $\partial_i$-direction, where this is independent of a choice of orthonormal vectors by the Pythagorean theorem.

### Exercises.
(a) Show that $\frac{1}{n}\Delta f(x)$ equals the spherical average of the second derivatives of $f$ along geodesics issuing from $x$,

$$\frac{1}{n} \Delta f(x) = \frac{1}{\text{vol}(S^{n-1})} \int_{S^{n-1}} \partial^2 f(x) ds$$

where $S^{n-1} \subset T_x(X)$ is the tangent unit sphere of $X$ at $x$.

(b) Express the integrals $\int_X \|\Delta f(x)\|^p$ by lifting $f(x)$ to a function $f(\tau)$ on the unit tangent bundle $UT(X)$, twice differentiating $f$ along the orbits of the geodesic flow and integrating $\|\partial^2 f(\tau)\|^p$ over $UT(X)$ with the Liouville measure (which is invariant under the geodesic flow).

(c) $f : X \to \mathbb{R}^N$ be an isometric embedding (immersion will do) and show then the Laplace operator coordinate-wise applied to $f = (f_1,\ldots,f_N)$ as $\Delta f = (\Delta f_1,\ldots,\Delta f_N)$ is equal the mean curvature vector field of $X \to \mathbb{R}^N$ interpreted as a map $X \to \mathbb{R}^N$ (by identifying the tangent spaces $T_y(\mathbb{R}^N)$ with $\mathbb{R}^N$ for all $y \in \mathbb{R}^N$),

$$\Delta f = \text{mean.curv}(X \to \mathbb{R}^N),$$

where the mean curvature vector of $X$ at $x \in X$ is the $n$-times spherical average of the Euclidean curvature vectors of geodesic in $X$ issuing from $x$.

For instance, the mean curvature of the sphere $S^n(R) \subset \mathbb{R}^{n+1}$ is $\frac{n}{R} \nu(s)$, where $\nu(s) \in T_s(\mathbb{R}^{n+1})$ is the outward looking unit normal vector to the sphere at $s \in S^n(R)$.

(d) Using this definition of the mean curvature, show that it is equal to the gradient of the function $f \mapsto \text{vol}_n(X)$ on $n$-submanifolds in $\mathbb{R}^n$.

Observe that $\Delta$ is a negative operator as

$$\int_X \langle \Delta f(x), f(x) \rangle dx = -\int_X \|df\|^2$$

on closed manifold by the Green’s formula (integration by parts) and the same is true for compact manifolds with boundaries for functions $f$ if either $f$ or $df$ vanish on $\partial X$.

Poincaré inequality concerns the smallest non-zero eigenvalue of the operator $-\Delta = -\Delta_X$ on the space of smooth $L_2$-functions $f$ on $X$ with either Dirichlet or Neumann boundary conditions, that is either for functions which vanish on the boundary $\partial X$ or with the gradient $\text{grad} f$ on the boundary $\partial X$ normal to $\partial X$.

If all connected components of $X$ are compact with nonempty boundaries then there is no zero eigenvalue: harmonic function zero on the boundary vanish.

Thus the first non-vanishing Dirichlet eigenvalue is the smallest one, denoted $\lambda_1(X)$.

But since constant functions satisfy Neumann’s condition, the first non-vanishing Neumann’s eigenvalue on a compact connected manifold with or without boundary is actually the second smallest one, which we denote $\lambda_2(X)$.\footnote{\textit{L}_2\textsuperscript{2} means that $\int_X f^2(s) ds < \infty$.}

\footnote{According to our notation, compact disconnected manifolds $X$ have Neumann’s $\lambda_2 = 0$.}
Examples. (a) The segment \([a,b]\) has \(\lambda_1 = 0\),

\[\lambda_2[a,b] = \frac{\pi^2}{(b-a)^2},\]

since the only bounded eigen functions of \(\Delta \equiv \frac{d^2}{dt^2}\) on the line are sin and cos,

and, for all \(i\),

\[\lambda_i[a,b] = \frac{(i-1)^2\pi^2}{(b-a)^2}\]

(b) Spectra of Riemannian Products One knows that the eigenvalues of \(-\Delta_{X \times Y}\) are the sums \(\lambda_i(X) + \lambda_j(Y)\).

In fact, this follows from general properties of elliptic selfadjoint operators (See below for a direct proof).

(c) The eigenvalues of the cube \([0,\pi]^n\) are the sums \(i_1^2 + i_2^2 + \ldots + i_n^2\). Thus, there are roughly \(R^n/2\) eigenvalues \(\lambda_i \leq R\).

Variational Principle and Green’s Formula. The (infinite dimensional in the present case) linear algebra tells you that the eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots\) of \(\Delta\) be they Dirichlet’s or Neumann’s ones, are equal to the critical values of the quadratic function

\[f \mapsto -(f, \Delta f) = \int_X f(x)\Delta f dx\]

on the unit sphere in the Hilbert space of \(L_2\)-functions \(f(x)\), where the corresponding eigen functions \(\phi_i\), or spaces if these for multiple \(\lambda_i\), are mutually orthogonal.

Green’s Formula. "Integration by Parts" shows that the above quadratic function is equal to the Dirichlet (energy) functional,

\[-(f, \Delta f) = \|df\|_{L_2}^2 = \int_X \|df\|^2 dx\]

for smooth functions \(f\) on compact manifolds, such that either \(f\) vanishes on the boundary \(\partial X\) or the normal derivative of \(f\) on \(\partial f\) vanish.

(In the case of disconnected boundary, one may have \(f\) vanishing on some components of \(\partial X\) and the normal derivative of \(f\) vanishing on the remaining components.)

Thus

\[\lambda_1(X) = \inf_{f \not\equiv 0} \frac{\|df\|_{L_2}^2}{\int_X f^2 dx}\]

and

\[\lambda_2(X) = \inf_{f \not\equiv 0} \frac{\int_X \|df(x)\|^2 dx}{\int_X \|f(x)\|^2 dx}\]

Remarks (a) Since the orthogonality to constants condition \(\int_X f(x) dx = 0\) (easily) implies that

\[\int f^2(x) dx = \frac{1}{2vol(X)} \int_{X \times X} |f(x_1) - f(x_2)|^2 dx_1 dx_2\]

probably there is a direct proof of the inequality \(\int f^{n/2} f^2(t) dt \leq \int f^{n/2} |f'(t)|^2 dt\) for functions \(f\) such that \(\int f^{n/2} f(t) dt = 0\), but I couldn’t find it.
one can define \( \lambda_2 \), as

\[
\lambda_2(X) = \frac{1}{2 \text{vol}(X)} \inf_{f} \int_{X} \frac{\left| f(x_1) - f(x_2) \right|^2 dx_1 dx_2}{\int_{X} \| f(x) \|^2 dx},
\]

where this definition makes sense for maps \( f \) from \( X \) to an arbitrary Riemannian (and Finsler) manifold with \( |a - b| \) understood as \( \text{dist}(a, b) \).

(b) The definition of \( \lambda_i \) via the Dirichlet functional \( \| df \|_{L_2}^2 \) without a direct reference to the Laplace operator makes sense for manifolds \( X \) with bounded measurable Riemannian metrics and it is quasi-invariant under bi-Lipschitz maps.

if \( \Phi : X_1 \to X_2 \) is a \( l \)-Lipschitz (i.e. \( \| d\Phi \| \leq l \)) homeomorphism, where \( \Phi^{-1} : X_2 \to X_1 \) is also \( l \)-Lipschitz, then, clearly, both Dirichlet and Neumann eigenvalues satisfy:

\[
\frac{1}{2^{n-1}} \lambda_1(X_1) \leq \lambda_2(X_2) \leq \frac{1}{2^{n-1}} \lambda_1(X_1)
\]

for all \( i = 1, 2, \ldots \).

**Exercises.** (a) Show that Dirichlet’s eigen values of subdomains \( X_0 \subset X \) satisfy \( \lambda_i(X_0) \geq \lambda_i(X) \)

(a) Let \( \Phi : X_1 \to X_2 \) be an \( l \)-Lipschitz map, where the (compact Riemannian) manifolds \( X_1 \) and \( X_2 \) may have different dimensions and let the pushforward of the Riemannian measure \( dx_1 \) be

\[
\Phi_*(dx_1) = \delta(x_2) \cdot dx_2
\]

for some positive function \( \delta(x_2) > 0 \) on \( X_2 \). (This makes \( \text{dim}(X_1) \geq \text{dim}(X_2) \).)

Show that the Neumann’s eigen values satisfy

\[
\lambda_i(X_1) \leq l^2 \sup_{x \in X_2} \delta(x_2) \cdot \lambda_2(X_2).
\]

(b) Show that, for all pairs of compact Riemannian manifolds \( X_1 \) and \( X_2 \) where \( \text{dim}(X_1) \geq \text{dim}(X_2) \) (e.g where \( X_1 \) is the \( n \)-cube \( [0,1]^n \) and \( X = X^n \) is arbitrary) there exist positive constants \( l > 0 \) and \( \delta > 0 \) depending on \( X_1 \) and \( X_2 \) and \( l \)-Lipschitz maps,

\[
\Phi : X_1 \to X,
\]

such that the pushforward measures \( \Phi_*(dx_1) \) are constant \( \delta \)-multiples of \( dx \).

**Remark.** Arbitrarily large balls \( B(R) \) in hyperbolic spaces may have arbitrarily large Dirichlet’s \( \lambda_1 \).

Less obviously, one can show, using *Riemann surface expanders*, that all compact manifolds \( X \) of dimension \( n \geq 3 \) (4 maybe) admit arbitrarily large Riemannian metrics with arbitrary large Neumann’s \( \lambda_2 \).

(c) Derive the above formula for the eigenvalues of \( \Delta_{X \times Y} \) from the variational principle.

(d) Combine this with (b) for \( X_1 = [0, \pi]^n \) and show that Neumann’s eigenvalues of compact connected manifolds \( X \) satisfy

\[
\lambda_i \geq \text{const}_X (i^{n/2} - 1).
\]

(e) Derive a similar bound from this for Dirichlet’s eigenvalues of compact connected \( n \)-manifolds with boundaries by applying (d) to a connected \( n \)-manifold.
Let $X$, which contains two copies of $X$ e.g. to the double of $X$ obtained by gluing two copies of $X$ across their boundaries.

(f) Let $X$ be a compact manifold with a boundary and show that the first Dirichlet’s eigenvalue of $X$ is equal to the supremum of numbers $\lambda$, such that $X$ admits a smooth positive function $f(x) > 0$, such that

$$-\Delta f(x) \geq \lambda f(x),$$

that is

$$\lambda_1(X) = \sup_{f > 0} \inf_{x \in X} \frac{-\Delta f(x)}{f(x)}.$$

Hint. Use the maximum principle for the first Dirichlet eigenfunction $f_1$.

(g) Divide a closed Riemannian $n$-manifold $X$ into two domains $X_-$ and $X_+$ with common smooth boundary $Y = X_- \cap X_+ \subset X$, let $\lambda_{\text{max}}(Y) = \max(\lambda_1(X), \lambda_1(X_+))$ and show that

$$\lambda_2(X) = \inf_{Y \subset X} \lambda_{\text{max}}(Y),$$

where ” sup” is taken over all smooth closed cooriented hypersurfaces $Y \subset X$.

**Laplacian on Riemannian Metric Measure Spaces.** This $\Delta$ is defined on $X = (X, g, \mu)$ as

$$\Delta (f) = \text{div}_\mu \text{grad}_g (f),$$

where the $\mu$-divergence of the vector field $\tau$ on $X$ is the ratio of the $\tau$-Lie derivative of $\mu$ by $\mu$, which makes sense for $\mu(x) = p(x) dx$ where $p(x) > 0$. In this case the $\mu$-Laplacian at least for smooth $p$ has the same properties as the pure Riemannian one which can be seen, for instance, by looking at the Riemannian Laplacian on the $S^1$-invariant functions on the warped product $X \times S^1$ with the metric $g_X + p^{2/n} ds^2$.

### 3.1 Spectra of Cubes, Spheres, Balls and Hyperbolic Cusps

#### 3.1.1 Rayleigh–Faber–Krahn Inequality

**Rewrite**

Let $f : X \to \mathbb{R}_+$ be a non-negative measurable function on a measure space $X = (X, \mu)$ e.g. on $X = \mathbb{R}^n$, which is supported on a subset with finite measure.

Recall that the $f$-pushforward of $\mu$ is the measure on $\mathbb{R}_+$, such that

$$\mu(S) = \mu f^{-1}(S)$$

for all Borel subsets (segments $[a, b]$ will do) will do) $S \subset \mathbb{R}_+$.

Define the $O(n)$-invariant model $f(x) = \int_{O(n)} (\|x\|) dx$ of $f$ as the radial function on $\mathbb{R}^n$, such that the $f$-pushforward of the Euclidean measure $dx$ is equal to the $f$ -pushforward of $\mu$, that is such that the Euclidean volumes of the balls $B^n(a) \subset \mathbb{R}^n$ for all $a \geq 0$, where the function $f$ is $\leq a$, are equal to the $\mu$-measures of the corresponding subsets in $X$, i.e. where $f(x) \leq a$.

**Remark.** In the case of the Faber-Krahn theorem, the functions $f$ and $\hat{f}$ are defined on the same space $\mathbb{R}^n$, but one also may compare a function $f$ on an arbitrary $n$-dimensional Riemannian manifold with its $O(n)$-symmetric model on $\mathbb{R}^n$ as well, more generally, on another $n$-manifold with $O(n)$-symmetry, e.g. the sphere $S^n$ or the hyperbolic $n$-space.


Now, similarly to the above proof of the Sobolev inequality, the Euclidean isoperimetric inequality
\[ \text{vol}_n(X)/\text{vol}_{n-1}(\partial(X)^{n/n-1}) \leq \text{vol}_n(B^n)/\text{vol}_{n-1}(S^{n-1})^{n/n-1} \]
and the coarea formula imply that the differentials of Euclidean \(O(n)\)-symmetric models \(f(x)\) of Lipschitz functions \(f\) on \(X = \mathbb{R}^n\) with bounded supports satisfy the following
\[ \|df\|_{L^p} \text{-Symmetrization Inequality.} \]
\[ \|df\|_{L^p} = \sqrt{\int_{\mathbb{R}^n} \|df(x)\|^p dx} \leq \sqrt{\int_{\mathbb{R}^n} \|df(x)\|^p} \]
for all \(p \geq 1\).

It follows that the infimum of the ratio
\[ \gamma_{p,q}(X) = \frac{\|df\|_{L^p}}{\|f\|_{L^q}}, p, q > 1 \]
over all Lipschitz functions on \(\mathbb{R}^n\) with supports in a given bounded domain \(X \subset \mathbb{R}^n\) is bounded from below by \(\gamma_{p,q}(B^n(a))\), where \(B^n(a) \subset \mathbb{R}^n\) is the ball with the volume equal to that of \(X\).

This, for \(p = q = 2\) is called the **Faber-Krahn inequality**, which, as explained below, shows that among all bounded domains in \(\mathbb{R}^n\) with a given volume, balls have the highest bottom oscillation frequency.

### 3.1.2 Bochner Formula and Lichnerowicz \(\lambda_2\)-Inequality

#### 3.2 ???

More specifically the "spherical Poincaré" says that
\[ \lambda_2(S^n(1)) = n \]
where the corresponding eigen functions are the linear ones. In fact a straightforward computation (see section ?? below) shows that the Euclidean coordinates \(x_i\) regarded as functions \(x_i(s)\) on
\[ S^n = \{x_0,...,x_n\} \sum_{i=1}^{n-1} \]
satisfy
\[ \Delta_{S^n} x_i(s) = x_i(s) \cdot \text{mean.curve}(S^n, s) = -nx_i(s), \]
which agrees with the identities
\[ \int_{S^n} \sum_i \|x_i(s)\|^2 ds = \int_{S^n} 1 ds = \text{vol}(S^n) \]
and
\[ \int_{S^n} \sum_i \|dx_i\|^2 = \int_{S^n} nds = n \cdot \text{vol}(S^n), \]
**Bochner Formula**
\[
\frac{1}{2} \Delta \|df\|^2 = \|\text{Hess} \ f\|^2 + (d\Delta f, df) + \text{Ricci}(df, df)
\]
\[
0 = \int_X \|\text{Hess} \ f\|^2 - \lambda(df, df) + \text{Ricci}(df, df)
\]
\[
\|\text{Hess} \ f\|^2 \geq \frac{1}{n} \|\Delta f\|^2
\]
\[
\text{Ricci}_{g^s} = (n - 1)g^s
\]
and Lichnerowicz \(\lambda_1\)-theorem 
\(\lambda_1\) for convex domains by needle integration and by needle decomposition.

### 3.3 Cheeger’s constant and Cheeger’s \(\lambda_2\) inequality

#### change!

Let \(X\) be Riemannian \(n\)-manifold (possibly with a boundary and let \(\text{che}(X)\) be the supremum of the numbers \(c > 0\), such that all smooth domains \(V \subset X\) with \(\text{vol}(V) \leq \frac{1}{2} \text{vol}(X)\) satisfy

\[
\text{vol} - n - 1(\partial V) \geq c \cdot \text{vol}_n(V).
\]

**3.3.A. Cheeger Inequality** [Che1969]. Let a smooth function on \(X\) satisfy one of the following conditions

- \(\bullet_1\) the volume of the support of \(f\) is finite and it is smaller than the volume of its complement, i.e. \(\text{vol}_n(\text{supp} \ f) \leq \frac{1}{2} \text{vol}(X)\);
- \(\bullet_2\) the function \(f\) is orthogonal to constants,

Then

\[
[\text{che}]_{1/4} \int_X \|df\|^2 dx \geq \frac{\text{che}^2}{4} \int_X f^2 dx.
\]

**Proof.** The \(\bullet_1\) case in the form

\[
\|f\|_{L_2} = \sqrt{\int_X f(x)^2 dx} \leq \frac{\text{che}}{2} \|df\|_{L_2} = \sqrt{\int_X \|df(x)\|^2 dx}
\]

follows from his Maz’ya-Cheeger conditional inequality (\(L_2\)-Example???) while

\(\bullet_2\) reduces to \(\bullet_1\) by the following.

\(\{\lambda_1 \implies \lambda_2\}\)-Lemma. If the first Dirichlet eigenvalues of all smooth domains \(V \subset X\) with \(\text{vol}(V) \leq \frac{1}{2} \text{vol}(X)\) satisfy \(\lambda_i(V) \geq c\), for some \(c = c(X) > 0\), then the second Neumann eigenvalue of \(X\) is also \(\geq c\).

In fact, let \(f\) be a non-zero Neumann’s \(\lambda\)-eigen-function in \(X\), let \(Y \subset X\) be the zero set of \(f\), which divide \(X\) into two regions, \(X_+\) and \(X_1\) with common boundary \(Y\).

Then

\[
\int_{X_+} df^2(x) dx = \int_{X_+} f(x) \Delta f = \lambda \int_{X_+} f^2 dx
\]

by the Green’s formula, since, in both regions, \(f\) vanishes on \(Y\) and its gradient is normal to the remaining boundaries of \(X_+\).
Since the minimum of the ratio
\[ \min_f \frac{\int_X df^2}{\int f^2} \]
on functions in \( X \) normal to the constants is achieved on a non-zero \( \lambda_2 \)-eigenfunction \( f_2 \), an application of \( \bullet_1 \) to the smallest of the two \( X \) yields \( \bullet_2 \).

**3.3.B. On Logic of Proofs of \( \bullet_1 \) and \( \bullet_2 \).** The reduction of \( \bullet_2 \) to \( \bullet_1 \) appears less elementary than the proof of \( \bullet_1 \).

For example, if \( X \) is a polyhedral domain and \( f \) is a piecewise linear function, the proof of the \( \bullet_1 \)-case of [che][1/4] in section ??? can be rendered in a purely algebraic (first order) language, namely that of ordered real fields and piecewise affine (or semialgebraic if you wish) function, while the reduction \( \bullet_1 \implies \bullet_2 \) depends on eigenfunction that are minima of the Dirichlet functional in the space all(!) functions.

Besides, while the proof of \( \bullet_1 \) given in ??? depends only on the isoperimetric properties of the domains \( A(t) \) bounded by the \( t \)-levels \( B(t) = f^{-1}(t) \) of our function \( f \), while, in the \( \bullet_2 \) case, one needs this for all domains in \( X \) with volumes \( \leq 1/2\text{vol}(X) \).

However, if one doesn’t care for sharpness of constants one can proceed as follows.

A function on a compact Riemannian \( n \)-manifold, possibly with a boundary, and let, as earlier,

\[ B(t) = f^{-1}(t) = \partial A(t) \subset X, \text{ for } A(t) = f^{-1}[t, \infty). \]

Let, for some \( t_0 \geq 0 \), all domains \( A(t) \subset X \) with \( t \geq 0 \) satisfy the same isoperimetric inequality as in 2.6.A:

\[ \text{vol}_{n}(A(t)) \leq c \cdot (\text{vol}_{n-1}B(t))^\nu \text{ for some } \nu \geq 1. \]

Then the \( L_p \) norm of \( f \) is bounded in terms of its \( L_1 \) norm, the \( L_p \)-norm of the differential of \( f \), where, as in and he volume \( v_0 \) of \( A(t_0) \), where

\[ \frac{1}{p} = 1 - \frac{\nu - 1}{\nu}, \]
as follows

\[ \| f \|_p \leq \frac{pc}{\nu} \| df \|_{L_q} + v_0^{\frac{1}{\nu} - 1} \| f \|_1, \]

Indeed, this follows from the Maz’ya-Cheeger inequality 2.6.A applied to the function \( h(t) = f(t) - t_0 \) restricted to \( A(t_0) \), where \( h \) is positive and vanishes on \( B(0) = \partial A(t) \), and where, clearly, \( \| df \|_{L_q} = \| dh \|_{L_q} \), and

\[ \| f \|_p \leq \| h \|_p + t_0 \sqrt[\nu]{v_0} \leq \| h \|_p + v_0^{\frac{1}{\nu} - 1} \| f \|_1. \]

Next, let \( \int f(x)dx = 0 \) as in \( \bullet_2 \) and let \( X_- \) and \( X_+ \) be the negative/positive regions of \( f \) of volumes \( v_- \) and \( v_+ \), and let \( f_-(x) = \min(f(x), 0) \). Then

\[ \| f \|_p \leq \frac{pc}{\nu} \| df \|_{L_q} + \left(\frac{v_-}{v_0}\right)^{1-\frac{1}{\nu}} \| f_- \|_p. \]
Finally, let \( v \leq v_0 \) and let the domains \( A_t(t) = f^{-1}(-\infty, t] \) for \( t \leq 0 \) satisfy the isoperimetric inequality

\[
\text{vol}_n(A^-(t)) \leq c(\text{vol}_{n-1}(\partial A^-(t)))^\nu,
\]

then

\[
\|f\|_p \leq \left( \frac{pc}{\nu} \left( 1 + \frac{1}{2} \right)^{1-\frac{1}{p}} \right) \|df\|_q.
\]

For instance, if \( \nu = 1 \) and \( p = q = 2 \), then

\[
\|f\|_2 \leq 2c\sqrt{\frac{3}{2}}\|df\|_2
\]

This proof uses the isoperimetric inequality only for the levels of \( f \), but the factor \( \sqrt{\frac{3}{2}} > 1 \) makes it non-sharp.

To make it sharp, let \( S = S_f(X) = (S, ds^2, b(t)ds) \) be the formal symmetrization of \( (X, f) \) (see section??), that is a one dimensional Riemannian manifold \( S = (S, ds^2) \) isometric to \( (\mathbb{R}, b(t)ds) \).

The above argument applies to \( (Sds^2, b(s)ds) \) — in fact, it applies to all metric measure spaces and yields Cheeger’s inequality for \( (S, f) \), but by applying Cheeger’s argument to the formal symmetrisation of \( f \)...

### 4 Minkowski and Brunn

**Divide and rule.** (Hadwiger-Ohmann?) cut Let two subsets \( X, Y \subset \mathbb{R}^n \) be divided by a hyperplane into \( X_- \cup X_+ = X \) and \( Y_- \cup Y_+ = Y \), then the Minkowski sums of corresponding "half-subsets" \( X_- + Y_- \) and \( X_+ + Y_+ \) in \( \mathbb{R}^n \) do not intersect — they lie in different half-spaces of \( \mathbb{R}^n \) divided by \( H \) — and, since

\[
X + Y \ni (X_- + Y_-) \cup (X_+ + Y_+),
\]

\[
\text{vol}(X + Y) \geq \text{vol}(X_- + Y_-) + \text{vol}(X_+ + Y_+).
\]

Therefore, if \( H \) equidivides both sets,

\[
\text{vol}(X_-) = \text{vol}(X_+) = \frac{1}{2} \text{vol}(X) \text{ and } \text{vol}(Y_-) = \text{vol}(Y_+) = \frac{1}{2} \text{vol}(Y),
\]

Then the Minkowski \(-\frac{1}{n}\)-inequalities for the pairs \((X_+, Y_+)\)

\[
\text{vol}(X_- + Y_-)^{-\frac{1}{n}} \geq \text{vol}(X_-)^{-\frac{1}{n}} + \text{vol}(Y_-)^{-\frac{1}{n}},
\]

and

\[
\text{vol}(X_+ + Y_+)^{-\frac{1}{n}} \geq \text{vol}(X_+)^{-\frac{1}{n}} + \text{vol}(Y_+)^{-\frac{1}{n}},
\]

[Che1969].
imply this inequality for \((X, Y)\)

\[
\text{vol}(X + Y)^{1/n} \geq (\text{vol}(X_+ + Y_+) + \text{vol}(X_- + Y_-))^{1/n} \geq \\
\left( (\text{vol}(X_-)^{1/n} + \text{vol}(Y_-)^{1/n})^n + (\text{vol}(X_+)^{1/n} + \text{vol}(Y_+)^{1/n})^n \right)^{1/n} = \\
\text{vol}(X)^{1/n} + \text{vol}(Y)^{1/n}
\]

Since all sets in \(\mathbb{R}^n\) can be translated to a position where they are equidivided by a given hyperplane \(H \subset \mathbb{R}^n\), the \(\sim^{1/n}\)-inequality for \((X, Y)\) reduces to those for two pairs of twice "thinner" sets \((X_+, Y_+), (X_-, Y_-)\).

Then by equidividing further and further we reduce \(\sim^{1/n}\) for \((X, Y)\) to inequalities for arbitrary thin sets, i.e. to intersections of \(X\) and \(Y\) with regions \(H^{n-1} \times [0, \epsilon] \subset \mathbb{R}^n\) between pairs of mutually \(\epsilon\)-close hyperplanes.

If \(X\) is a smooth or piecewise domain and \(H\) is transversal to \(\partial X\) then

\[
\text{vol}_n(X \cap H^{n-1} \times [0, \epsilon]) = \epsilon \cdot \text{vol}_n(X \cap H^{n-1}) + o(\epsilon),
\]

i.e.

\[
|\text{vol}_n(X \cap H^{n-1} \times [0, \epsilon]) - \epsilon \cdot \text{vol}_n(X \cap H^{n-1})|/\epsilon \to 0 \text{ for } \epsilon \to 0.
\]

Moreover, the intersection \(X \cap H^{n-1} \times [0, \epsilon] \subset \mathbb{R}^n\) is "physically" \(o(\epsilon)\)-close to the product

\[(X \cap H^{n-1}) \times [0, \epsilon] \subset \mathbb{R}^n,
\]

that is the volume of the symmetric difference between the two subsets is \(o(\epsilon)\).

It follows, in the limit for \(\epsilon \to 0\), that the \(\sim^{1/n}\)-inequality for \((X, Y)\) reduces to \(\sim^{1/n}\) for pairs of products \(X' \times [0, \epsilon], Y' \times [0, \delta] \subset X', Y' \mathbb{R}^{n-1}\), which is obviously equivalent to the \(\sim^{(n-1)/n}\)-inequality for \((X', Y')\).

Then he Minkowski \(\sim^{1/n}\)-inequality for piecewise smooth domains follows by induction on \(n\) and, if you care, this inequality for general measurable subsets follows by approximation.

Isoperimetric Corollary. Isoperimetric inequality for smooth and piecewise smooth domains \(X \subset \mathbb{R}^n\) follows from \(\sim^{1/n}\) applied to \(X\) and the \(\epsilon\)-ball \(B^n(\epsilon)\) for \(\epsilon \to 0\).

Indeed, since

\[
\text{vol}_n(X + B^n(\epsilon)) = \text{vol}_n(X) + \epsilon \text{vol}_{n-1}(\partial X) + o(\epsilon),
\]

the \(\sim^{1/n}\) implies that

\[
(\text{vol}_n(X) + \epsilon \text{vol}_{n-1}(\partial X))^{1/n} + o(\epsilon) \geq \text{vol}_n(X)^{1/n} + \epsilon \text{vol}_n(B^n(1))^{1/n} \implies \\
\epsilon \text{vol}_{n-1}(\partial X) \geq \text{vol}_n(X) + n \text{vol}_n(X)^{(n-1)/n} \epsilon \text{vol}_{n-1}(B^n(1))^{1/n} + o(\epsilon) \implies \\
\text{vol}_{n-1}(\partial X) \geq n \text{vol}_n(X)^{(n-1)/n} \text{vol}_{n-1}(B^n(1))^{1/n} \implies \\
\frac{\text{vol}_{n-1}(\partial X)}{\text{vol}(X)^{(n-1)/n}} \geq n \cdot \text{vol}_{n-1}(B^n(1))^{1/n},
\]

\text{---}\)

\text{Here } \mathbb{R}^n \text{ is represented as } H^{n-1} \times \mathbb{R}.\]
which is recognizable for the constant-wise sharp isoperimetric inequality, as it becomes equality for $X = B^n(1)$:

$$\frac{\text{vol}_{n-1}(S^n(1))}{\text{vol}(B^n(1))^{(n-1)/n}} = n \cdot \text{vol}_{n-1}(B^n(1))^{1/n}.$$

(This proof, unbelievably primitive, sharply contrasts with the elegance of the ABP-proof, but, at the bottom, the two arguments rely on the same principle.)

**Exercises.** (a) Prove $-1/n$ for connected solvable simply connected $n$-dimensional unimodular Lie groups $G$. Hint. Use a normal codimension one subgroup in $G$ for $H$.  

(b) Show that If the Haar measures of subset $X$ and $Y$ in a compact topological group $G$ satisfy $\text{mes}(X) + \text{mes}(Y) > \text{mes}(G)$, then $X \cdot Y = G$.

**Remark.** (a) There are similar results for infinite discrete groups $G$, e.g. Mann’s theorem for $G = \mathbb{Z}$, but the proofs of these are more subtle. (b) If $G$ is an infinite, e.g. discrete non-abelian group then it may satisfies much stronger isoperimetric and Minkowski type inequalities for sufficiently large and/or dense subsets as we shall see in section ???

**Projection Concavity Theorem.** A measure $\mu = \mu(x)dx$ on $\mathbb{R}^n$ with Borel measurable density $\mu(x)$, $x \in X$, is called $*^{1/\lambda}$-concave if the function $\mu(x)^{1/\lambda}$ is concave on the support of $\mu$.

**BRUNN’S THEOREM.** Let $\mu$ be a $*^{1/\lambda}$-concave measure on $\mathbb{R}^n$ with convex support $V \subset \mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a surjective linear map. Then the pushforward measure $\pi_*\mu$ on $\mathbb{R}^{n-k}$ is $*^{1/\lambda+k}$-concave.

**Example.** If $V \subset \mathbb{R}^n$ is a convex subset and $\mu(x)$ is constant on $V$, this says that the function

$$y \mapsto (\text{vol}_{n-1}(\pi^{-1}(y))^{1/\lambda}, y \in \mathbb{R}^{n-k},$$

is concave on the image $\pi(V) \subset \mathbb{R}^{n-k}$.

### 4.1 Needle Decomposition

Let a convex set $W \subset \mathbb{R}^n$ be divided by a hypersurface $Y \subset W$ into two parts,

$$W = W_1 \cup W_2, \quad W_1 \cap W_2 = Y = \partial_W W_1 = \partial_W W_2,$$

such that $\text{vol}(W_1)/\text{vol}W = \lambda \leq 1/2$.

**Payne-Weinberger "Needle" Inequality.**

$$\text{vol}_{n-1}(Y) \geq c_\lambda \frac{\text{vol}(W)}{\text{diam}(W)} \text{ for } c_\lambda = 1.$$  

**Remark.** This inequality is sharp, for instance if $W$ is a "thin" (think of a needle) cylinder,

$$W = V \times [0, 1], \quad V \subset \mathbb{R}^{n-1}, \quad \text{diam}(V) \leq \frac{1}{2\pi}.$$

---

32The Haar measure is biinvariant.

33See https://link.springer.com/article/10.1007/s00039-023-00647-6 for such an inequality for more general locally compact groups.

34$\partial_W$ stands for the topological boundary of a subset in $W$.
but if \( W \) has strictly convex boundaries, then (this will be clear with the proof below) \( c_\lambda > 1 \) for all \( \lambda \).

**Proof.** Cut \( W \) be by a hyperplanes \( H \subset \mathbb{R}^n \)

\[ W = W' \cup W'' \quad \text{such that} \quad H \text{ cuts the volumes of} \quad W_1 \quad \text{and} \quad W_2 \quad \text{in equal proportions, that is} \]

\[ W_i = W'_i \cup W''_i \quad \text{where} \quad \frac{\text{vol}(W'_i)}{\text{vol}(W''_i)} = \lambda = \frac{\text{vol}(W'_1)}{\text{vol}(W''_2)}, \]

and where we assume for the simplicity sake that \( H \) is transversal to \( Y \) and thus divides

\[ Y = Y' \cup Y'', \]

such that

\[ \text{vol}_{n-1}(Y') + \text{vol}_{n-1}(Y'') = \text{vol}_{n-1}(Y). \]

One knows in this regard that such a cut always exists. In fact an obvious continuity argument (the trivial case of the Borsuk-Ulam sandwich theorem) shows that there always exists a hyperplane \( H \), which divides \( W_1 \) and \( W_2 \) into equal parts, i.e. with \( \text{vol}(W'_i) = \text{vol}(W''_i) \).

Then observe that, given such a cut,

the lower (isoperimetric) bounds on \( Y' \) and \( Y'' \)

\[ \text{vol}_{n-1}(Y') \geq c_\lambda \frac{\text{vol}(W')}{\text{diam}(W)} \quad \text{and} \quad \text{vol}_{n-1}(Y'') \geq c_\lambda \frac{\text{vol}(W'')}{\text{diam}(W'')} \]

imply the same inequality for \( Y \),

\[ \text{vol}_{n-1}(Y) \geq c_\lambda \frac{\text{vol}(W)}{\text{diam}(W)}. \]

Keep cutting the resulting convex subsets into equal parts,

\[ W'_1 \xrightarrow{\text{cut}} \{(W'_1)', (W'_1)''\} \xrightarrow{\text{cut}} \{((W'_1)'), ((W'_1)'')', ((W'_1)'')'', ((W'_1)''')', ((W'_1)'''')'' \} \xrightarrow{\text{cut}} \ldots \]

\[ W''_1 \xrightarrow{\text{cut}} \{(W''_1)', (W''_1)''\} \xrightarrow{\text{cut}} \{((W''_1)'), ((W''_1)'')', ((W''_1)'')'', ((W''_1)''')', ((W''_1)'''')'' \} \xrightarrow{\text{cut}} \ldots \]

and thus reduce the inequality

\[ \text{vol}_{n-1}(Y) \geq c_\lambda \frac{\text{vol}(W)}{\text{diam}(W)}, \]

with the desirable constant \( c_\lambda = n \) to such inequalities for "infinitesimally narrow needle shaped" convex subsets \( W^\circ \subset \mathbb{R}^n \), where each such \( W^\circ \) is contained in the \( o(1) \)-neighbourhood \(^{35}\) of a straight segment \([x_1, x_2] \subset W\), and where the corre-\(^{35}\) This \( o(1) \) means \( \varepsilon \to 0 \)
sponding hypersurface $Y^\circ \subset W^\circ$ equals the intersection of $W^\circ$ with a hyperplane
normal to the segment \([x_1, x_2]\).\(^{36}\)

\(^{36}\)See http://www.math.tau.ac.il/~klartagb/papers/seoul_slides.pdf for more explanations and references.

4.2 Reparameterization, Knöte Map and Prekopa-Leindler inequality

4.3 Mass Transportation, Brenier Maps...

4.4 Alexandrov-Fenchel and Hodge Inequalities

4.5 Minkowski Inequality in Arakelov Geometry

5 Filling Inequalities

5.1 Non-Sharp Federer Fleming Filling Inequality in \(\mathbb{R}^N\)

???.A. Euclidean Filling-by-Collapsing with Codimension one Theorem. Let \(Y = Y^{n-1} \subset \mathbb{R}^N\) be a smooth submanifold.

Then there exist constants \(C_N\) and \(D_N\) and a smooth homotopy \(\Phi_t(y)\) of the embedding \(Y \rightarrow \mathbb{R}^N\), that is a smooth map

\[\Phi: Y \times [0, 1] \rightarrow \mathbb{R}^N, \quad \Phi_{y,0} = y,\]

with the following properties.

- \(\partial_t\) the partial t-derivative, of \(\Phi\) is bounded by:

\[|\partial_t\Phi| \leq D_N \text{vol}_n(X)\frac{1}{\text{vol}_n}\); \(\text{vol}_n\)

- The \(L_{n-1}\) norm of the differential of the map \(\Phi\) on the subset \(Y_n = \text{supp} \wedge^n df \subset Y\), where \(\text{rank}(\Phi) = n\) is \(C_n\)-bounded

\[\int_{Y_n} |df(y)|^{n-1}dy \leq C_N^{n-1}.

Consequently, the \(n\)-dimensional volume of the map \(\Phi^{37}\) satisfies

- \(\text{vol}_n(\Phi) \leq D_N C_N^{n-1} \text{vol}_{n-1}\).

- \(\text{vol}_n(\Phi) \leq D_N C_N^{n-1} \text{vol}_{n-1}\).

**Remark.** If \(N = n\) and \(Y \subset \mathbb{R}^n\) is closed hypersurface, then the above implies a non-sharp isoperimetric inequality, since the domain bounded by \(Y\) is, because of \(\text{vol}_n\), contained in the image of the (cylinder) map \(\Phi\).

(A natural candidate for a collapsing \(\Phi\)-like map. from the boundary \(\partial X\) of a closed hypersurface \(X \subset \mathbb{R}^n\) is the canonical retraction of \(X\) to the cut locus of \(Y = \partial X\) in \(X\), but there is no apparent simple geometric map, which would also satisfy \(\text{vol}_n\) (and or \(\text{vol}_{n-1}\)).

In fact, \(\text{vol}_n\) and \(\text{vol}_{n-1}\) show that closed submanifolds \(Y_{n-1} \subset \mathbb{R}^N\) (this effortlessly extends to quite general \((n-1)\)-cycles) bound "cylindrical" \(n\)-chains with volumes bounded by \(\text{const}_N\text{vol}_{n-1}\).

**Proof of ???.A.** Let \(N > n - 1\), the integral

\[\int_{[x]^{n-1}} \frac{1}{[x]^{n-1}} dx\]

converges at zero in \(\mathbb{R}^n\), the mean of the \(\text{dist}^{n-1}\)-function in the ball \(B^N(r) \subset \mathbb{R}^N\) satisfies,

\[\frac{1}{\text{vol}_N(B^N(r))} \int_{B^N(r)} |x-x_0|^{-(n-1)}dx \leq \text{const}_N/r^n\]

for all \(x_0 \in \mathbb{R}^N\).

It follows, that for all submanifolds (not necessarily of dimension \(n-1\)) in \(\mathbb{R}^N\), there exist points \(x \in B^N(r)\), such that

\[\int_Y \|y-x\|^{-(n-1)}dy \leq \frac{-\text{const}_N\text{vol}(Y)}{r^n}.

\]
Therefore, the radial projection $\psi_x$, $x \in B^N(r)$, of $Y = Y^{n-1}$ from such a point $x$ to the boundary of a convex subset $V \supset B^N(2r)$ satisfies:

$$\text{vol}_{n-1}(\psi_x(Y \cap V)) \leq \int_{Y \cap V} |d\psi(y)||y|^{n-1}dy \leq \frac{d(\text{diam}(Y) \cdot \text{const}'_N \text{vol}_{n-1}(Y \cap V))}{r^{N}} \cdot 38$$

Now, let us partition $\mathbb{R}^N \supset Y$ into parallel translates of the cube $[0,d]^N \subset \mathbb{R}^N$, where $d$ is much larger than $\text{vol}_{n-1}(Y)1/n-1$, say

$$d = (10N)^{10N^2} \text{vol}_{n-1}(Y)^{1/n-1}.$$  

Then radially project the intersections of $Y$ with these cubes, say $Y \cap [0,d]^N + x_i$, where $x_i$ are vectors in the lattice $d\mathbb{Z}^N \mathbb{R}^N$, to the boundaries of these cubes with a controlled increase of their volumes, say by factors $(2N)^N$.

Then apply the same to the images of these maps intersected with the $(N-1)$-faces of the cubes and continue until you land up in the $n-1$-faces.

Since the cubes were chosen so large, none of these maps is onto, hence their images can be radially projected to the $(n-2)$-dimensional boundaries of these faces.

Then composition of all these radial projections naturally included into a homotopy of maps satisfies $\phi_1 \cdot \phi_2^{-1} = \phi_3$ and $\phi_2^{-1}$. QED.

Questions. (a) What are the optimal constants $C_N$ and/or $D_N$ in $\phi_1 \cdot \phi_2^{-1}$ and $\phi_3$? Do they depend on $n$ rather then on $N$? (Compare with ???C below).

(b) Does ???A hold for submanifolds $Y$ in complete simply connected spaces with $\text{sect.curb} \leq 0$?

Does ???A hold with logarithmic bound on $\|\phi_2\|$ and on $\|\phi_2\|_{L_{n-1}}$ in symmetric spaces with $\text{sect.curb} \leq 0$ and with $\text{rank} \leq n - 1 = \dim(Y)$?

Does ???A hold in complete $N$-dimensional manifolds with $\text{sect.curb} \geq 0$ (and singular Alexandrov spaces) with volume growth $\text{const}R^N$?

\subsection{Slicing by Parallel "Planes"}

Let us show that an isoperimetric (filling) inequality for maps – we allow non-embeddings $Y \to Z$ but keep notation as if these are embedding – for manifolds (or cycles) of dimensions $n - 1$ and $n - 2$ in a Riemannian manifold $Z$, e.g. $Z = \mathbb{R}^{n-1}$, implies a similar inequality for maps $Y^{n-1} \to Z \times \mathbb{R}$.

In fact, given $Y \to Z \times \mathbb{R}$, the mean/mean value argument, implies, as earlier, that, for all $d$ there a "greed of parallel $Z$-planes within distance $d$ one from another", that is a subset $Z' = Z \times (d\mathbb{Z} + t)$. for some $t \in \mathbb{R}$, such that

$$\text{vol}_{n-2}(Y \cap Z') \leq \text{vol}_{n-1}(Y)/d.$$  

Let $d = \text{vol}_{n-1}(Y)^{1/n-1}$, let is fill in all intersections $Y \cap Z \times \{di + t\}$, $i \in Z$ by $(n-1)$-chains in the "planes" $Z_i = Z \times \{di + t\}$ and thus decompose the $(n-1)$-cycle (represented by) $Y$ into the sum $Y = \bigcup_i Y_i$, where each $Y_i$ is contained in the "$d$-band between two "planes",

$$Y_i \subset Z \times \{di + t, di + 1\} + t.$$  

Then, fill in all $Y_i$ by firstly normally projecting them to $Z_i = Z \times \{di + t\}$ and then filling them in these $Z_i = Z$.

Example. Let $Z = \mathbb{R}^2$ and $Y \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ be a surface with unit area. Then $d = 1$ and the total sum of the planar domains in all $Z_i$ encompassed by their intersections with $Y$ is $< 1/2$ by the (rough) 2-dimensional isoperimetric inequality. Then we project each $Y_i$ to $Z_i$ which needs $d$-area of filling 3-volume, while nothing is added in $Z_i$ since $\dim(Z_i) = \dim Y_i$ in the present case.

Thus we conclude that the domain $X \subset \mathbb{R}^3$ bounded by $Y$ satisfies

$$\text{vol}_3(X) \leq C_4 \text{area}(Y)^{2/3}$$  

with $C_4 = 2$ instead of $C_3 = \frac{4\pi}{6\sqrt{3}} = \frac{4\pi/3}{(4\pi/3)^{2/3}}$ of the sharp inequality.

Question. Does the Euclidean type filling inequality hold for $(n-1)$-cycles $Y$ in product manifolds $Z_1 \times Z_2$, where $Z_1$ and $Z_2$ are complete contractible Riemannian manifolds, such that $\text{sect.curb}(Z_1) \leq 0, \text{sect.curb}(Z_2) \geq 0$ and where $Z_2$ has (at least) $\text{const}_2 \text{R}^{\dim(Z_2)}$ volume growth?

Remark. The $(n-1)$-cycles $Y \subset Z_1$, are known (see ???) and section ???) to bound chains $X$ with

$$\text{vol}_{n-1}(X) \leq \text{const}_1 \text{vol}_{n-1}(Y)^{n/n+1}, \ n_1 = 2, 3, ...$$
Proof. Let \( B_0^+ \subset V \) be as above, apply Vitali’s lemma to the covering of \( V \) by \( Y^{n_2-1} \subset Z_2 \) bound \( X \) with
\[
\text{vol}_{n_2}(X) \leq \frac{\text{const}_{n_2}}{\text{const}_{n_2}^2 N} \text{vol}_{n_1} - \frac{n_2^2}{n_2}, \quad n_2 = 2, 3, \ldots,
\]
(see section ?? in [??]).

Such inequalities with \( n_1 = \dim Z_1 \) and \( n_2 = \dim Z_2 \) imply the corresponding Euclidean-type isoperimetric inequality for hypersurfaces in \( Z_1 \times Z_2 \) by the (formal) Schwarz symmetrization argument (see section ?? below, [grigorian???95], section 9 in [waists???2003], [morgan???2006] and references therein) but a similar product property is unlikely to hold, in general, for cycles of higher codimension.

However, this may work in the (annoyingly eclectic) \( \{\text{curve} \leq 0\} \times \{\text{curve} \geq 0\} \) case.

5.2 Filling Lipschitz Cycles in Riemannian Manifolds

Let \( X \) be a Riemannian manifold and
\[
\sigma = (C_0(X, F, \partial)) = ((\partial_i : C_i \rightarrow C_{i-1})_{i=0, 1, \ldots, n=\dim(X)}
\]
be the complex of Lipschitz singular chains: the \( i \)-chains are finite sums \( c = \sum_j f_j \sigma_j \), where \( \sigma_j : \Delta^i \rightarrow X \) are Lipschitz maps of the standard \( i \)-simplex to \( X \) and \( f_j \in F \).

If \( F \) comes with a norm (e.g. \( F \) equals \( \mathbb{R} \), \( \mathbb{Z} \), or \( Z_2 = Z/2Z \)), then each chain \( c \in C^i \) is given the Riemannian \( i \)-volume norm, \( \text{vol}_i(c) = \|c\|_{\text{vol}_i} = \sum_j \|f_j\|_{\text{vol}_i}(\sigma_j) \).

Clearly \( \|\partial_i\| = \infty \) for all \( i \geq 2 \) (and also for \( i = 1 \) if the norm in \( F \) is unbounded). The corresponding minimal norm on the homology \( H_i(X) = \ker \partial_i/\ker \partial_{i+1} \), that is the infimum of the volumes of the cycles \( c \in \ker \partial_{i+1} \) representing an \( h \in H_i(X) \) is called the volume or the mass norm,
\[
\text{vol}_i(h) = \inf_{h_{\neq 0}} \text{vol}_i(h), \quad h \in H_i(X).
\]

5.2.1 Riemannian Federer-Fleming

A minor modification of Federer-Fleming’s “filling-by-collapsing” argument from section ?? yields the following general filling inequality for “small” Lipschitz chains in all Riemannian manifolds with bounded Lipschitz geometries.

Let \( X \) be a Riemannian manifold and
\[
\sigma = (C_0(X, F, \partial)) = ((\partial_i : C_i \rightarrow C_{i-1})_{i=0, 1, \ldots, N=\dim(X)}
\]
be the complex of Lipschitz singular chains: the \( i \)-chains are finite sums \( c = \sum_j f_j \sigma_j \), where \( \sigma_j : \Delta^i \rightarrow X \) are Lipschitz maps of the standard \( i \)-simplex to \( X \) and \( f_j \in F \).

If \( F \) comes with a norm (e.g. \( F \) equals \( \mathbb{R} \), \( \mathbb{Z} \), or \( Z_2 = Z/2Z \)), then each chain \( c \in C^i \) is given the Riemannian \( i \)-volume norm, \( \text{vol}_i(c) = \|c\|_{\text{vol}_i} = \sum_j \|f_j\|_{\text{vol}_i}(\sigma_j) \).

Clearly \( \|\partial_i\| = \infty \) for all \( i \geq 2 \) (and also for \( i = 1 \) if the norm in \( F \) is unbounded). The corresponding minimal norm on the homology \( H_i(X) = \ker \partial_i/\ker \partial_{i+1} \), that is the infimum of the volumes of the cycles \( c \in \ker \partial_{i+1} \) representing an \( h \in H_i(X) \) is called the volume or the mass norm,
\[
\text{vol}_i(h) = \inf_{h_{\neq 0}} \text{vol}_i(h), \quad h \in H_i(X).
\]

An \( N \)-dimensional Riemannian manifold (possibly with a boundary) \( X \) is \( \text{bLg} \) (bounded Lipschitz geometry) if there exist numbers \( r_0 > 0 \) and \( \lambda > 0 \), such that all ball \( B(r) \subset X \), \( r \leq r_0 \) admit \( \lambda \)-bi-Lipschitz embeddings \( B(r) \rightarrow \mathbb{R}^N \).

For instance compact manifolds and those admitting co-compact isometry groups are \( \text{bLg} \). Exercise. Show that the mass/volume norm does not vanish in the \( \text{bLg} \) manifolds: \( \inf_{h_{\neq 0}} \text{vol}_i(h) > 0 \), for \( h \neq 0 \).
??? A. Theorem Let $X$ be an $N$-dimensional Riemannian $bLg$ manifold. Then there exists a positive constant $\beta_0 = \beta_0(X) > 0$, such that if $n - 1 \geq 1$, then every $(n - 1)$-cycle $b$ in $X$ with $\|b\| \leq \beta_0$ satisfies
\[ \|b\|_{f_1} \leq \text{const} \|b\|^\alpha, \text{ where } \text{const} = \text{const}(N, r_0, \lambda), \]
where the $n$-fillings $c_b$, $\varepsilon > 0$ of a $(n - 1)$-cycle $b$, i.e. chains $\partial c_b$, such that $\partial c_b = b$ and $\|[c]\| \leq \text{const} \|b\|^\alpha = \varepsilon$, $\varepsilon \to 0$, are (implemented by) the cylinders of Lipschitz map $\Phi_b$ from the support of $b$ to $X$, such that $\text{dist}(x, \Phi_b(x)) \leq \text{const} + \varepsilon$ for all $x$ in the support of $b$ and such that the rank of the differential of $\Phi_b$ is almost everywhere $\leq n - 2$.

If, moreover, $X$ is compact, then all $b$ homologous to zero satisfy
\[ \|b\|_{f_1} \leq \text{const} X \|b\|. \]

mostly move to ??? From const$(N)$ to const$(n)$. A Riemannian manifold $X$ is unlc (uniformly locally Lipschitz contractible) if there exist positive constants constants $r_0 > 0$ and $\lambda \geq 0$ such that all balls $B(r) = B_x(r) \subset X$, $x \in X$, with $r \leq r_0$ admit $\lambda$-Lipschitz homotopies $H_t$, $t \in [0, 1]$ to points in $X$:

these homotopies are $\lambda$-Lipschitz maps $H : B(r) \times [0, 1] \to X$, such that $H_0 : B(r) \to X$ are equal to the original embeddings $B(r) \to X$ and $H_1$ are constant maps, $B(r) \to x^0(x) = H_1(B_x(r))$.

??? B. Theorem. Let $X$ be an $Unlc$ Riemannian manifold. Then there exists a positive constant $\beta_0 = \beta_0(X) > 0$, such that if $n - 1 \geq 1$, then every $(n - 1)$-cycle $b$ in $X$ with $\|b\| \leq \beta_0$ satisfies
\[ \|b\|_{f_1} \leq \text{const} \|b\|^\alpha, \text{ where } \text{const} = \text{const}(n, r_0, \lambda), \]
If, moreover, $X$ is compact, then all $b$ homologous to zero satisfy
\[ \|b\|_{f_1} \leq \text{const} X \|b\|. \]

Remark. Unlike the above theorem ???A, it is unclear if there are $n$-fillings $c$ of $b$, which can be implemented by the cylinders of $\text{dim}(b)$-controllably Lipschitz homotopies $\Phi_b$ with $\text{dist}(x, \Phi_b(x)) \leq \text{const}(n) \|b\|^{1/n - 1}$ and with the ranks of the differentials of $\Phi_b$ almost everywhere bounded by $n - 2$.

5.3 Vitali Decomposition of Submanifolds and Measures into $\varepsilon$-Round Pacies

???A. Vitali Covering Lemma. Let a metric space $X$, e.g. $X \subset \mathbb{R}^n$, be covered by finitely many subsets $B_i \subset X$, $i \in I$, of diameters $\delta_i$ (e.g. by balls of radii $r_i = \delta_i/2$). Then there exists a subset $J \subset I$ such that the sets $B_j$, $j \in J$, do not pairwise intersect and such that the closed $\delta_i$-neighbourhoods $U_{\delta_i}(B_j)$ (e.g. the concentric balls $B(B_j)$ in the case of $r_j$-balls) cover $X$.

Proof. Let $B_1$ be the subset with the largest diameter, let $B_2$ be the largest subset which doesn’t intersect $B_1$, let $B_3$ be the largest one, which doesn’t intersect $B_1 \cup B_2$, and, in general, let $B_{i+1}$ be the largest subset, among $B_i$, which don’t intersect the union $B_1 \cup \ldots \cup B_j$.

Since each $B_i$ intersects some $B_j$ with $\delta_j = \text{diam}(B_j) \geq \text{diam}(B_i)$
\[ U_{\delta_j}(B_j) \supset B_i, \]
and the proof follows.

Corollary "Round" (Quasi) Decomposition of Measures and Submanifolds.

Let $V$ be a metric space, e.g. $V \subset \mathbb{R}^N$ with the Euclidean metric (distance function)) and $\mu$ be a Borel measure in $v$, e.g. the $n$-volume measure on a smooth $n$-dimensional submanifold in $V \subset \mathbb{R}^N$ and let $\mu(v, r, \varepsilon) \geq 0$, be a positive monotone increasing function, e.g. $\mu(v) = \varepsilon r^n$.

A Borel (e.g. closed) subset $B \subset V$ is called $\varepsilon$-round with respect to $\mu$ if
\[ \mu(V) \geq \varepsilon \cdot \text{diam}(V). \]

More specifically an $n$-dimensional submanifold $V \subset X$ is called $\varepsilon$-round if
\[ \mu(V) \geq \varepsilon \cdot \text{diam}(V)^n \]
For instance the Euclidean $n$-balls $B^n(r)$ in $\mathbb{R}^n$ are $\varepsilon_n$-round with $\varepsilon_n > n^{-n}$, while the cylinders $B^{n-1}(r) \times [0, R]$ are $\varepsilon(n, r, R)$-round with $\varepsilon(n, r, R) \to 0$ for $R/r \to \infty$ as well as for $R/r \to 0$.

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$U_j = U_d(B'_j)$, for $d_j = \text{diam}(B'_j)$ and let $U_k = U_d(B'_k) \subset V$ be Vitali’s disjoint subsets, which, observe, cover at least $\text{vol}_{n}(V)/3^n$.

Then let $\rho_k(v)$ be the distance functions on $U_k$,

$$\rho_k(v) = \text{dist}(v, B'_k).$$

let $S^+_k = \rho^{-1}(r_k) \subset U_k$ be the level of the function $\rho_k$, which minimizes $\text{vol}_{n-1}(\rho^{-1}(r))$ for $0 \leq r \leq d_k = \text{diam}(B'_k)$ and observe that the subsets $B_k^+ = \rho_k^{-1}[0, r_k] \subset U_k \subset V, k \in K \subset J,$

satisfy the requirements of ???.

5.3.1 From $\text{const}(\mathbb{N})$ to $\text{const}(\mathbb{n})$ by Cutting off Bubbles on Narrow Necks.

Let $Z$ be a Riemannian manifold and $A, B, d$ be positive numbers, such that

- $A$ $(n - 2)$-cycles $Y'$ in a Riemannian manifold $Z$ bound $(n-1)$-chains $X'$ with volumes

$$\text{Vol}_{n-2} \quad \text{vol}(X') \leq A \cdot \text{vol}_{n-1}(Y')^{\frac{n-1}{n}};$$

- $B$ $(n-1)$-cycles $Y$ with diameters $D$, bounds chains $X$ in $Z$ with

$$D \cdot \text{vol}_{n-1} \quad \text{vol}(X) \leq BD \cdot \text{vol}(Y)^{n/n-1};$$

- $d$ $(n-1)$-cycles $Y$ in the $\delta$-balls $B'_\delta \subset Z$ for $\delta \leq d$ bound $n$-chains $X \subset B'_\delta$ of volumes

$$\delta \cdot \text{Vol}_{n-1} \quad \text{vol}_n(X) \leq B \delta \cdot \text{vol}(Y)^{n/n-1}$$

???. Let $V$ be covered by finitely many $\nu$-round subsets $B_i \subset V, i \in I$. If

$$\nu(3r) \leq C \nu(r) \text{ for all } r > 0 \text{ and some } C > 0,$$

(e.g. $\nu(r) = \varepsilon r^n$, where $C = 3^n$). Then there exist finitely many disjoint $\nu$-round subsets $B'_j \subset V, j \in J$, which contain a "substantial amount" of the measure of $V$, that is

$$\mu\left( \bigcup_{j \in J} B'_j \right) \geq \frac{\mu(V)}{C}.$$

Proof. Let $\rho^*_i \geq 0$ be the maximum of the numbers $\rho \geq 0$ such that the $\rho$-neighbourhood $U_\rho(B_i) \subset V$ is $\nu$-round and apply Vitali’s lemma to the covering of $V$ by $B'_i = U_{\rho^*_i}(B_i), i \in I.$ ???. This argument, I guess, is used everywhere in analysis; I learned it from the geometric paper [We??].

???. Let $X$ be a Riemannian Manifold $V \subset X$ a compact $n$-dimensional submanifold with a (possibly empty) boundary $\partial V$ let $\mu = \text{meas}$ be the Riemannian measure of $V$, let $\nu(r) = \varepsilon r^n$ and let $B_i \subset X$ be a covering of $V$ by $\nu$-round subsets.

Then there exist finitely many disjoint $\nu(\theta)\theta^n$-round $n$-dimensional submanifolds $B'^{+}_k \subset V, k \in K$, such that

$$\mu\left( \bigcup_{i \in I} B'^*_i \right) = \text{vol}_n\left( \bigcup_{k \in K} B'^*_k \right) \geq \frac{\mu(V)}{9^n} = \frac{\text{vol}_n(V)}{9^n}$$

and such that the $(n - 1)$-dimensional volumes off the boundaries of $B'^*_k \subset V$ satisfy

$$\text{vol}_{n-1}(\partial V_B'^*_k) \leq \frac{2^n \text{vol}_n(B'^*_k)}{\text{diam}(B'^*_k)}.$$
and \((n-2)\)-cycles \(Y' \subset B_z(\delta)\) bound \((n-1)\)-chains \(X' \subset B_z(\delta)\) of volumes
\[ \delta \cdot Vol_{n-2} \]
\[ vol_{n-1}(X) \leq B\delta \cdot vol(Y)^{n-1/n-2} \]

Then there exists a constant \(C\), which depends only on \(n, A, B\) (but not on \(d\)), such that \((n-1)\)-cycles in \(Z\) bound chains \(X\) with
\[ vol_n(X) \leq vol_{n-1}(Y)^{n/n-1}. \]

Remark. The condition \(\bullet_d\) with some \(d > 0\) is satisfied by all compact manifolds \(Z\) and also non-compact complete ones with locally bounded geometries.

Idea of the Proof. Property \(\bullet_A\) allows a decomposition of \(Y\) into a sum \(Y = Y_1 + Y_2\), such that \(\bullet_B\) applies to ("connected components" of) \(Y_D\) and where \(vol(Y_2) \leq (1 - \epsilon)vol(Y_1)\). Then this applies to \(Y_2\), etc and eventually reduces the problem to cycles of volumes \(< d^{n-1}\), where \(\bullet_d\) leads to termination of the iteration process.

5.3.2 References


[Weng2003] Stefan Wenger, Isoperimetric inequalities of euclidean type in metric

[Weng2007] Stefan Wenger, A short proof of Gromov’s filling inequality

Questions.
(Hyperbolic case? sharp constant \(\pi/2\)?
2. proof induction with hyperplanes.
3. Induction with spheres, Banach spaces

5.4 Dehn-Levy-Almgren Local-to-Global Argument

Here we are mostly concerned with explicite bounds on the constants \(const_n\) and \(const_X\) for ‘simple’ manifolds and we start with

Sharp evaluation of \(|\partial_{t+1}^*\|_{\text{rand}}\) for round spheres. If \(X\) is the round Euclidean sphere \(S^n\) with the \(O(n+1)\)-invariant \(i\)-volumes normalized such that the equatorial spheres \(S^i \subset S^n\) have volume 1, then the average \((i+1)\)-volume of the geodesic cones from the points \(s \in S^N\) over an equator \(S^i \subset S^n\) (with respect to every probability measure on \(S^n\)), obviously, equals \(1/2\). Since the group \(O(n+1)\) is transitive on the set of tangent \(i\)-planes in \(S^n\), the equality \(\|S^i\|_{\text{rand}} = vol(S^i)/2\) for the averages with respect to the \(O(n+1)\)-invariant measure on \(S^n\) implies that \(||c||_{\text{rand}} = vol(c)/2\) for all \(i\)-chains \(c\), hence
\[ ||\partial_{i+1}^{-1}||_{\text{rand}}(\beta) = 1/2 \text{ for all } i = 1, 2, ..., n-1 \text{ and } \beta \geq 0. \]

Notice that the resulting bound \[ \|\partial_{i+1}^{-1}\|_{\text{int}}(\beta) \leq 1/2 \] is not sharp unless \( \beta = 1 \), but, as one everybody would guess,

If \( X \) is either \( \mathbb{R}^n \) or \( S^n \), then round (umbilical) \( i \)-spheres of volume \( v_0 \) (filled-in by flat \( (i+1) \)-discs) have maximal filling volumes (i.e. \[ \|...\|_{\text{int}} \]) among all \( i \)-cycles \( b \) in \( X \), with \( \text{vol}(b) = v_0 \).

If \( X = \mathbb{R}^n \) this is due to Almgren [?] and the spherical case was reduced to \( \mathbb{R}^{n+1} \supset S^n \) by Bruce Kleiner (private communication.)

Remarks. If \( X = S^n \), this leaves open the sharp bound on \( \|\partial_{i+1}^{-1}\|_{\text{int}}(\beta) \) for \( \beta > 1 \) that may depend on \( n \) (if \( n > i + 1 \)) and on the coefficient field \( \mathbb{F} \) in a rather complicated manner.

Also, the "filling extremality" of round \( i \)-spheres (filled-in by flat \( (i+1) \)-discs) remains unproven in the hyperbolic spaces \( X \); but the Almgren-Levy argument provides rather sharp bounds on the filling volume of cycles in manifolds with lower bounds on curvatures (see below).

Local-to-Global Variational Principle: \[ \|...\|_{\text{int}} \leq \|...\|_{\text{loc}} \]. Let \((C_*, \partial_* \mathcal{H})\) be normed chain complex, \( h \in \mathcal{H}(C_*) \) a homology class and let \( B = B_r(h) \subset C_\ast \) be the space of \( i \)-cycles in the class of \( h \) with the filling metric \( \text{dist}_B(b_1, b_2) = \|b_1 - b_2\|_{\text{int}} \). Define the supremum norm of the "downstream gradient" of the function \( b \mapsto \|b\| \) on \( B \) as follows,

\[ \|b\|_{\sup} = \limsup_{||c|| \to 0} \frac{||b| - ||b + \partial_{i+1}(c)||}{||c||} \text{ for } c \in C_{i+1} \setminus \{0\}. \]

Observe that this norm on smooth submanifolds \( Y \) representing cycles \( b \) in Riemannian manifolds \( X \) equals the supremum of the norm of the mean curvatures of \( Y \), denoted \( \text{sup}_y \|\mathfrak{M}(Y)\|_{\gamma} \).

Let \( m(v) = \inf_{\|b\| = \beta} \|\mathfrak{M}(b)v\|_{\sup} \), define

\[ \|\partial_{i+1}^{-1}\|_{\text{int}}(\beta) = \int_{0}^{\beta} v \cdot m^{-1}(v)dv \]

and say that \( \partial_{i+1} \) satisfies local-to-global principle if

\[ \|\partial_{i+1}^{-1}\|_{\text{int}} \leq \|\partial_{i+1}^{-1}\|_{\text{loc}}. \]

This would hold if we had a gradient flow \( b(v) \) in \( B \) parameterized by \( v = \|b(v)\| \) starting from \( b \) with \( \|b\| = \beta \) and terminating with \( b = 0 \). One can not, in general, guarantee such flows; yet,

if \( C_\ast \) is the complex of Lipschitz chains in a smooth Riemannian manifold \( X \) with (possibly empty) \( i \)-mean convex boundary \( \partial X \) (i.e. where the traces of the second fundamental form are non-negative on all tangent \( i \)-planes in \( \partial X \)), then

\[ \|\partial_{i+1}^{-1}\|_{\text{int}} \leq \|\partial_{i+1}^{-1}\|_{\text{loc}} \]

for all \( i = 1, 2, ... \).

In fact, there is the following better bound. Call a compact \( i \)-dimensional subvariety (rectifiable set) \( Y \) in \( X \) \emph{quasiregular} if the subset \( \text{reg}(Y) \subset Y \) of \( C^2 \)-smooth
points has full $i$-measure in $Y$ and such that the function $d_2(y) = dist_X(x, y)$ assumes its minimum in $Y$ at a regular point $y \in \text{reg}(Y) \subset Y$ for almost all $x \in X$. Set

$$m_{\text{reg}}(v) = \inf_{[Y]=v} \sup_{y \in [Y]} \|M(Y)\|_y \text{ (this is } \geq m(v))$$

where the supremum is taken over all compact quasiregular $i$-dimensional subvarieties $Y$ in $X$ with $[Y] = \text{df} \ vol_i(Y) = v$, where $\|M\|_y$ denotes the norms of the mean curvature vectors at regular points $y \in \text{reg}(Y)$ and where the supremum is taken over all $y \in \text{reg}(Y)$. Define

$$\|\partial_{i+1}^{-1}\|_{f_{i+1}}^r(\beta) = \int_0^\beta v \cdot m_{\text{reg}}^{-1}(v) dv \text{ (this is } \leq \|\partial_{i+1}^{-1}\|_{f_{i+1}}^r(\beta))$$

and conclude, appealing to the geometric measure theory, to the (intuitively obvious but technically non-trivial)

Local-to-Global Inequality. ————-

$$\|\partial_{i+1}^{-1}\|_{f_{i+1}} \leq \|\partial_{i+1}^{-1}\|_{f_{i+1}}^r$$

for all $i = 1, 2, \ldots$.

Remark. The local-to-global principle is ubiquitous in the geometric measure theory, albeit it is rarely stated explicitly (see [? and references therein). It holds for complete non-compact $X$ with "decent" behavior at infinity e.g. for $\varepsilon$-locally contractible $X$ with some $\varepsilon > 0$ where every ball of radius $\varepsilon$ is contractible in the concentric unit ball. (This principle seems to hold for many classes of non-Riemannian $X$, e.g. for Alexandrov spaces with a lower bound on curvatures and for smooth strictly locally convex Finsler spaces.)

Example: Dehn’s Lemma. Let $X$ admit a family of properly immersed cooriented smooth hypersurfaces $S_r$, $r > 0$, such that

(a) the $i$-mean curvatures $M_{i-1}(S_r)$ of all $S_r$, i.e. the traces of the restrictions of the second fundamental form of $S_r$ to all tangent $i$-planes to $S_r$, are bounded from below by a positive constant $m_0$.

(b) There exists a locally compact space $\tilde{X}$, a proper continuous function $p: \tilde{X} \to X$ and a continuous function $f: \tilde{X} \to \mathbb{R}_{+}$, such that

(1) $p^{-1}(S_r) = f^{-1}(r)$ for all $r > 0$;

(2) the map $p$ properly embeds the $0$-level $f^{-1}(0) \subset \tilde{X}$ to $X$, where the image is a rectifiable set and where either $\text{dim}(f^{-1}(0)) \leq i-1$ (e.g. $f^{-1}(0)$ is empty) or $f^{-1}(0)$ is contractible of dimension $i$.

Then every quasiregular $Y$, that is not contained in $p(f^{-1}(0))$, (obviously) has $\sup_y \|M(Y, y)\| \geq m_0$; hence,

$$\|\partial_{i+1}^{-1}\|_{f_{i+1}}(\beta) \leq m_{0}^{-1}$$

for all $\beta \geq 0$.

Example. The concentric $r$-spheres $S_r$ in the hyperbolic $n$-space $X$, and in every complete simply connected manifold of curvature $\leq -1$, have $M_i(S_r) \geq i$; thus, $\|\partial_{i+1}^{-1}\|_{f_{i+1}}(\beta) \leq i^{-1}$ for all $i \geq 1$ in these $X$.

Remark. The Dehn inequality is never sharp (at least in the natural examples) and the true value of $\|\partial_{i+1}^{-1}\|_{f_{i+1}}$ remains unknown in most cases, even in the hyperbolic $n$-space for $3 \leq i \leq n-1$.
5.5 Tube Formulas and Generalized Levy-Almgren Inequalities move tubes to an appendix

Given a submanifold $Y \subset X$, possibly with a boundary $\partial Y$, denote by $U_{\varepsilon}^Y(Y) \subset X$ the subset of those $x \in X$ for which $\text{dist}_X(x, Y) \leq R$ and such that all distance minimizing segments $[x, y] \subset X$ (of lengths $= \text{dist}(x, Y)$) have their $Y$-ends $y$ in the interior $Y \setminus \partial Y$ of $Y$.

Observe that if $Y$ has no boundary, then $U_{\varepsilon}^Y(Y)$ equals the $R$-neighbourhood $U_R(Y)$ and, thus, $\text{vol}_n(U_{\varepsilon}^Y(Y)) = \text{vol}_n(U_R(Y)) \geq \text{vol}_n(U_{\varepsilon}(y))$ for $n = \dim(X)$ and all $y \in Y$ (where $U_{\varepsilon}(y)$ is the $r$-ball around $y$).

If $X = X^n(\kappa)$ is the complete simply connected $n$-dimensional manifold of constant sectional curvature $\kappa$ and $Y = Y^i(M)$ is a round (umbilical) $i$-sphere of mean curvature $m$, let $\text{VL}_i(R; m, \kappa) = \text{vol}_n(U_R(Y))/\text{vol}(Y)$.

The Hermann Weyl tube formula implies that every quasiregular $Y = Y^i \subset X^n(\kappa)$ with $\text{supp}_{\text{reg}}(Y) \|M(Y)\|_y \leq m$ has

$$\text{vol}_n(U_R(Y) \leq \text{VL}_i(R, m, \kappa) \cdot \text{vol}(Y).$$

If $\kappa = 1$ and $R = 2$ this, combined with the local-to-global-principle, immediately yields the Almgren-Kleiner result on filling extremality of round submanifolds in $S^n$ and a similar filling inequality in $X = S^n/G$ for finite isometry groups $G$ (fixed points are allowed) of order $|G|:$

the filling norm in $X$ is bounded by that in $S^n$ as follows:

$$\|\partial_{X^n}^1 \|_{\text{fil}}(\beta) \leq \|\partial_{S^n}^1 \|_{\text{fil}}(|G| \cdot \beta) \text{ for all } \beta \leq \text{vol}_i(S^n)/|G|.$$

If $\kappa = 0$ this yields, with $R \to \infty$, Almgren’s sharp filling inequality in $\mathbb{R}^n$ and similar inequalities in the quotient spaces $\mathbb{R}^n/G$.

Question. Does this bound on $\|\|_{\text{fil}}$ remain valid if $\text{vol}_{i+1}(b)$ is substituted by the measure of the set $A_{\text{reg}}(b) \subset A_{\text{fin}}_{i+1}(\mathbb{R}^n)$ of the $(n-i+1)$-dimensional affine subspaces $\gamma$ in $\mathbb{R}^n$ that intersect $b$? (See the last three lines in 5.7 of [?] for such bound for hypersurfaces.)

Let $X$ be an $n$-dimensional Riemannian manifold, and $V_\varepsilon = V_{\varepsilon}^i \subset X$ an $\varepsilon$-germ of a smooth $i$-submanifold at a point $v_0 \in X$.

The normal $R$-tube around $V_\varepsilon$, denoted $V_\varepsilon + R \subset X$, is the union of the geodesic segments $\gamma = [v, x] \subset X$ normal to $V_\varepsilon$, such that, every $\gamma$ has length $\gamma \leq R$ and such that no point in $\gamma$ is focal with respect to $V_\varepsilon$, (which is essentially equivalent to $\text{dist}(x, V_\varepsilon) = \text{dist}(x, v) = \text{length}(\gamma)$).

If $\varepsilon \to 0$, then the volume of the tube depends on the second jet of $V_\varepsilon$ at $v_0$,

$$\text{vol}_n(V_\varepsilon + R) = \text{vol}_n(V_\varepsilon) \cdot \text{VL}_i(X, R, v_0, \tau_0, K_0) + o(\text{vol}_n(V_\varepsilon))$$

where $\tau$ is the tangent space to $V_\varepsilon$ at $v_0$ and $K_0$ the second fundamental form of $V_\varepsilon$ at $v_0$.

Denote by $MVL_i(X, R, m)$ the supremum of $\text{VL}_i(X, R, u_0, \tau_0, K_0)$ over all $K_0$ with the norm of the trace (mean curvature) bounded by $m$, over all tangent $i$-planes $\tau$ at $v_0$ and over all $v_0 \in X$.

For instance, if $X$ is a complete simply connected space of curvature $\kappa$ then $MVL_i(X, R, m, \kappa)$ equals the above $\text{VL}_i(R, m, \kappa)$, since the supremum of $\text{VL}_i(X, R, u_0, \tau_0, K_0)$ is assumed on umbilic submanifolds $V_\varepsilon$. 47
Clearly, every quasiregular $Y$ with mean curvature bounded by $m$ satisfies

$$\text{vol}_i(Y)/\text{vol}_n(X) \geq \text{MVL}_i(X, R, m)$$

for all $R \geq \text{diam}(X)$ and so an upper bound on MVL can be used in a conjunction with the local-to-global-principle same way as the Weyl tube formula.

On the other hand, the function MVL can be evaluated in a variety of cases. This provides lower bounds on the volumes of quasiregular $Y \subset X$ in terms of the mean curvature, namely $\sup_{\gamma} |M(Y)|_g$. However, the such bounds are sharp only in rather special cases.

**Examples.** (A) If $X$ is a symmetric space (where the equation for Jacby fields along geodesics $\gamma$ in $X$ satisfy a linear ODE-system with constant coefficients) it satisfies a Weil type formula and MVL is, in principle computable. The resulting lower bound on $\text{vol}(Y)$ by $\sup_{\gamma} |M(Y)|_g$ is sharp for manifolds of constant curvature.

Also, if $X = CP^k$ is a complex projective space and $Y$ is minimal, i.e. $M(Y) = 0$, then the above indicated bound is sharp: every $2j$-diminsional quasiregular $Y \subset CP^k$ has $\text{vol}_{2j}(Y) \geq \text{vol}_{2j}(CP^k)$.

However, the corresponding sharp bound is unknown for odd dimensional $Y$. For instance, if $Y \subset CP^k$ is a hypersurface, $\text{dim}(Y) = 2k - 1$, one expects that its volume is bounded from below by the volume of some homogeneous $Y_0 \subset CP^k$ with $\sup_{\gamma} |M(Y_0)| = \sup_{\gamma} |M(Y)|_g$, where “homogeneous” means that the isometry group of $CP^k$ preserving $Y_0$ is transitive on $Y_0$.

On the other hand, if, for instance, $M(Y) = 0$ (i.e. $M$ is minimal), then a potential $Y_0$ with minimal volume guarantied by the tube formula would be a totally geodesic submanifold. But there is no odd dimensional totally geodesic $Y_0 \subset CP^k$ for $\text{dim}(Y_0) > 1$.

It follows by compactness argument, that, there is a non-zero correction term to the lower bound on $\text{vol}_i(Y)$ improving the bound with the tube formula, but it still leaves far from a sharp bound.

Also, one does not know if domains $U \subset CP^k$ solving the isoperimetric problem have homogeneous boundaries, unless they have (very) small volumes.

(B) If $Ricci(X) \geq (n - 1)\kappa$ and $i = n - 1$, then

$$\text{MVL}_i(X, R, m) \leq Vl_i(R, m, \kappa),$$

by the Paul Levy tube bound.

(C) If $X$ is a Riemannian manifold where the sectional curvatures are bounded from below by $\kappa$, then

$$\text{MVL}_i(X, R, m) \leq Vl_i(R, m, \kappa),$$

for all $i$ by Buyalo-Heintze-Karcher comparison theorem [7], [7]

(D) If $Ricci(X) \geq (n - 1)/\rho^2 > 0$, then, $\text{MVL}_i(X, R, m) \leq \text{MVL}_i(X, \rho R, m)$ for all $R$. and if, moreover, $\text{curv}(X) \geq \kappa$, then $\text{MVL}_i(X, R, m) \leq Vl_i(\rho R, m\kappa)$.

These, together with the local-to-global principle, provide bounds on the filling volumes in $X$, e.g. as follows.

Let $X = X^n$ be a complete no-compact Riemannian manifold with sectional curvature $\geq 0$ and let the $R$-balls around some (and, hence each) point satisfy,

$$\limsup_{R \to \infty} \text{vol}_n(B(R; X))/R^n \geq c \cdot \text{vol}_n(B(1; \mathbb{R}^n)).$$
Then, for each \( i = 1, 2, ..., \dim(X) - 1 \), every \( i \)-cycle of volume \( c \cdot \alpha \) bounds an \((i + 1)\)-chain of volume \( c \cdot \beta \) where \( \beta = \beta(\alpha) \) equals the volume of the \((i + 1)\)-dimensional Euclidean ball \( B \) with \( \vol(\partial B) = \alpha \). Furthermore, if \( i = \dim(X) - 1 \), then the condition \( \curv \geq 0 \) can be relaxed to \( \Ricci \geq 0 \).

Remarks and Questions. (a) If \( X = \mathbb{R}^n \) this reduces to Almgren’s inequality.

(b) If the sectional curvatures of \( X \) are bounded from below by \( 1 \), then the corresponding filling inequality generalizes that of Kleiner for \( S^n \), where the case \( i = n - 1 \) and \( \Ricci \geq n - 1 \) goes back to Paul Levy.

(c) Does every quasi-regular \( Y^i \) in a complete simply connected manifold \( X \) with non-positive curvature (or in any \( CAT(0) \) space for this matter) and with \( \Ricci_{i+1}(X) \leq -1 \) have \( \sup_{\text{perg}(Y)} \|M(Y)\|_y \) greater or equal than the mean curvature of a round (umbilical) sphere \( S^i \) with \( \vol_i(S^i) = \vol_i(Y) \) in the hyperbolic space of constant curvature \(-1\)? (The lower bound on \( \sup_{\text{perg}(Y)} \|M(Y)\|_y \) and the issuing bound the filling inequality issuing from Weyl’s formula are non-sharp in the hyperbolic spaces of constant curvature \( \kappa < 0 \).)

(b) Can one "hybridize" Dehn’s and Almgren’s inequalities, e.g. for Cartesian products of manifolds of positive and of negative curvatures?

(c) If \( X = X^\infty \) is an infinite dimensional Riemannian manifold that densely and isometrically contains an increasing union of finite dimensional submanifolds, \( X^\infty \supset \cdots \supset X^{n+N} \supset \cdots \supset X^n \), such that all \( X^{n+N} \), \( N = 1, 2, ..., \), have \( \|\partial^i \|_{fil} (\beta_0) \leq \delta_0 \), for some \( i(\leq \infty) \), then, obviously, \( X^\infty \) also has \( \|\partial^i \|_{fil} (\beta_0) \leq \delta_0 \). This applies, for example, to the Hilbert space \( \mathbb{R}^\infty \), to the Hilbertian sphere \( S^\infty \subset \mathbb{R}^{\infty+1} \) and to other infinite dimensional symmetric spaces of "compact type", where the argument depends on the \( N \)-asymptotic of the \((n+N)\)-volumes of \( X^{n+N} \).

Is there a dimension free proof applicable to more general \( X^\infty \) (e.g. to \( S^\infty \) divided by an infinite discrete isometry group)?

(d) Is there a (sufficiently) sharp generalization of Almgren’s filling bound in \( \mathbb{R}^n \) to non-Euclidean Banach-Minkowski spaces \( X^n \) in the spirit of the Brunn-Minkowski inequality (corresponding to \( i = n - 1 \))? Are there such inequalities in the metric spheres in these \( X^n \) and other flag (e.g. Grassmannian) manifolds?

(e) Does the variational method apply to \( \Delta(V) \) and similar measurable complexes, and improve the bound \( \|((\partial^i)^{-1})_{fil} \leq 1 \)?

(f) Is there an algebraic/topological version of \( \|...\|_{fil}^{\infty} \) in the context of our chapter 4?

6 Isoperometry on Submanifolds

6.1 \( n \)-Divergence, Mean Curvature, Minimal Surfaces and Allard-Michael\&Simon Inequality

\[ \text{The } n \text{-Divergence } \text{div}^{[n]}(\tau) = \text{div}^{[n]}(\tau, V) \text{ of a vector field } \tau \text{ in a Riemannian manifold } X \text{ is the rate of increase (decrease) of the volumes of } n \text{-submanifolds } V = V^n \subset X \text{ under the flow generated by } \tau. \]

That is, if the the-\( \varepsilon \)-initial \( \tau \)-flow moves
\[ V \xrightarrow{\text{flow}} (1 + \varepsilon \tau)(V) \subset X, \]

then
\[ \text{div}^{[n]}(\tau, V) = \partial_{\tau} \vol_n(V) = \lim_{\varepsilon \to 0} \frac{\vol_n((1 + \varepsilon \tau)(V)) - \vol_n(V)}{\varepsilon}, \]
where, for smooth fields $\tau$, $\text{div}^{[n]}(-\tau) = -\text{div}^{[n]}(\tau, V)$ by the basic rules of the calculus.

**Examples.** (a) If $n = N$, this is the ordinary divergence,

$$\text{div}^{[n]}(\tau) = \text{div}(\tau),$$

(b) if $X = \mathbb{R}^N$ and $\tau$ is the radial field $x \mapsto \tau_x \in T_x(\mathbb{R}^N)$, $x \in \mathbb{R}^n$, that is $\tau = \text{grad} ||x||^2$, then

$$\text{div}^{[n]}(\tau) = n,$$

since the multiplicatively written one-parameter group, generated by this field is the group of homotheties, $x \mapsto tx$, which expands the volumes of all $n$-submanifolds by the factor $nt$.

(c) Let $X$ be a Riemannian manifold, where the inverse exponential map $\exp^{-1}_x : X \rightarrow T_x(\mathbb{R}^N)$ is a homeomorphism onto a (star convex) domain in the tangent space $T_x(\mathbb{R}^N) = \mathbb{R}^N, N = \text{dim} X$, i.e. all $x \in X$ are joint with $x_0 \in X$ by a unique geodesic segment $[x_0, x] \subset X$.

Let $\tau_0(x)$ be the field tangent to $[x_0, x]$ at $x$ with norm $||\tau(x)|| = \text{length}[x_0, x]$, that is $\tau_0$ is equal to the gradient of the squared distance function to $x_0$,

$$\tau_0 = \text{grad} \text{dist}(x, x_0)^2.$$

If $\text{sect.curv}(X) \leq 0$, then

$$\text{div}^{[n]}(\tau_0) \geq n$$

and if $\text{sect.curv}(X) \leq -1$ then the corresponding unit field $\tau_* = \frac{\tau_0}{||\tau_0||}$ satisfies a similar inequality

$$\text{div}^{[n]}(\tau_*) \geq n.$$

**First variation of the $n$-Volume.** Let $X$ be a Riemannian manifold and $V \subset X$ an $n$-dimensional submanifold with a boundary $S = \partial V$, e.g. a curve or surface in the Euclidean space, and let $\tau : V \rightarrow T_V(X)$ be an $X$-tangent field along $V$. Then the $\tau$-derivative of the volume of $V$ is equal to the sum of two terms:

$$\partial_\tau \text{vol}_n(V) = \int_S (\tau(s), \nu(s))ds + H^* (V, \tau),$$

where:

- $S$ the field $\nu : S \rightarrow TS(V) \cap T(S)$ is the external looking unit normal field to $S$ in $V$, thus, $(\tau(s), \nu(s))$ is the (signed) length of the normal projection of $\tau$ to $V$ along $S$;
- the second term $H^*(V, \tau) = \text{mean.curv}(V)$ is the mean curvature of $V$, that is a cotangent vector field along $V$, $\text{mean.curv} = H^* : V \rightarrow T^*_V(X)$.

\[42\] Here $\tau_x$ is the tangent vector corresponding to the point (Euclidean vector) $x \in \mathbb{R}^n$ under the obvious identification $T_x(\mathbb{R}^N) = \mathbb{R}^N$. 

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which is the differential of the $n$-volume function(al) on the space of submanifolds in $X$ obtained by normal deformations\(^{43}\) of $V$, which is customary represented by the scalar product of tangent fields $\tau$ with a normal (also called mean curvature) field

$$H : V \to T^1(V) = T_V(X) \oplus T(V),$$

which represents the "normal gradient" of the volume function. Thus, the value of $H^*$ on the tangent fields $\tau : V \to T_V(X)$ is:

$$H^*(V, \tau) = \int_V H^*(v) dv = \int_V (\tau, H(v)) dv^{44}$$

1d Example. If $V^1 \to \mathbb{R}^N$ is a curve parametrized by the arclength $t$, then the mean curvature vector $H(v)$ is equal to the second derivative of $v(t)$

$$H(t) = \frac{d^2 v(t)}{dt^2}.$$  

"From-1-to-n" Example. If $V^i \subset V = V^n \subset \mathbb{R}^N, i = 1, \ldots, n$, are mutually normal geodesic (with respect to the Riemannian metric induced from $\mathbb{R}^N \ni V$) lines in $V$ at $v \in V$, then

$$H(V, v) = \sum_{i=1}^n H(V^i, v).$$

"Spherical" sub-Example. The (normal) mean curvature field of the sphere $S^n(R) \subset \mathbb{R}^{n+1} \subset \mathbb{R}^N$ is contained in (tangent to) $\mathbb{R}^{n+1} \ni S^{n-1}$ and the norm of this field is everywhere

$$\|\text{mean.curv}(S^n(R), s)\| = n/ R$$

"Closed" Example. Let $V \subset B^{n}(R) \subset \mathbb{R}^N$ be a closed (i.e. compact without boundary) $n$-submanifold in the $R$-ball. Then (the mean value of) the norm of the mean curvature of $V$ is bounded from below by that of the sphere $S^n(R) = \partial B^{n+1}(R),$

$$\int_V R \|\text{mean.curv}(V, v)\| dv \geq n \cdot \text{vol}_n(V).$$

In fact,

$$\partial_\tau \text{vol}_n(V) \leq \int_V \|\tau\| \cdot \text{mean.curv}(V, v) \| dv$$

for all fields $\tau$, and the proof follows by applying this to the above radial field $\tau(x) = \text{grad} \|x\|^2$.

??? D. Example "with a Boundary". Let $V \subset B^{n}(R) \subset \mathbb{R}^N$ be a compact manifold with a boundary $S = \partial V$. then

$$\int_V R \|\text{mean.curv}(V, v)\| dv + R \cdot \text{vol}_{n-1}(S) \geq n \cdot \text{vol}_n(V).$$

Proof. Argue as above with $\tau = \|x\|^2$ and observe that the $\int_S$-term in the first variation formula (•$S$ in ???C) is bounded by

$$\int_S \langle \tau(s), \nu(s) \rangle ds \leq R \cdot \text{vol}_{n-1}(S).$$

\(^{43}\)This means "along vector fields normal to $V".  

\(^{44}\)A priori, the mean curvature is a vector valued measure on $V$ but it is represented by a (co)vector valued function on $V$, i.e. a section of the bundle $T_V(X) \to V$, in the present case of a Riemannian $X$ and smooth $V$ and $\tau$.  

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**Minimality: Definition/Exercise.** A smooth submanifold \( V \subset X \) is called minimal if one of the following four equivalent conditions is satisfied.

- \( \nu \partial_\nu \text{vol}_n(V) = 0 \) for all fields \( \nu \) normal to \( V \);
- \( \tau \partial_\tau \text{vol}_n(V) = 0 \) for all fields \( \tau \) with compact supports away from the boundary of \( V \), \( \text{supp}(\tau) \subset \bigcup_{\partial V} \); 
- \( H \) the mean curvature of \( V \) is zero;
- \( \text{min}_V \) is locally volume minimizing: all points \( v \in V \setminus \partial V \) admit neighborhoods, \( U = U(v) \subset V \), such that all \( n \)-submanifolds \( U' \subset X \) with \( \partial U' = \partial U \) have larger volumes than \( U \), 

\[ \text{vol}_n(U') \geq \text{vol}_n(U), \]

where this inequality is strict unless \( U' = U \).

**Miranda-Allard-Michael-Simon Inequality.** Let \( X \) be a Riemannian manifold and let \( \delta > 0 \) and \( R_0 \geq 0 \) be constants, such that all balls \( B^N_x(r), x \in X, r < R_0 \) admit vector fields \( \tau = \tau_{x,r} \), such that

\[ \text{div}_V([n](\tau)) \leq -\delta \]

and the norms of \( \tau \) on the boundaries of these balls are bounded by \( r \). (E.g. \( X = \mathbb{R}^N, R_0 = \infty \), and \( \delta = n \).)

Let \( V = V^n \) be a smooth compact \( n \)-dimensional submanifold with a (possibly empty) boundary. Then either \( \text{vol}_n(V) \geq R^n_0/\delta^n \) or

\[ \int_V \|\text{mean.curv}(V,v)\|dv + \text{vol}_{n-1}(S) \geq \text{const}_n \cdot \text{vol}_n(V)^{n-1/n} \]

for some strictly positive \( \text{const} = \text{const}(n,\delta) > 0 \).

**Proof.**

6.2 Mass Transportation: Castillon-Brendle Inequality

7 Bolzmann-Gibbs-Shannon Entropic Inequalities

7.1 Hölder Inequality via Tensorisation.

We introduce below the Gibbs tensorisation trick and then use it for the proof of the Shannon inequalities relating the entropy of a measure and its pushforwards under the maps (partitions) in a given family.

**Hölder Inequality.** The log of the integral

\[ \int_X \prod_{i \in I} f_i(x)^{\beta_i} dx \]

is a convex function of \( \beta = \{\beta_i\} \in \mathbb{R}^I \) for arbitrary positive functions \( f_i \) on \( X \).

**Proof.** The inequality

\[ \log \left( \int_X \prod_{i \in I} f_i(x)^{\alpha_i \beta_i} dx \right) \leq \sum_{i \in I} \alpha_i \log \left( \int f_i(x)^{\beta_i} dx \right) = \log \prod_{i \in I} \left( \int f_i(x)^{\beta_i} dx \right)^{\alpha_i} \]

for \( \sum \alpha_i = 1, \alpha_i \geq 0 \) is (trivially) true if the functions \( f_i(x) \) are constant on the intersection \( S \subset X \) of their supports (with the equality for functions with a common
support $S \subset X$, where all $f_i$ are constant) and the general case reduces to this by the law of large numbers via the tensorisation.

This argument also shows by how much the inequality deviates from equality. Denote $\mu_i = g_i dx$ for $g_i = f_i^\beta$, and let $\mu = (\prod_{i \in I} g_i) dx$. Then

$$\log \int \prod_{i \in I} g_i^{\alpha_i} dx \leq \sum \alpha_i \text{ent}_\mu(\mu) \leq \sum \alpha_i \log(\mu_\nu(X)) = \log \prod_{i \in I} \left( \int_X g_i dx \right)^{\alpha_i}.$$ 

The Hölder Inequality can be equivalently stated as follows

let $\nu$ be a measure on the linear space $X$, and let $Y$ be the linear dual to $X$.

Then the function

$$\Psi(y) = \Psi_\nu(y) = \log \left( \int_X \exp(x,y) d\nu \right)$$

is convex on $Y$, where the entropy of a measure with the density function $\exp(\varphi(x))$, $x \in \mathbb{R}$ equals the derivative $\psi'(y = 1)$ for $\psi(y) = \int_{\mathbb{R}} \exp(x,y) dx$ by the Boltzmann formula.

This appears in the Gibbsian thermodynamics as the concavity of the entropy of the ideal gas and represents a tiny instance of Boltzmann’s and Gibbs’ ideas (see [?]).

Remarks. (a) The information theoretic rendition of the Gibbs argument is often presented as a chat between Alice and Bob. (See [?] and references therein.)

(b) The differential $D\Psi: Y \rightarrow X$ injectively sends $Y$ to $X$, where the closure of the image equals the convex hull of the support of $\mu$.

Thus, if $X = Y = \mathbb{R}^n$, then the volume of this hull equals the integral of the determinant of the Hessian of the (convex!) function $\Psi$, where the $\mathbb{R}_+$-valued map

$$\Psi \mapsto M(\Psi) \overset{\text{def}}{=} \int_Y \det(\text{Hess}(\Psi(y))) dy$$

obeys non-trivial convexity relations: the Minkovski inequality, $M^2(\Psi_1 + \Psi_2) \geq M^2(\Psi_1) + M^2(\Psi_2)$, and the Alexandrov-Fenchel-Hodge inequality. (See [?] for a survey and references.)

Let $(X, \lambda)$ be a Borel measure space, where $\lambda$ is regarded as a background measure and where we use the notation $|Y| = |Y|_\lambda = \lambda(Y)$ for all $Y \subset X$.

The basic examples are given by countable spaces $(X, \lambda)$ with the unitary measures, where all atoms have unit weights (thus, $|Y| = \text{card}(Y)$) and by the Euclidean spaces with the Lebesgue or with the Gaussian measures $\lambda$.

Consider measures $\mu = f(x) \lambda$ for (non-strictly) positive measurable functions $f$ on $X$ and first define the entropy of such a $\mu$ where $f(x)$ is constant on its (essential) support $S = \text{supp}(f) \subset X$ by

$$\text{ent}_\lambda(\mu) = \log|S| = \log(\mu(S)) - \mu(S)^{-1} \int_S \log(f) d\mu,$$

where, observe, $f \equiv \mu(S)/|S|$.

Then, for a general $\mu = f \lambda$, let $|\mu|_\lambda = \lambda_\nu(\mu)$ denote the infimum of the $\lambda$-measures of the subsets $S_\nu \subset X$ with $\mu(S_\nu) \geq (1 - \varepsilon) \mu(X)$, where we assume that $\mu$ has finite total mass, $|\mu|_\lambda = \mu(X) < \infty$.

Take the Cartesian (tensordial) powers $(X^N, \lambda^N = \lambda^\otimes N, \mu^N = \mu^\otimes N)$ and with $\lambda^\otimes N$ for the background measures on $X^N$. Set

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ent(\mu^N - [\varepsilon]) = \liminf_{N \to \infty} \frac{1}{N} \log |\mu^N|_\varepsilon

and

\text{ent}(\mu) = \text{ent}_\lambda(\mu) = \lim_{\varepsilon \to 0} \text{ent}(\mu^N - [\varepsilon]).

Observe that, this entropy is invariant under scaling of \mu, that is \text{ent}(c \cdot \mu) = \text{ent}(\mu), while \text{ent}(c\lambda) = \text{ent}(\lambda) + \log|\mu|/|c|.

If \mu is a probability measure with a \lambda-measurable density function f = d\mu/d\lambda and with the support denoted S \subset X, then \text{ent}_\lambda(\mu) \leq \log \lambda(S) with equality (only) for \mu = \lambda(S). On the other hand, \text{ent}_\lambda(\mu) \geq \log(\sup_{x \in S} f(x)) - 1.

We shall use the above definition only for log-LLN-measures \mu, i.e. where \mu = f\lambda for a \lambda-measurable function f, such that \log|f| is summable on its support S, e.g. if |\log(f)| is bounded on S.

The Law of Large Numbers. The \mu^N measure of the subset Y(\varepsilon, N) \subset X^N of the points y \in X^N, where

\frac{1}{N} \log(f^{\otimes N}(y)) - \int_S \log(f) d\mu \geq \varepsilon

satisfies

(\text{LLN}) \quad \mu^{\otimes N}(Y(\varepsilon, N)) \to 0 \text{ for } N \to \infty.

One knows that (\text{LLN}) is satisfied if and only if the function |\log(f)| is summable on its support S, e.g. if |\log(f)| is bounded on S.

If \mu is not log-LLN, one can LLN-regularize it, e.g. by cutting away the part of the support of f where |\log(f)| approaches infinity and then define a suitable regularized entropy with such an approximation.

Cartesian Additivity of the Entropy. Observe that LLN ensures the additivity of the entropy under the Cartesian product of measure spaces and yields the celebrated Boltzman Formula. All log-LLN-measures \mu satisfy,

\text{ent}_\lambda(\mu) = \log|\mu| - |\mu|^{-1} \int_S \log(f) d\mu = \log|\mu| - |\mu|^{-1} \int_S f \log(f) d\lambda

(for |\mu| denoting the total mass \mu(X) = \mu(S)).

In other words,

the \mu-average of \log(f) plus \text{ent}_\lambda(\mu) equals the log of the total mass of \mu.

In particular, the entropy of a probability measure \mu is expressed by the Boltzmann integral,

\text{ent}(\mu) = \int_S \log \frac{1}{f} d\mu = \int_S f \log \frac{1}{f} d\lambda.

This formula is customary taken for the definition of the entropy without assuming LLN, but only the convergence of the Boltzmann integral, possibly to \pm \infty. This definition is equivalent to the above “regularized entropy” but in all our applications we can (and do) assume that \mu is log-LLN.
7.2 Entropic Profiles and Stable $E_\alpha$ Functions of Families of Partitions.

Given a finite mass measure $\mu$ on a Borel measure space $X = (X, \lambda)$ with a family of partitions $P_i$, we denote by $\mu_i = \mu/P_i$ = the pushforward of $\mu$ to $X_i = X/P_i$, call this the $P_i$-reduction of $\mu$, and write

$$\text{ent}(\mu/P_i) = \text{ent}(\mu_i) = \text{ent}_{\lambda_i}(\mu_i)$$

for the background measures $\lambda_i$ in $X_i$.

For example, if $\mu$ equals the restriction of the background measure $\lambda$ on $X$ to a subset $Y \subset X$, then the value of the density function of $\mu_i$ with respect to the background measure $\lambda_i$ on $X/P_i$ at each point $x_i \in X/P_i$ equals the Fubini mass of the corresponding $P_i$-slice of $Y$.

Denote by $\mu_{x_i}$, $x_i \in X_i = X/P_i$ the measure $f dP(x)$ on the slice $P^{-1}(x_i)$ for the background Fubini measure $dP(x)$ on this slice and $f = d\mu/d\lambda$ and let $\text{ent}_{x_i}$ be the entropy of $\mu_{x_i}$ with respect to $dP(x)$ on this slice. Define the entropic profile of $(X, \mu)$ over $X_i$, also denoted $\text{ent}(P_i)$ as the average

$$\text{ent}(P_i) = \mu(X)^{-1} \int_{X_i} \text{ent}_{x_i} d\mu.$$

It is obvious (but significant) that

Entropy is additive.

$$\text{ent}(P_i)^+ = \text{ent}(P) + \text{ent}(\mu/P) = \text{ent}(\mu).$$

Finite Example. Let $P$ be a partition of $X$, a finite set with the unitary atoms and take a subset $Y \subset X$. Denote by $|P(y)|$ the cardinality of the $P$-slice of $Y$ through $y \in Y$, and observe with the Boltzman (and Shannon in the finite case) formula that

$$\text{ent}(P|Y) = \log \prod_{y \in Y} |P(y)|^{\text{ent}(x\epsilon_y)}.$$

Entropic Profile. Consider a family $\mathcal{P}$ of partitions $P_i$, $i \in I$, of $X$, where we usually assume that the single slice partition, corresponding to the map of $X$ to a single point, is among our $P$. Every LLN measure $\mu$ on $X$ defines the point $\varepsilon(\mu) = \{\text{ent}(P_i)\} \in \mathbb{R}^I$; the set $\text{ENT}(\mathcal{P})$ of these points for all $\mu$ is called the entropic profile of $\mathcal{P}$. In what follows we shall evaluate the conical convex hull of $\text{ENT}(\mathcal{P}) \subset \mathbb{R}^I$ in the simple cases.

The definition of the entropy and the slice removal lemma from 4.4 imply the following

Sliced Tensorisation Lemma. Given a finite family $\mathcal{P}$ of partitions $P_i$, $i \in I$, of $X$ and an LLN measure $\mu$ on $X$, there exists, for every $\varepsilon > 0$, an integer $N_0 = N_0(\varepsilon, \mu, \mathcal{P})$ and a subset $Y = Y_N$ in the Cartesian power $X^N$, for every $N \geq N_0$, such that

$$Y \subset \text{supp}(\mu^\otimes N),$$

where $\mu^\otimes N(Y) \geq (1 - \varepsilon)\mu^\otimes N(X)$,

and the Fubini measures $\phi_i^N = \lambda^\otimes N/L_i^\otimes N$ of the $P_i^N$-slices of $Y$ satisfy,

$$N(\text{ent}(P_i^N) - \varepsilon) \leq \log(\phi_i^N(P_i^N \cap Y)) \leq N(\text{ent}(P_i^N) + \varepsilon).$$
for all \( y \in Y \) and all \( i \in I \).

Next, observe that the \( E_\circ \)-functions of Cartesian powers of partitions \( P_i \) of \( X \), satisfy,

\[
E_\circ(n_i^{N_1}; P_i^{N_1}) \cdot E_\circ(n_i^{N_2}; P_i^{N_2}) \geq E_\circ(n_i^{N_1+N_2}; P_i^{N_1+N_2})
\]

and define

\[
E_\infty(n_i; P_i) = \lim_{N \to \infty} (E_\circ(n_i^N; P_i^N))^{\frac{1}{N}}.
\]

The above lemma implies the following Shannon \( E_\infty \)-Inequality. Let \( P = \{ P_i \} \) be a finite family of partitions on \( X \) and \( \mu \) a measure of finite mass on \( X \). Then the entropies \( \text{ent}(P_i) = \text{ent}(\mu) - \text{ent}(\mu/P_i) \) of \( P_i \) with respect to \( \mu \) satisfy,

\[
\text{ent}(\mu) \geq \log E_\infty(\exp(\text{ent}(P_i)); P).
\]

Remark on Hölder. The tensorisation lemma also implies the Hölder version of the above inequality.

Let \( f_i \geq 0 \) be measurable functions on \( X_i = X/P_i \), let

\[
|f_i|^p = \left( \int_{\text{supp}(f_i)} f_i^p \right)^{\frac{1}{p}}
\]

and let \( |\Pi_P f_i|_1 \) denote the integral of the product of the pullbacks of \( f_i \) to \( X \). Then

\[
|\Pi_P f_i|_1 \geq E_\infty(|\Pi_P f_i|_1/|f_i|_{p_i}; P)
\]

for all \( \{ p_i \} \in \mathbb{R}_+^I \).

If all \( P_i \) are single slice partitions, this reduces to the Hölder inequality from 5.1 with positive \( p_i \) (and with no entropic correction term).

### 7.3 Shannon and Harper Inequalities for the Coordinate Line and Plane Partitions.

Let \( (X, \lambda) = \times_i (X_i, \lambda_i) \), \( i = 1, 2, \ldots, k \). Then the partitions \( P_i \) of \( X \) into the "coordinate lines" with the slices isomorphic to \( X_i \) and corresponding to the projections \( P_i: X \to X_i = (X_i, \lambda_i) = \times_{j \in \{i\}} (X_j, \lambda_j) \) satisfy

\[
Sh_1 \quad \text{ent}(\mu) \geq \sum_i \text{ent}(P_i),
\]

or, equivalently,

\[
\text{ent}(\mu) \leq \frac{1}{k-1} \sum_i \text{ent}(\mu/P_i)
\]

for all measures \( \mu \) on \( X \). Furthermore, the partitions \( P_j \) of \( X \) into the fibers of the projections \( X = X_1 \to X_{1 \cup J} = \times_{i \in 1 \cup J} X_i \) (with "\( J \)-plane" slices representing \( X_J = \times_{i \in J} X_i \)) satisfy
Sh\(\alpha\), \(\text{ent}(\mu) \geq \sum_{J \subset I} \alpha_J \cdot \text{ent}(P_J)\)

for all partitions of unity \(\alpha_J\) of \(I\) (see 4.3).

**Proof.** Here, obviously, \(E_{\infty} = E_{\circ}\) and the above applies.

**Loomis-Whitney Inequality.** This is an upper bound on \(|Y| = \lambda(Y)\) for sub-

sets \(Y \subset X\) in terms of the background measures of \(Y/P_i\), (assuming these are measurable) written as if it were a lower bound,

\(|Y| \geq \prod_i |Y|(\lambda_i(Y/P_i))^{-1} \).

This follows from the Shannon Inequality, since \(\text{ent}_{\lambda_i}(\mu_i) \leq \lambda_i(supp(\mu_i))\) and \(\text{ent}(\mu) = \log(\lambda(Y))\) for \(\mu = \lambda|Y|\).

Similarly one derives the Shearer Inequality that is the bound on \(\log|Y|\) by \(\log|Y| - \log(\lambda_J(Y/P_J))\) substituting \(\text{ent}(P_J)\) in \(\text{Sh}_{\infty}\). (The role of the entropy in such inequalities was pointed out to me by Noga Alon.)

If \(X_i\) are countable sets with the atoms of unit weights, then the Shannon

inequality for subsets \(Y \subset X = \times_i \times X_i\) with the restricted product unitary measures reads,

**Combinatorial Shannon Inequality for the Coordinate Line Partitions.** Let \(\prod_i |P_i(y)|\), \(y \in Y\), denote the cardinality of the \(P_i\)-slice of \(Y\) through \(y\). Then the geometric means \(|MP_i| = \left( \prod_{y \in Y} |P_i(y)| \right)^{1/|Y|}\)

satisfy

\[ \prod_i |MP_i| \leq |Y| . \]

**Harper Inequality.** The Shannon inequality, when applied to the vertex set \(X\) of the edge graph of a Euclidean \(n\)-cube with the edges for slices, says that the vertex and the edge numbers of every subgraph \(Y\) in the cubical graph satisfy,

\[ (4^N) \quad N_{\text{vert}} \geq 4^N_{\text{edge}}/N_{\text{vert}}. \]

For example, if all vertices in \(Y\) have the valency (degree) at least \(d\), then \(|Y| \geq 2^d\).

Another corollary of the combinatorial Shannon inequality is the following (well

known) relation between the three numbers: the cardinality \(|Y|\), the number \(N\) of the slices of \(Y\) with respect to all \(P_i\) and the sum \(C\) of the cardinalities of all these slices.

**A\(B\)-Inequality.** Let \(A = C/N\) and \(B = C/|Y|\). Then \(|Y| \geq A^B\).

**Proof.** Since the function \(s^a\) is log-convex, \(\log(s^a)' = 1/s\),
A^B \leq \prod_S |S|^{|S|} \leq |Y|,

where the product is taken over all slices $S$ of the partitions $P_i$.

### 7.4 Strict Concavity of the Entropy and Refined Shannon Inequalities.

A probability measure $\mu$ on $X_1 \times X_2$ can be regarded as a family of probability measures $\mu_x$, on $X_2$ parametrized by $x \in X_1$, where the density $f_{\mu_x}(x_2)$ of (almost) every measure $\mu_x$ on $X_2$ equals the restriction of the density of $\mu$ to $x_1 \times X_2 \subset X_1 \times X_2$ divided by $p(x_1) = \int_{X_1} f_{\mu_x}(x_2) \, d\lambda_2$.

The Shannon inequality written as $\text{ent}(\mu/P_2) \leq \text{ent}(P_1)$ says that the entropy is a concave function on the space of probability measures on $X_2$, since the measure $\mu_2$ on $X_2$, that is the pushforward of $\mu$, equals the $p(x_1)$-weighted convex combination of the probability measures $\mu_x$, while the entropy is (defined as) the corresponding convex combination of the entropies of $\mu_x$.

In fact, the entropy is strictly concave as follows from the Boltzmann formula and the strict convexity of the function $t \cdot \log(t)$. (This is the common way for deriving the Shannon inequality). Then the quantity $\text{ent}(\mu) - \text{ent}(P_1) - \text{ent}(P_2) \geq 0$ tells us how far $\mu$ is from equilibrium, i.e. a probability measure $\mu'$ on $X_1 \times X_2$, for which the probability measures $\mu_x'$ on $X_2$ are mutually equal for all $x \in X_1$, or equivalently all $\mu_x'$ on $X_1$ are equal.

Here is another characteristic of (non-)equilibrium for measures $\mu$ on product spaces $X = \times_i X_i$, $i \in I$.

The index set $I \cup I$, (disjoint union of $I$ with itself) and, hence, the Cartesian power $X^2$ of $X$, is naturally acted by the Mendelian recombination group $\mathbb{Z}_2^I = (\mathbb{Z}/2\mathbb{Z})^I$ generated by $|I|$ coordinate involutions on $I \cup I$ and/or on $X_i \times X_i$ for all $i \in I$. By strict convexity, a measure $\mu$ on $X$ is at equilibrium, where (by definition if you wish) all Shannon inequalities $\text{Sh}$ becomes equalities, if and only if the measure $\mu^{\otimes 2}$ on $X^2$ is invariant under $\mathbb{Z}_2^I$ and (where, observe, $\mu^{\otimes 2}$ is invariant under the diagonal involution on $X^2$ for all $\mu$ on $X$.)

We introduce the entropic displacement of $\mu^{\otimes 2}$ by $z$,

$$|\mu^{\otimes 2} - z(\mu^{\otimes 2})|_{\text{ent}} = \text{def} \ \text{ent}\left(\frac{1}{2}(\mu^{\otimes 2} + z(\mu^{\otimes 2})) - \frac{1}{2}(\text{ent}(\mu^{\otimes 2}) + \text{ent}(z(\mu^{\otimes 2})))\right) \geq 0$$

and then identify involutions $z \in \mathbb{Z}_2^I$ with subsets $J \subset I$ by

$$z \leftrightarrow J = \text{supp}(z) \subset I$$

where the support of $z$ is defined by $z(i) \neq i$.

The composition of involutions corresponds to the symmetric difference of subsets that we denote $J_1 \cap J_2 = \text{def} \ (J_1 \cup J_2) \setminus (J_1 \cap J_2)$. We also abbreviate by writing

$$|J_i|_{\text{ent}}(\mu) = |\mu^{\otimes 2} - z(J_i)(\mu^{\otimes 2})|_{\text{ent}},$$

where, observe, $|J_i|_{\text{ent}}(\mu) = |J_i|_{\text{ent}}(\mu)$, for $J_i = I \setminus J$. A measure $\mu$ on $X$ satisfies the equality
\[ \text{ent}(P_J) + \text{ent}(P_{J_i}) = \text{ent}(\mu) \]

if and only if \( \mu^{\otimes 2} \) is \( z(J) \)- (or, equivalently \( z(J^i) \))-invariant; this is also equivalent to

\[ |J_i|_{\text{ent}}(\mu) = 0. \]

Since the entropy is strictly concave, the function \( |J_i|_{\text{ent}}(\mu) \) of \( J \subset I \) satisfies some triangle-type inequalities,

\[ (\Delta) \quad |J_1 \cdot J_2|_{\text{ent}}(\mu) \leq \Delta_\mu(|J_1|_{\text{ent}}(\mu), |J_2|_{\text{ent}}(\mu)), \]

where \( \Delta_\mu(0,0) = 0 \) for all \( \mu \) and \( \Delta_\mu(a,b) \) is uniformly continuous in \((a,b)\) with the modulus of continuity \( \delta \) depending on \( \mu \). Moreover, \( \delta \) is uniformly bounded on certain (compact in a suitable sense) classes of measures \( \mu \).

For example, if the density function \( f \) of \( \mu \) satisfies

\[ \int_X |\log(f(x))| d\mu \leq \text{const} < \infty, \]

then \( \delta \) is bounded by some universal \( \delta_{\text{const}} \) as a simple continuity argument shows.

This is useful, for instance, if \( \int f(x) \leq 0 \), e.g. if \( X \) is a discrete space with unitary atoms, where \( (\Delta) \) becomes a relation between the entropies of \( P_J \) depending only on \( \text{ent}(\mu) \).

\[ (\Delta) \quad |J_1 \cdot J_2|_{\text{ent}}(\mu) \leq \Delta_{\text{ent}(\mu)}(|J_1|_{\text{ent}}(\mu), |J_2|_{\text{ent}}(\mu)), \]

for some function \( \Delta_\varepsilon(a,b) \) that is continuous in \( a, b \) and \( \varepsilon \) and such that \( \Delta_\varepsilon(0,0) = 0 \).

Remarks. (a) All this is, apparently, well known but I could not find a reference; nor do I know a specific sufficiently “elegant” \( \Delta_\varepsilon(a,b) \). I guess, there are sharp “mixed symmetric mean inequalities” for measures on \( \times X_i \), similar to the classical Muirhead’s inequalities, such as the mixed discriminant inequality of Alexandrov (that is \( GL(k) \)- rather than just \( S_k \)-symmetric).

(b) The above generalizes to the Cartesian powers \( X^N \) with the Cartesian products of \( I \)-copies of the permutation group \( S_N \) acting on it. The resulting inequalities become, in a sense, asymptotically sharp for \( N \to \infty \) due to the law of large numbers (applied to convolution of measures on the spaces of measures).

A possible framework for this is suggested by the Mendelian dynamics

### 7.5 Equipartitions, Tensorization and the Hölder-Loomis-Whitney-Shearer Inequality


Recall the classical Hölder inequality

\[ \int_X \prod_{i \in I} f_i(x)^{p_i} \leq \prod_{i \in I} \left( \int_X f_i(x) \right)^{\beta_i}, \beta_i \geq 0, \sum_{i \in I} \beta_i = 1, \]

which takes more familiar form for \( \beta_i = 1/p_i \).
Exercise. Show that the Hölder inequality implies log-convexity of the function

$$H(p_i) = H(f_i)(p_i) = \int_X f_i(x)^{p_i}$$

in \((p_i) \in \mathbb{R}^I\) for all positive measurable functions \(f_i(x)\) on a measure space \(X\), that is convexity of \(\log H(p_i)\).

The Hölder-Looms-Whitney-Shearer Inequality refines classical Hölder as follows.

Let \(K\) be a finite or countable set, let \(X_k, k \in K\), be measure spaces, e.g. \(X_k = \mathbb{R}\) or \(X_k = \{0, 1\}\), let \(J_i \subset K\) be subsets indexed by a finite or countable set \(I \ni i\) and let \(\beta_i \geq 0\) satisfy the following partition of unity condition:

$$\sum_{i \in I} \beta_i 1_{J_i}(k) = 1,$$

where \(1_{J_i}(k)\) are the characteristic functions of the subsets \(J_i \subset K\).

For instance, if all \(J_i = K\), this becomes \(\sum_{i \in I} \beta_i = 1\).

Let \(Y = \times_{k \in K} X_k\) and let \(f_i(y), y = (x_k) \in X\) be positive measurable functions, such that \(f_i\) depends only on the variables \(x_k\) for \(k \in J_i\).

In other words \(f_i\) is equal to the pullback of a function on \(\times_{k \in J_i} X_k\) under the projection

$$Y = \times_{k \in K} X_k \to \times_{k \in J_i} X_k.$$

Then

$$[HLWS] \int_Y \prod_{i \in I} f_i^{p_i \beta_i} \leq \prod_{i \in I} \left( \int_Y f_i^{p_i} \right)^{\beta_i},$$

Proof. Observe that the integral \(\int_Y a(y)\) is multiplicative under product of measure spaces

$$\int_{Y \times Z} a(y) b(z) = \int_Y a(y) \int_Z b(z)$$

Therefore [HLWS] for the functions \(f_i(y)\) is equivalent to this inequality for the corresponding product functions

$$f_{i,N}(y_1, ..., y_N) = f_i(y_1) \times ... \times f_i(y_N)$$

and/or their geometric means \(\sqrt[N]{f_{i,N}}\) on the \(N\)th power spaces

$$Y^N = Y \times Y \times ... \times Y.$$ 

Next, let \(X_k\) be finite sets with atoms of unit weight, observe that the general case of [GLWS] reduces to that by an obvious approximation argument.\(^{45}\) and also observe. that if a function \(f(y) > 0\) satisfies \(\int_Y f(y) = \sum_{y \in Y} f(y) = 1\), then, the law of large numbers for the sums of independent random variables

$$\sum_{j=1}^N \log f(y_j)$$

\(^{45}\)This is unnecessary, if you are comfortable with the abstract measure theoretical terminology.
on the power probability spaces \((Y, f(y))^N\), yields the following.

**Bernoulli Approximation Theorem.** There exist (automatically measurable under our assumptions) subsets \(V_N \subset Y^N\), such that some constant multiples of the characteristic functions of these subsets are asymptotically equivalent to functions \(f_N\), according to the following definition.

**Definition.** Two sequences of probability measures \(\phi_N\) and \(\psi_N\) defined by positive functions \(\phi_N(y)\) and \(\psi_N(y)\) on finite sets \(Y\) are asymptotically equivalent if there exist subsets \(V'_N \subset Y_N\), such that both functions \(\phi_N(y)\) and \(\psi_N(y)\) are strictly positive on \(Y_N\) and

- \(\phi_N(V'_N) \to \phi_N(Y_N) = 1\) and \(\psi_N(V'_N) \to \psi_N(Y_N) = 1\) for \(N \to \infty\);
- \(\sup_{y \in V} \frac{\log \phi(y) - \log \psi(y)}{\log \text{card}(V)} \to 0\) for \(N \to \infty\).

**Conclusion.** [HLWS] reduces to [LWS], that is where the functions \(f_i\) are characteristic functions of measurable subsets \(V_i \subset Y\).

This settles the problem for \(\text{card}(K) \leq 1\) (the classical Hölder inequality) and \(\text{card}(K) = 2\), where [LWS] is obvious but if \(Y = \times_{k \in K} X_k\) and \(\text{card}(K) \geq 3\), which, geometrically, is the most interesting case, one needs to use the law of large numbers for the second time as follows.

**Lemma.** Let \(X_1 = \{(x_1, dx_1)\}\) and \(X_2 = \{(x_2, dx_2)\}\) be measure spaces, let \(V \subset X_1 \times X_2\) be a measurable subset with measure one and let \(\chi_N\) be the characteristic functions of power subsets \(V^N \subset X_1^N \times X_2^N\). Then there exists subsets \(V_1,N \subset X_1^N\) and \(V_{2,N} \subset X_2^N\), such that the characteristic functions of the products

\[ V_{1,N} \times V_{2,N} \subset X_1^N \times X_2^N \]

are asymptotically equivalent to \(\chi_N\).

**Proof.** Apply Bernoulli approximation theorem to \(\phi_1 = \phi_1(y_1)\) and \(\phi_2 = \phi_2(y_2)\) that are the pushforwards of the measure \(\chi(x_1, x_2)dx_1dx_2\) to \(X_1\) and to \(X_2\) under the projections maps \(X_1 \times X_2 \Rightarrow X_1, X_2\).

**Proof of [LWS].** Apply lemma to the subsets \(V_i \subset Y = X_K = \times_{k \in K} X_k\) and the splittings \(Y = X_{J_i} \times X_{J_2}\) for all decomposition \(K = J_1 \cup J_2\) and thus reduce the general [LWS] to the trivial case, where all \(V_i\) are products sets, \(V_i = \times_{k \in K} V_{i,k}\), \(V_{i,k} \subset X_k\). Q.E.D

**[LW]-Corollary** (Loomis-Whitney theorem V from section 1) The volumes of subsets \(V \subset \mathbb{R}^n\) are bounded by the volumes of their \(n\) projections \(V_i\) to the coordinate hyperplanes as follows:

\[ \text{vol}_n(V) \leq \prod_{i=1}^{n} \text{vol}_{n-1}(V_i)^{1/n-1}. \]

Indeed, this is [LWS] for subsets \(J_i \subset K\), which are complements to points \(k \in K\).

**Non-Sharp Isoperimetric Subcorollary.** Let \(V \subset \mathbb{R}^n\) be a domain with a smooth boundary. Then

\[ \text{vol}(V) \leq \left( \frac{1}{2n} \text{vol}_{n-1}(\partial V) \right)^{n/(n-1)}. \]

In fact, \(\text{vol}(V) \leq \frac{1}{2} \text{vol}_{n-1}(\partial V)\) and the arithmetic/geometric mean inequality applies.
Exercises. (a) Derive [HLWS] from the classical Hölder by induction on card(K).

(b) Prove the following Bolloás-Thomason Box Theorem. Given a bounded measurable subset \( V \subset \mathbb{R}^n \), there is a rectangular parallelepiped \( U \) of the same volume as \( V \), such that the projection of \( U \) onto any coordinate subspace is at most as large as that of the corresponding projection of \( V \).

Apology. I couldn’t find the above "Bernoulli proof" of [HLWS] in the literature and recorded it in [???] and [???]. My apologies to the person who was the first to use it.

Remark. Besides Bernoullian, there are other "equalization techniques" such as Knöte map, Brenier’s solution to Monge-Kantorovich transportation problem in the proof of Bracamp-Lieb refinement of the Shannon-Loomis-Whitney-Shearer inequality (see [?] and references therein) and invertibility of some Hodge operators on toric Kähler manifolds as in the analytic rendition of Khovanski-Teissier proof of the Alexandrov-Fenchel inequality for mixed volumes of convex sets [?]. It is tempting to try to find "quantum counterparts" to these proofs.

Also it is desirable to find more functorial and more informative proofs of "natural" inequalities in geometric (monoidal?) categories. (See [?],[?] for how it goes along different lines.)

7.5.1 Reverse Loomis-Whitney Inequality

Let \( V_{i,d} \subset V \subset \mathbb{R}^n \) be the union of straight segments \( I \), contained in \( V \), which are parallel to the \( i \)-th coordinate axes in \( \mathbb{R}^n \) and such that \( \text{length}(I) = d \).

Observe that the volume of the complement to \( V_{i,d} \) is bounded by the \((n-1)\)-volume of the boundary of \( V \) as follows,

\[
[\text{Vol} < d] \quad \text{vol}(V \setminus V_{i,d}) \leq d \cdot \text{vol}(\partial V).
\]

Next, given numbers \( d_i \geq 0 \), \( i = 1, \ldots, n \), let

\[
\epsilon_i = \frac{\text{vol}(V \setminus V_{i,d_i})}{\text{vol}(V)}
\]

and rewrite \([\text{Vol} < d, \ldots]\) as

\[
[\text{Vol} < d_i/\epsilon_i, \ldots] \quad \text{vol}(V) \leq \frac{d_i}{\epsilon_i} \text{vol}_{n-1}(\partial V), \ i = 1, \ldots, n.
\]

Let

\[
V \sqcup = \bigcap_i V_{i,d_i} \subset V;
\]

observe that

\[
[\text{Vol} \geq \ldots] \quad \text{vol}(V \sqcup) \geq \left(1 - \sum_{i=1}^{n} \epsilon_i\right) \text{vol}(V).
\]
and that, by the mean value theorem, there exists an affine hyperplane \( A = A^{n-1} \subset \mathbb{R}^n \) parallel to the first \( n - 1 \) coordinate axes, such that

\[
vol_{n-1}(V \cap A) \geq \left(1 - \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \right) vol_{n-1}(V \cap A).
\]

Since

\[
V \cap A_{i,d_i} \supset V \cap A_{i=1,...,n-1},
\]

this shows that

\[
vol_{n-1}(V \cap A)_{i,d_i} \geq \left(1 - \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \right) vol_{n-1}(V \cap A).
\]

our \( V \) contains a large part of the volume, say more than one half, of a Euclidean rectangular \( \times_1 d_i \)-solid

\[
\times_{i=1}^{n} [a_i,b_i] \subset \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}, \quad [a_i,b_i] \subset \mathbb{R}, \quad b_i - a_i = d_i.
\]

It follows, that

\[
d_{i_o} = \min_i d_i \leq 2\sqrt{vol(V)}
\]

and \( [Vol < d_i/\varepsilon_i... ] \) for \( i = i_o \) shows that

\[
vol(V) \leq \frac{2\sqrt{vol(V)}}{\varepsilon_{i_o}} vol_{n-1}(\partial V),
\]

that is the isoperimetric inequality with an albeit depending only on \( n \), but an comfortably large constant,

\[
isop:C_0
\]

\[
vol(V)^{n-1/n} \leq C_0 vol_{n-1}(\partial V) \text{ for } C_0 = 2/\varepsilon_{i_o}.
\]

Here is a justification of ■ and [isop:C_0].

**Proposition.** There

Let \( \boxtimes^n = \times_{i=1}^{n} [0,a_i] \) be the rectangular solid and \( V \subset \boxtimes^n \) be an open subset. and let the (topological) boundary of \( V \) in \( \boxtimes^n \) is bounded by

\[
vol_{n-1}(\partial V) \leq \varepsilon vol_n(V), \text{ for some } \varepsilon > 0.
\]
Then $V$ contains a product subset

$$V_\square = \bigtimes_{i=1}^n S_i, S_i \subset [0, a_i],$$

such that the volume of $V_\square$ is bounded from below by

$$\text{vol}_n(V_\square) \geq (1 - n\varepsilon)\text{vol}_n(V)$$

Let $\hat{V}_i \subset V$, $i = 1, \ldots, n$, be the union of $[0, a_i]$-segments which are fully contained in $V$,

$$\hat{V}_i = \{x_1, \ldots, x_i, \ldots, x_n\}, \text{ such that } (x_1, \ldots, x_i, \ldots, x_n) \in V, \text{ for all } x'_i \in [0, a_i].$$

If the $(n-1)$-volume of the boundary of $V$ in $\square^N$ is $\varepsilon$-small compare to the volume of $V$

$$\text{vol}_{n-1}(\partial V) \leq \varepsilon \text{vol}_n(V), \varepsilon > 0$$

Second Isoperimetric Application of [LW]

Let $A, B \subset Y = \times_{k \in K} X_k$, $k \in K$, where $X_k = (X_k, \mu_k)$ are be probability measure spaces, be measurable subsets, such that the images of $A$ and $B$ under the projections

$$\pi_k : Y \to Y_k = \bigtimes_{j \in K \setminus \{k\}} X_j$$

do not intersect $Y_K$ for all $k \in K$.

**Lemma.** If the (product probability) measure of the union of $A$ and $B$ in $Y$ is bounded from below by

$$\mu(A \cup B) \geq 1 + \delta$$

then

$$\min(\mu(A), \mu(B)) \leq C_n \delta$$

where $C_n \leq ???, n = \text{card}(K)$.

**Proof.**

**Corollary.** Let $U \subset [0, 1]^n$ be a subset in the unit cube with $\text{vol}(U) \leq \delta$. Then there exist a connected subset in the complement of $U$, say $V \subset [0, 1]^n \setminus U$, with volume

$$V \geq 1 - D_n \delta,$$

where $D_n \leq ???. Proof.??$

**Rewrite**... be disjoint as well $\Gamma_1$-disjoint subsets, i.e. there is no edge in $\Gamma_1$ between their points. Let $\mu_1^* (M_\square) \geq \mu_1^* (M_\varepsilon)$ and let $\mu_1^* (M_\square \cup M_\varepsilon) \geq 1 \delta$. Then

$$\mu_1^* (M_\varepsilon) \leq C_n \delta,$$

where $C_n \leq ???. Proof. ??$

let the pullbacks of points by the natural maps (projections) $Y \to Y_k = \times_{j \in K \setminus \{k\}} X_j$, be called $X_k$-*lines* or just *lines*, and where the set of $X_k$-*lines* is naturally identified with $Y_k$.

---

46This is just to simplify notation.
Let $\Gamma_L$ be the (naturally $K$-colored) graph of "lines" with the vertex set

$$L = \bigsqcup_{k \in K} Y_k, k \in K,$$

where the pairs of intersecting lines in $Y$ are taken for the edges.

**Preisometric Lemma** Let $K = \{1, \ldots, n\}$, let $(X_k, \mu_k)$ be probability measure spaces, i.e. $\mu_k(X_k) = 1$, let the product spaces $Y_k$ be given the corresponding product measures, denoted $\mu_k^n$, and let $M_k \subset Y_k$ be measurable subsets with measures

$$\mu_k^n(M_k) > 1 - \varepsilon_k \geq 0$$

Then there exists a connected subgraph in $\Gamma_L$ with the vertex set $M'_1 \subset M_1$, where

$$\mu_1^n(M'_1) \geq 1 - A_n \sum_{k \in K} \varepsilon_k$$

for $A_n \leq \frac{1}{n}$.

**Proof.** Let is introduce another ($\{2, \ldots, n\}$-colored) graph $\Gamma_1$ now on the vertex set $Y_1$ where two points are joint by a $k$-colored edge if they lie on the projection of an $X_k$-"line" to $Y_1$.

**Sublemma.** Let $M^>, M^< \subset Y_1$ be disjoint as well $\Gamma_1$-disjoint subsets, i.e. there is no edge in $\Gamma_1$ between their points. Let $\mu_1^n(M^>) \geq \mu_1^n(M^<)$ and let $\mu_1^n(M^> \cup M^<) \geq 1 - \delta$. Then

$$\mu_1^n(M^<) \leq C_n \delta,$$

where $C_n \leq \frac{1}{n}$.

**Proof.** Let

$$\pi_{1,k} : Y_1 \rightarrow Y_{1,k} = \prod_{j \in K \setminus\{k\}} X_j, k = 2, \ldots, n$$

be the natural maps (projections) and let $a^>_k$ and $a^<_k$ be the measures of the images of $M^>$ and of $M^<$ under these maps.

$\Gamma_1$-disjointness of $M^>$ and of $M^<$ says that these images are disjoint and the [LW]-inequality shows that

$$\left( \sqrt[n]{\prod_{k=1}^{n-1} a^>_k} \right)^{n/n-1} + \left( \sqrt[n]{\prod_{k=1}^{n-1} a^<_k} \right)^{n/n-1} \geq 1 + \delta.$$

Since

$$\sqrt[n]{\prod_{k=1}^{n-1} a^>_k} + \sqrt[n]{\prod_{k=1}^{n-1} a^<_k} \leq 1$$

by the geometric-arithmetic mean inequality, and since $(1 - t)^{n/n-1} \geq 1 - tn/n - 1, 0 \leq t \leq 1$,

$$\mu_k(M_1) \leq \left( \sqrt[n]{\prod_{k=1}^{n-1} a^<_k} \right)^{n/n-1} \leq (n - 1) \delta.$$

QED.

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47 This is just to simplify notation.
Let $\Gamma$ be a finite edge colored graph on a set $V$, where the set of colors $k$ is denoted $K \ni k$ of a finite set $K$, let $V_k \subset V$ be the sets of ends of $k$-colored edges and $V_k^\hat{}$ be the sets of connected components of $V_k$, and let $\pi_k : V \rightarrow V_k^\hat{}$ be the natural (quotient) maps.

Let $V$ and $V_k^\hat{}$ be endowed with measures $\mu$ and $\mu_k$ structures, such that the subsets $V_k \subset V$ are measurable (e.g. $V_k = V$)) and the maps $\pi_k : V \rightarrow V_k^\hat{}$ are measure preserving.

Let $c^i = c^i(\Gamma)$ is the infimum of the numbers $c \geq 0$ with the following property;

Given measurable subsets $U_k \subset V_k^\hat{}$ there exists a connected subgraph in $\Gamma$ on a measurable vertex subset $U^\perp$ such that

- the $\pi_k$-images of $U^\perp$ lie in the complements to $U_k$,

\[ \pi_k(U_k) \subset V_k \setminus U_k; \]

- the inequality

\[ \mu(V \setminus U^\perp) \leq c \sum_{k \in K} \mu_k(U_k) \]

Cubical Example. Let $K = \{1, \ldots, n\}$, let $V = [0, 1]^n$, let $p_k$ be the projections of the cube to its coordinate faces $V_k$,

\[ p_k : (x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \]

such that the edges of the graph $\Gamma$ are pair of vertices $(v_1, v_2)$ on same coordinate lines in the cube, i.e. $p_k(v_1) = p_k(v_2)$ for some $k$.

Then, given $U_k \subset V_k$, let $U^\perp$ be the union of all lines which meet through the intersection $\cap_k \pi_k^{-1}(V_k \setminus U_k)$.

Thus,

\[ \mu(U^\perp) \geq \mu(\bigcap_{k \in K} \pi_k^{-1}(V_k \setminus U_k)) \geq \sum_{k \in K} \mu(U_k) \]

and all lines in $\mu(U^\perp)$ meet other lines in the remaining $(n-1)$ directions.

The latter implies the connectedness of the subset $U^\perp$; thus, the inequality $c^i \leq 1$ for this $\Gamma$.

Isoperimetric Corollary. (Compare with ??? in section 3.3.) Let $Y \subset [0, 1]^n$ be a hypersurface, which divides the cube into two parts, say $W_1$ and $W_2$. Then

\[ \min(\text{vol}(W_1), \text{vol}(W_2)) \leq n \cdot \text{vol}_{n-1}(Y). \]

Proof. Apply the above to the projections $U_k = \pi_k \subset V_k$ and show that

\[ \max(\text{vol}(W_1), \text{vol}(W_2)) \geq 1 - n\text{vol}_{n-1}Y. \]

7.5.2 Combinatorial Shannon and Harper Inequalities.

7.5.3 Linearized Loomis-Whitney-Shearer Inequality

Let

\[ Y \subset X_K = \bigtimes_{k \in K} X_k, \quad K = \{1, \ldots, n\}, \]
be a finite set let $L_k$ be the linear spaces of functions on $X_k$, let $L_0 = L_Y$ be the space of functions on $Y$ and let

$$\Phi(l_0, l_1, \ldots, l_k) = \sum_{y \in Y} l_0(y) \cdot l_1(x_1) \cdot \ldots \cdot l_k(x_k).$$

be an $(n + 1)$-linear form in the variables $l_i$, $i = 0, 1, \ldots, n$, where every subset $J \subset K = \{1, \ldots, n\}$ turns this $\Phi$ into a bilinear form $\Phi_J$ between tensor product spaces, $L_J$ and $L_{J^\perp}$,

$$L_J = \bigotimes_{i \in J} L_i \text{ and } L_{J^\perp} = L_0 \otimes (\bigotimes_{i \in K \setminus J} L_i).$$

Since $\text{rank}(\Phi_J) = \text{card}(Y_J)$, where $Y_J \subset X_J = \times_{i \in J} X_j$ is the projection of $Y$ to $X_J$ (check it!) the [LWS] inequality (for functions constant on their supports $Y_J$) says in this terms that

$$(P_{\Phi}) \quad \prod_{J_i \subset K, i \in I} (\text{rank}(\Phi_{J_i}))^{\beta_i} \geq \text{rank}(Y)$$

for all partitions of unity $(J_i, \beta_i)$, $i \in I$ on $K$.

**Linearized** [LWS] claims that this inequality holds true for all $(n + 1)$-linear forms $\Phi(l_0, l_1, \ldots, l_N)$.

This can be reduced (this is easy) to the original combinatorial [LWS] by using a suitable basis in $L_K$ or proven by the Bernoulli approximation argument applied to $L_K^{\otimes N}$, with $N \to \infty$ (see ???[structure], [action]).

**Example:** The linearized Loomis-Whitney 3D-isoperimetric inequality for ranks of bilinear forms associated with a 4-linear form $\Phi = \Phi(l_0, l_1, l_2, l_3)$ reads

$$|\Phi_{0,123}|^2 \leq |\Phi_{01,23}| \cdot |\Phi_{02,13}| \cdot |\Phi_{03,12}|$$

where $|\ldots|$ stands for $\text{rank}(\ldots)$.

**Remark.** Probably, the linear [LWS]-inequalities are the only universal relations between the ranks of $\Phi_J$, but there are further inequalities of this type for particular polynlinear forms, e.g. defined by the $\vee$-product in the cohomology algebra of certain manifolds (see [expanders, singularities]) and also in spaces of sections and (cohomologies in general) of holomorph vector bundles such e.g. as in the Khovanski-Teissier theorem and in the Esnault-Viehweg proof of the generalized Dyson-Roth lemma, but a direct link between all such inequalities is yet to be found.

### 7.6 Isoperimetry in the Exterior Algebras

### 7.7 Strong Subadditivity of the von Neumann Quantum Entropy

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### 8 Fixed Points, Amenability, T-property and Isoperimetry in Groups and Algebras

*von-Neumann*
8.1 "Parallel" Mass Transport in Groups and Saloff-Coste bound on the Fölner-Vershik function

8.2 Kazhdan’s $T$-Property, Margulis’ Expanders, Spectral Logic, Garland theorem, High dimensional Expanders

$X \subset B^N(R) \Rightarrow Vol(X) \leq 1/nvol(\partial X)$ In fact,

$vol(X) = \int_{Y=\pi_v(X-x_0)} dy$ $\pi_v$ is the projection to the normal line to $Y$ at $y$.

(This integral doesn’t depend on $x_0$.)

Average intersection of $Y$ with the $(n-1)$ faces of an $\varepsilon$-cubilation of $\mathbb{R}^n$

9 Measure Concentration

9.1 Talagran Inequality

9.2 Poincare Concentration Inequalities for Mapping to Wirtinger and other Spaces

9.3 Stability of Matter

10 Waist Inequalities

11 Isoperimetry Settings and Directions of Generalizations

1. Given Euclidean vector bundles over a Riemannian manifold

$V_0, V_1, \ldots, V_k \to X,$

and linear differential operators on spaces of sections $X \to V_i$.

$D_i : C^\infty(V_0) \to C^\infty(V_i), i = 1, \ldots, k$

evaluate (the size of) the set of values of the $L_{p_i}$-norms of these sections for given $p_i$,

$\mathcal{F} = \left\{ \left( \int_X D_i f(x)^{p_i} dx \right)^{1/p_i} \right\}_{f \in C^\infty(V_0)} \subset \mathbb{R}_+^k$

For instance, decide when $\mathcal{F} \neq \mathbb{R}_+^k$.

More generally, study possibilities for the joint distribution of $\|D_i f\|$ regarded as random variables on $X$.

Example
12 Isoperimetry for Families, Spectra and Morse

13 Poincare-Hahn Banach duality

14 Isoperimetry Problems Inspired by Biology

14.1 Micella, Nash Blow up and Higher Order Soap Bubbles

14.2 Viral Isoperimetry: Minimization of Information for Building the Wall around the Carrier of this Information

15 Appendices

15.1 Basics on Curvature

§2 https://link.springer.com/article/10.1007/BF02925201

We enlist in this section several classical formulas of Riemannian geometry and indicate their (more or less) immediate applications.

15.2 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) Riemannian Variation Formula. Let $h_t, t \in [0, \varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold $Y$ and let us incorporate $h_t$ to the metric $g = h_t + dt^2$ on $Y \times [0, \varepsilon]$.

Notice that an arbitrary Riemannian metric on an $n$-manifold $X$ admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The $t$-derivative of $h_t$ is equal to twice the second fundamental form of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0, \varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_t h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_v h = 2A^*,$$

where $\nu$ is the unit normal field to $Y$ defined as $\nu = \frac{\nu}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the principal values $\alpha_i^*(y), i = 1, ..., n-1$, of the quadratic form $A_i^*$ on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal
vectors $\tau_i^* \in T_i(Y)$, which diagonalize $A^*$, are called the principal curvatures of $Y$, and that the sum of these is called the mean curvature of $Y$,

$$\text{mean.curv}(Y, y) = \sum_i \alpha_i^*(y),$$

where, in fact,

$$\sum_i \alpha_i^*(y) = \text{trace}(A^*) = \sum_i A^*(\tau_i)$$

for all orthonormal tangent frames $\tau_i$ in $T_y(Y)$ by the Pythagorean theorem.

**Sign Convention.** The first derivative of $h$ changes sign under reversion of the $t$-direction. Accordingly the sign of the quadratic form $A^*(Y)$ of a hypersurface $Y \subset X$ depends on the coorientation of $Y$ in $X$, where our convention is such that the boundaries of convex domains have positive (semi)definite second fundamental forms $A^*$, also denoted $\text{II}_Y$, hence, positive mean curvatures, with respect to the outward normal vector fields.\footnote{At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.}

\footnote{At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.}

**(2.1.B) First Variation Formula.** This concerns the $t$-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$\partial_t \text{vol}_{n-1}(U_t) = \frac{dh_t}{dt} \text{vol}_{n-1}(U_t) = \int_{U_t} \text{mean.curv}(U_t) dy_t$$

where $dy_t$ is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi_U = \psi \cdot \nu$ for $C^1$-smooth functions $\psi = \psi(y)$ as follows

$$\partial_{\psi_U} \text{vol}_{n-1}(Y_t) = \int_{Y_t} \psi(y) \text{mean.curv}(Y_t) dy_t$$

Now comes the first formula with the Riemannian curvature in it.

**15.3 Gauss’ Theorema Egregium**

Let $Y \subset X$ be a smooth hypersurface in a Riemannian manifold $X$. Then the sectional curvatures of $Y$ and $X$ on a tangent 2-plane $\tau \subset T_y(Y) \subset T_y(X)$, $y \in Y$, satisfy

$$\kappa(Y, \tau) = \kappa(X, \tau) + \lambda^2 A^*(\tau),$$

where $\lambda^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form $A^* = A^*(Y) \subset X$ restricted to the plane $\tau$,

$$\lambda^2 A^*(\tau) = \alpha_1^2(\tau) \cdot \alpha_2^2(\tau).$$

\footnote{This come with the minus sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal) 2019].}

\footnote{This come with the minus sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal) 2019].}

\footnote{This remains true for Lipschitz functions but if $\psi$ is (badly) non-differentiable, e.g. it is equal to the characteristic function of a domain $U \subset Y$, then the derivative $\partial_{\psi_U} \text{vol}_{n-1}(Y_t)$ may become (much) larger than this integral.}
This, with the definition the scalar curvature by the formula \( Sc = \sum \kappa_{ij} \), implies that
\[
Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^*(y)\alpha_j^*(y) - \sum_i \kappa_{\nu,i},
\]
where:
- \( \alpha_i^*(y) \), \( i = 1, ..., n - 1 \) are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors \( \tau_i \) in \( T_y(Y) \);
- \( \alpha^* \)-sum is taken over all ordered pairs \((i, j)\) with \( j \neq i \);
- \( \kappa_{\nu,i} \) are the sectional curvatures of \( X \) on the bivectors \((\nu, \tau_i)\) for \( \nu \) being a unit (defined up to \( \pm \)-sign) normal vector to \( Y \);
- the sum of \( \kappa_{\nu,i} \) is equal to the value of the Ricci curvature of \( X \) at \( \nu \),
\[
\sum_i \kappa_{\nu,i} = \text{Ricci}_X(\nu, \nu).
\]

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of \( Y \) and that in the case of \( Y = S^{n-1} \subset \mathbb{R}^n = X \) this gives the correct value \( Sc(S^{n-1}) = (n-1)(n-2) \).

Also observe that
\[
\sum_{i \neq j} \alpha_i\alpha_j = \left( \sum_i \alpha_i \right)^2 - \sum_i \alpha_i^2,
\]
which shows that
\[
Sc(Y) = Sc(X) + (\text{mean.curv}(Y))^2 - ||A^*(Y)||^2 - \text{Ricci}(\nu, \nu).
\]

In particular, if \( Sc(X) \geq 0 \) and \( Y \) is minimal, that is \( \text{mean.curv}(Y) = 0 \), then
\[
(Sc \geq -2\text{Ric}) \quad Sc(Y) \geq -2\text{Ricci}(\nu, \nu).
\]

Example. The scalar curvature of a hypersurface \( Y \subset \mathbb{R}^n \) is expressed in terms of the mean curvature of \( Y \), the (point-wise) \( L_2 \)-norm of the second fundamental form of \( Y \) as follows.
\[
Sc(Y) = (\text{mean.curv}(Y))^2 - ||A^*(Y)||^2
\]
for \( ||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2 \), while \( Y \subset S^n \) satisfy
\[
Sc(Y) = (\text{mean.curv}(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n\max_i (\alpha_i^*)^2.
\]

It follows that minimal hypersurfaces \( Y \) in \( \mathbb{R}^n \), i.e. these with \( \text{mean.curv}(Y) = 0 \), have negative scalar curvatures, while hypersurfaces in the \( n \)-spheres with all principal values \( \leq \sqrt{n-2} \) have \( Sc(Y) > 0 \).

Let \( A = A(Y) \) denote the shape that is the symmetric on \( T(Y) \) associated with \( A^* \) via the Riemannian scalar product \( g \) restricted from \( T(X) \) to \( T(Y) \),
\[
A^*(\tau, \tau) = \langle A(\tau), \tau \rangle_g \text{ for all } \tau \in T(Y).
\]

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15.4 Variation of the Curvature of Equidistant Hypersurfaces and Weyl’s Tube Formula

(2.3.A) Second Main Formula of Riemannian Geometry. Let $Y_t$ be a family of hypersurfaces $t$-equidistant to a given $Y = Y_0 \subset X$. Then the shape $A_t = A(Y_t)$ satisfy:

$$\partial_t A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where $B_t$ is the symmetric associated with the quadratic differential form $B^*$ on $Y_t$, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the sectional curvature of $g$ at (the 2-planes spanned by) the bivectors $(\tau, \nu = \frac{d}{dt})$.

Remark. Taking this formula for the definition of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic Riemannian comparison theorems along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry.

Tracing this formula yields

(2.3.B) Hermann Weyl’s Tube Formula.

$$\text{trace} \left( \frac{dA_t}{dt} \right) = -\|A^\ast\|^2 = \text{Ricci}_g \left( \frac{d}{dt}, \frac{d}{dt} \right),$$
or

$$\text{trace}(\partial_t A) = \partial_t \text{trace}(A) = -\|A^\ast\|^2 = \text{Ricci}(\nu, \nu),$$

where

$$\|A^\ast\|^2 = \|A\|^2 = \text{trace}(A^2),$$

where, observe,

$$\text{trace}(A) = \text{trace}(A^\ast) = \text{mean.curv} = \sum_i \alpha_i^\ast,$$

and where $\text{Ricci}$ is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_y(X)$ is equal to the trace of the above $B^*$-form (or of the $\mathcal{B}$) on the normal hyperplane $\nu^\perp \subset T_y(X)$ (where $\nu^\perp = T_y(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$\text{Sc}(X) = \text{trace}(\text{Ricci})$$

and that the above formula $\text{Sc}(Y, y) = \text{Sc}(X, y) + \sum_{i \neq j} \alpha_i^\ast \alpha_j^\ast - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$\text{Ricci}(\nu, \nu) = \frac{1}{2} \left( \text{Sc}(X) - \text{Sc}(Y) - \sum_{i \neq j} \alpha_i^\ast \cdot \alpha_j^\ast \right).$$

\[51\] The first main formula is Gauss’ Theorema Egregium.

\[52\] Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl’s Tube Formula", appears in [Dahos (Gravitation einsteinienne) 1927] with the reference to Darboux’ textbook of 1897.
\[
\frac{1}{2} \left( Sc(X) - Sc(Y) - (\text{mean.curv}(Y))^2 + \|A^*\|^2 \right)
\]

where, recall, \( \alpha_i^* = \alpha_i^*(y), y \in Y, i = 1, ..., n-1 \), are the principal curvatures of \( Y \subset X \), where \( \text{mean.curv}(Y) = \sum_i \alpha_i^* \) and where \( \|A^*\|^2 = \sum_i (\alpha_i^*)^2 \).

15.5 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface \( Y \subset X \) is called \textit{umbilic} if all principal curvatures of \( Y \) are mutually equal at all points in \( Y \).

For instance, spheres in the \textit{standard} (i.e. complete simply connected) \textit{spaces} \( \text{with constant curvatures} \) (spheres \( S^n_{\alpha,0} \), Euclidean spaces \( \mathbb{R}^n \) and hyperbolic spaces \( H^n_{\alpha,0} \)) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let \( Y = (Y, h) \) be a smooth Riemannian \((n-1)\)-manifold and \( \varphi = \varphi(t) > 0, t \in [0, \varepsilon] \) be a smooth positive function. Let \( g = h_t + dt^2 = \varphi^2 h + dt^2 \) be the corresponding metric on \( X = Y \times [0, \varepsilon] \).

Then the hypersurfaces \( Y_t = Y \times (t) \subset X \) are umbilic with the principal curvatures of \( Y_t \) equal to \( \alpha_i^*(t) = \frac{\varphi'(t)}{\varphi(t)^2}, i = 1, ..., n-1 \) for

\[
A_t^* = \frac{\varphi'(t)}{\varphi(t)} h_t \quad \text{for} \quad \varphi' = \frac{d\varphi(t)}{dt} \quad \text{and} \quad A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}.
\]

The Weyl formula reads in this case as follows.

\[
(n-1) \left( \frac{\varphi'}{\varphi} \right)' = - (n-1)^2 \left( \frac{\varphi'}{\varphi} \right)^2 - \frac{1}{2} \left( Sc(g) - Sc(h_t) - (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2 \right).
\]

Therefore,

\[
Sc(g) = \frac{1}{\varphi^2} Sc(h) - 2(n-1) \left( \frac{\varphi'}{\varphi} \right)' - n(n-1) \left( \frac{\varphi'}{\varphi} \right)^2 =
\]

\[
= \frac{1}{\varphi^2} Sc(h) - 2(n-1) \frac{\varphi''}{\varphi} - (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2,
\]

where, recall, \( n = \text{dim}(X) = \text{dim}(Y) + 1 \) and the mean curvature of \( Y_t \) is

\[
\text{mean.curv}(Y_t) \subset X = (n-1) \frac{\varphi'(t)}{\varphi(t)}.
\]

**Examples.** (a) If \( Y = (Y, h) = S^{n-1} \) is the unit sphere, then

\[
Sc(g) = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1) \frac{\varphi''}{\varphi} - (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2,
\]

which for \( \varphi = t^2 \) makes the expected \( Sc(g) = 0 \), since \( g = dt^2 + t^2 h, t \geq 0 \), is the Euclidean metric in the polar coordinates.

If \( g = dt^2 + \sin t^2 h, -\pi/2 \leq t \leq \pi/2 \), then \( Sc(g) = n(n-1) \) where this \( g \) is the spherical metric on \( S^n \).
(b) If $h$ is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(H^n_{n-1}).$$

(c) What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_t^{\frac{t}{\tan \frac{\pi}{n}}} - \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where $h$ is flat, is constant positive, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation\(^{53}\)

Cylindrical Extension Exercise. Let $Y$ be a smooth manifold, $X = Y \times \mathbb{R}_+$, let $g_0$ be a Riemannian metric in a neighbourhood of the boundary $Y = Y \times (0) = \partial X$, let $h$ denote the Riemannian metric in $Y$ induced from $g_0$ and let $Y$ has constant mean curvature in $X$ with respect to $g_0$.

Let $X'$ be a (convex if you wish) ball in the standard (i.e complete simply connected) space with constant sectional curvature and of the same dimension $n$ as $X$, let $Y' = \partial X'$ be its boundary sphere, let, let $Sc(h) > 0$ and let the mean and the scalar curvatures of $Y$ and $Y'$ are related by the following (comparison) inequality.

$$\frac{|mean.curv_{g_0}(Y)|^2}{Sc(h, y)} < \frac{|mean.curv(Y')|^2}{Sc(Y')}\quad \text{for all } y \in Y.$$

Show that

if $Y$ is compact, there exists a smooth positive function $\varphi(t), 0 \leq t < \infty$, which is constant at infinity and such that the the warped product metric $g = \varphi^2 h + dt^2$

has

the same Bartnik data as $g_0$, i.e.

$$g(Y = h_0 \text{ and } mean.curv_y(Y) = mean.curv_{g_0}(Y).$$

Then show that

one can't make $Sc(g) \geq Sc(X')$ in general, if $[<]$ is relaxed to the corresponding non-strict inequality, where an example is provided by the Bartnik data of $Y' \in X'$ itself.\(^{54}\)

Vague Question. What are "simple natural" Riemannian metrics $g$ on $X = Y \times \mathbb{R}_+$, with given Bartnik data $(Sc(Y), mean.curv(Y))$, where $Y \subset X$ is allowed variable mean curvature, and what are possibilities for lower bound on the scalar curvatures of such $g$ granted $|mean.curv(Y, y)|^2/Sc(Y, y) < C$, e.g. for $C = |mean.curv(Y')|^2/Sc(Y')$ for $Y'$ being a sphere in a space of constant curvature.

Curvature Formulas for Manifolds and Submanifolds.

\(^{53}\)See §12 in [GL(complete) 1983].

\(^{54}\)It follows from [Brendle-Marques(balls in $S^n$)](2011) that the the cylinder $S^{n-1} \times \mathbb{R}_+$ admits a complete Riemannian metric $g$ cylindrical at infinity which has $Sc(g) > n(n-1)$, and which has the same Bartnik data as the boundary sphere $X_0'$ in the hemisphere $X'$ in the unit $n$-sphere. But the non-deformation result from [Brendle-Marques(balls in $S^n$)](2011) suggests that this might be impossible for the Bartnik data of small balls in the round sphere.
15.5.1 Comparison Inequalities

15.6 Carnot-Caratheodory Spaces

16 Amenability and Isoperimetry in Groups and Algebras

17 references

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**FOUR ISOPERIMETRIC PROPERTIES OF HOMOGENEOUS SPHERICAL MEMBRANES JOSEPH HERSCH**
Abstract. With the help of a “conformal mapping” [1] of the coordinate functions \( x^+, y^+, z^+ \) of the sphere, we will show four inequalities for inhomogeneous vibrating membranes: the first on a surface of the type of a sphere, the second on a Jordan domain, the third on a “bilatere” and the fourth on a ”trilatere” [2]
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