

A Rigidity Lemma and the Kirszbraun Theorem

1 A basic identity

Let $x_1, \dots, x_N$ be vectors in a Euclidean space E . For numbers

$$\lambda_i \geq 0, \quad \sum_i \lambda_i = 1,$$

one has

$$\frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x_i - x_j|^2 = \sum_i \lambda_i |x_i|^2 - \left| \sum_i \lambda_i x_i \right|^2.$$

Indeed,

$$|x_i - x_j|^2 = |x_i|^2 + |x_j|^2 - 2\langle x_i, x_j \rangle,$$

and therefore

$$\begin{aligned} \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x_i - x_j|^2 &= \sum_i \lambda_i |x_i|^2 - \sum_{i,j} \lambda_i \lambda_j \langle x_i, x_j \rangle \\ &= \sum_i \lambda_i |x_i|^2 - \left| \sum_i \lambda_i x_i \right|^2. \end{aligned}$$

In particular, if

$$\sum_i \lambda_i x_i = 0,$$

then

$$\boxed{\frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x_i - x_j|^2 = \sum_i \lambda_i |x_i|^2.}$$

2 The spherical rigidity lemma

Lemma 1. *Suppose that*

$$x_1, \dots, x_N \in S^{n-1}$$

and that

$$0 \in \text{conv}\{x_1, \dots, x_N\}.$$

Thus there exist numbers

$$\lambda_i > 0, \quad \sum_i \lambda_i = 1, \quad \sum_i \lambda_i x_i = 0.$$

Let

$$x'_1, \dots, x'_N \in S^{m-1}$$

be another collection of unit vectors, and suppose that

$$|x'_i - x'_j| \geq |x_i - x_j| \quad \text{for all } i, j.$$

Then

$$|x'_i - x'_j| = |x_i - x_j| \quad \text{for all } i, j,$$

and

$$\sum_i \lambda_i x'_i = 0.$$

Proof. Applying the identity above to $x'_1, \dots, x'_N$ gives

$$\frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x'_i - x'_j|^2 = \sum_i \lambda_i |x'_i|^2 - \left| \sum_i \lambda_i x'_i \right|^2.$$

Since $|x'_i| = 1$,

$$\frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x'_i - x'_j|^2 = 1 - \left| \sum_i \lambda_i x'_i \right|^2 \leq 1.$$

On the other hand,

$$\frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x_i - x_j|^2 = \sum_i \lambda_i |x_i|^2 = 1,$$

because

$$\sum_i \lambda_i x_i = 0$$

and $|x_i| = 1$.

Since

$$|x'_i - x'_j| \geq |x_i - x_j|,$$

we obtain

$$1 \leq \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x'_i - x'_j|^2 \leq 1.$$

Hence equality holds throughout. In particular,

$$\sum_i \lambda_i x'_i = 0.$$

Also,

$$\sum_{i,j} \lambda_i \lambda_j (|x'_i - x'_j|^2 - |x_i - x_j|^2) = 0.$$

Every summand is nonnegative, and every λ_i is positive. Therefore

$$|x'_i - x'_j|^2 = |x_i - x_j|^2$$

for every i, j , and hence

$$|x'_i - x'_j| = |x_i - x_j|.$$

□

3 Variable lengths

Proposition 1. *Let*

$$x_1, \dots, x_N \in E$$

satisfy

$$0 \in \text{conv}\{x_1, \dots, x_N\}.$$

Choose

$$\lambda_i > 0, \quad \sum_i \lambda_i = 1, \quad \sum_i \lambda_i x_i = 0.$$

Let

$$x'_1, \dots, x'_N \in E'$$

satisfy

$$|x'_i - x'_j| \geq |x_i - x_j|$$

for all i, j , and

$$|x'_i| \leq |x_i|$$

for every i .

Then

$$|x'_i| = |x_i|$$

and

$$|x'_i - x'_j| = |x_i - x_j|$$

for all i, j . Moreover,

$$\sum_i \lambda_i x'_i = 0.$$

Proof. We have

$$\begin{aligned} \sum_i \lambda_i |x_i|^2 &= \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x_i - x_j|^2 \\ &\leq \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |x'_i - x'_j|^2 \\ &= \sum_i \lambda_i |x'_i|^2 - \left| \sum_i \lambda_i x'_i \right|^2 \\ &\leq \sum_i \lambda_i |x'_i|^2 \\ &\leq \sum_i \lambda_i |x_i|^2. \end{aligned}$$

Thus every inequality is an equality. Hence

$$\sum_i \lambda_i x'_i = 0,$$

and, since all $\lambda_i > 0$,

$$|x'_i| = |x_i|$$

for every i . Likewise,

$$|x'_i - x'_j| = |x_i - x_j|$$

for every i, j . □

Corollary 1. *Suppose*

$$0 \in \text{conv}\{z_1, \dots, z_N\},$$

and let $u_1, \dots, u_N$ be vectors such that

$$|z_i - z_j| \leq |u_i - u_j|.$$

If

$$\sum_i \lambda_i z_i = 0, \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0,$$

then

$$\boxed{\sum_i \lambda_i |z_i|^2 \leq \sum_i \lambda_i |u_i|^2.}$$

Proof. Since

$$\sum_i \lambda_i z_i = 0,$$

we have

$$\sum_i \lambda_i |z_i|^2 = \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |z_i - z_j|^2.$$

Hence

$$\begin{aligned} \sum_i \lambda_i |z_i|^2 &= \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |z_i - z_j|^2 \\ &\leq \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j |u_i - u_j|^2 \\ &= \sum_i \lambda_i |u_i|^2 - \left| \sum_i \lambda_i u_i \right|^2 \\ &\leq \sum_i \lambda_i |u_i|^2. \end{aligned}$$

□

4 Kirschbraun's theorem

Theorem 1 (Kirschbraun). *Let*

$$A \subset \mathbb{R}^n$$

and let

$$f : A \longrightarrow \mathbb{R}^m$$

be 1-Lipschitz:

$$|f(a) - f(b)| \leq |a - b| \quad \text{for all } a, b \in A.$$

Then f extends to a 1-Lipschitz map

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

Proof. It is enough to show that one can extend f by one point.

Let

$$x \in \mathbb{R}^n \setminus A.$$

For points $a_i \in A$, put

$$y_i = f(a_i), \quad r_i = |x - a_i|.$$

We want to find

$$y \in \mathbb{R}^m$$

such that

$$|y - y_i| \leq r_i$$

for every i .

Equivalently, we want to prove

$$\bigcap_i B(y_i, r_i) \neq \emptyset.$$

For a finite family $a_1, \dots, a_N$, consider

$$\Phi(y) = \max_i \{ |y - y_i|^2 - r_i^2 \}.$$

Let y_0 be a point where Φ attains its minimum, and write

$$R = \Phi(y_0).$$

If

$$R \leq 0,$$

then

$$|y_0 - y_i|^2 \leq r_i^2$$

for all i , and we are done.

Assume, therefore, that

$$R > 0.$$

Let I be the set of active indices:

$$I = \{i : |y_0 - y_i|^2 - r_i^2 = R\}.$$

At a minimum of the maximum of these quadratic functions,

$$0 \in \text{conv}\{y_0 - y_i : i \in I\}.$$

Therefore there exist

$$\lambda_i > 0, \quad \sum_{i \in I} \lambda_i = 1,$$

such that

$$\sum_{i \in I} \lambda_i (y_i - y_0) = 0.$$

Set

$$z_i = y_i - y_0$$

and

$$u_i = a_i - x.$$

Then

$$\sum_{i \in I} \lambda_i z_i = 0.$$

Since i is active,

$$|z_i|^2 = |y_i - y_0|^2 = r_i^2 + R = |a_i - x|^2 + R = |u_i|^2 + R.$$

Hence

$$\sum_i \lambda_i |z_i|^2 = \sum_i \lambda_i |u_i|^2 + R.$$

On the other hand, since f is 1-Lipschitz,

$$|z_i - z_j| = |y_i - y_j| \leq |a_i - a_j| = |u_i - u_j|.$$

By the preceding corollary,

$$\sum_i \lambda_i |z_i|^2 \leq \sum_i \lambda_i |u_i|^2.$$

Therefore

$$\sum_i \lambda_i |u_i|^2 + R \leq \sum_i \lambda_i |u_i|^2,$$

which implies

$$R \leq 0,$$

contrary to the assumption $R > 0$.

Thus

$$R \leq 0.$$

Hence there exists y_0 such that

$$|y_0 - y_i| \leq |x - a_i|$$

for every i .

Thus one may define

$$F(x) = y_0,$$

and the extended map remains 1-Lipschitz.

Repeating the one-point extension and using the standard compactness argument gives an extension

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

satisfying

$$|F(x) - F(x')| \leq |x - x'|$$

for all $x, x' \in \mathbb{R}^n$. □

5 The case of Lipschitz constant L

If

$$|f(a) - f(b)| \leq L|a - b|,$$

then applying the theorem to $L^{-1}f$ gives an extension F satisfying

$$|F(x) - F(x')| \leq L|x - x'|.$$

6 Generalization to Spheres and Hyperbolic Spaces

$$S^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}, \quad d_S(x, y) = \arccos\left(\sum_{i=1}^{n+1} x_i y_i\right).$$

Let $A \subset S^n$. If $f : A \rightarrow S^n$ satisfies

$$d_S(f(x), f(y)) \leq d_S(x, y) \quad \text{for all } x, y \in A,$$

then there is a map $F : S^n \rightarrow S^n$ such that

$$F(x) = f(x) \quad (x \in A), \quad d_S(F(x), F(y)) \leq d_S(x, y) \quad (x, y \in S^n).$$

$$H^n = \left\{x \in \mathbb{R}^{n+1} : x_0 > 0, \quad x_0^2 - \sum_{i=1}^n x_i^2 = 1\right\}, \quad d_H(x, y) = \operatorname{arcosh}\left(x_0 y_0 - \sum_{i=1}^n x_i y_i\right).$$

Let $A \subset H^n$. If $f : A \rightarrow H^n$ satisfies

$$d_H(f(x), f(y)) \leq d_H(x, y) \quad \text{for all } x, y \in A,$$

then there is a map $F : H^n \rightarrow H^n$ such that

$$F(x) = f(x) \quad (x \in A), \quad d_H(F(x), F(y)) \leq d_H(x, y) \quad (x, y \in H^n).$$

These are the spherical and hyperbolic extension statements attributed to Valentine.

For each $\kappa < 0$, define

$$X_\kappa = \left\{x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 > 0, \quad x_0^2 - \sum_{i=1}^n x_i^2 = \frac{1}{-\kappa}\right\}.$$

Equip X_κ with the distance

$$d_\kappa(x, y) = \frac{1}{\sqrt{-\kappa}} \operatorname{arcosh}\left((- \kappa) \left(x_0 y_0 - \sum_{i=1}^n x_i y_i\right)\right).$$

This is the n -dimensional space of constant curvature κ .

Extension theorem. Let $\kappa' \leq \kappa < 0$, let $A \subset X_\kappa$, and let $f : A \rightarrow X_{\kappa'}$ satisfy

$$d_{\kappa'}(f(x), f(y)) \leq d_\kappa(x, y) \quad \text{for all } x, y \in A.$$

Then f extends to a map $F : X_\kappa \rightarrow X_{\kappa'}$ satisfying

$$F|_A = f, \quad d_{\kappa'}(F(x), F(y)) \leq d_\kappa(x, y) \quad \text{for all } x, y \in X_\kappa.$$

Fix $\kappa > 0$. Define the sphere of curvature κ by

$$X_\kappa = \left\{x \in \mathbb{R}^{n+1} : |x| = \frac{1}{\sqrt{\kappa}}\right\}, \quad d_\kappa(x, y) = \frac{1}{\sqrt{\kappa}} \arccos(\kappa \langle x, y \rangle).$$

For $p \in X_\kappa$, its open and closed hemispheres are

$$H_\kappa^\circ(p) = \left\{ x \in X_\kappa : d_\kappa(p, x) < \frac{\pi}{2\sqrt{\kappa}} \right\}, \quad H_\kappa(p) = \left\{ x \in X_\kappa : d_\kappa(p, x) \leq \frac{\pi}{2\sqrt{\kappa}} \right\}.$$

1. Condition on the values. Let $0 < \kappa' \leq \kappa$, let $A \subset X_\kappa$, and let $f : A \rightarrow X_{\kappa'}$ satisfy

$$d_{\kappa'}(f(x), f(y)) \leq d_\kappa(x, y) \quad (x, y \in A).$$

If, for some $q \in X_{\kappa'}$,

$$f(A) \subset \left\{ y \in X_{\kappa'} : d_{\kappa'}(q, y) \leq \frac{\pi}{2\sqrt{\kappa}} \right\},$$

then f extends to a map $F : X_\kappa \rightarrow X_{\kappa'}$ satisfying

$$d_{\kappa'}(F(x), F(y)) \leq d_\kappa(x, y) \quad (x, y \in X_\kappa).$$

When $\kappa' = \kappa$, the condition on $f(A)$ says precisely that its values lie in a closed hemisphere. When $\kappa' < \kappa$, the required set is a ball smaller than a hemisphere of $X_{\kappa'}$.

2. Condition on the initial points. Again let $0 < \kappa' \leq \kappa$. Suppose instead that

$$A \subset H_\kappa^\circ(p)$$

for some $p \in X_\kappa$. Then every map $f : A \rightarrow X_{\kappa'}$ satisfying

$$d_{\kappa'}(f(x), f(y)) \leq d_\kappa(x, y) \quad (x, y \in A)$$

extends to such a map $F : X_\kappa \rightarrow X_{\kappa'}$, with no hemisphere condition imposed in advance on $f(A)$.

In the first statement, the ball condition supplies room for the value assigned to each new point while keeping all required distance inequalities. In the second statement, the strict inequality defining the open hemisphere supplies that room on the source side.

For $\kappa' = \kappa$ and equal source and target dimensions, there is also the stronger spherical extension theorem: every such $f : A \rightarrow X_\kappa$ extends to X_κ , without either hemisphere condition.

The radius in statement 1 is $\pi/(2\sqrt{\kappa})$, determined by the source curvature. Thus, when $\kappa' < \kappa$, merely saying that the image lies in a target hemisphere would give a weaker condition than the one used in that statement. The two hemisphere conditions and the special equal sphere result are distinct formulations.

Why the unrestricted extension theorem fails for spheres. Let

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\},$$

and let $A = \{x \in S^2 : x_3 = 0\}$ be its equator. Identify A with the unit circle

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\},$$

and define

$$f : A \longrightarrow S^1, \quad f(x_1, x_2, 0) = (x_1, x_2).$$

With the spherical distances on S^2 and S^1 , this map preserves distances:

$$d_{S^1}(f(x), f(y)) = d_{S^2}(x, y) \quad (x, y \in A).$$

Nevertheless, f has no distance-nonincreasing extension $F : S^2 \rightarrow S^1$. Indeed, the restriction of such an F to the closed northern hemisphere would be a continuous retraction of a disk onto its boundary $A \cong S^1$. No such retraction exists. The same argument shows that f cannot be extended even from A to the closed northern hemisphere.

Thus an unrestricted extension statement for maps into spheres fails when the target sphere has smaller dimension. This example does not contradict the extension theorem for maps $A \subset S^n \rightarrow S^n$ when source and target spheres have the same dimension and curvature.

Correction. The unrestricted extension theorem for maps into spheres is *wrong* if the dimensions of the source and target spheres are allowed to differ. In particular, the statement

$$\text{“Every distance-nonincreasing map } f : A \subset S^2 \longrightarrow S^1 \text{ extends to } S^2\text{”}$$

is false.

Take A to be the equator of S^2 , and let $f : A \rightarrow S^1$ be its natural identification with the unit circle. This map preserves spherical distances. An extension $F : S^2 \rightarrow S^1$ would restrict on the closed northern hemisphere to a continuous retraction of a disk onto its boundary. Such a retraction does not exist.

This counterexample also rules out extension from the equator to the closed hemisphere. It does not rule out the separate theorem for $f : A \subset S^n \rightarrow S^n$, where the two spheres have the same dimension and curvature.

Third correction. My second correction was still misleading. There is no unrestricted extension theorem valid for maps

$$f : A \subset S^m \longrightarrow S^n$$

for arbitrary dimensions $m \geq 0$ and $n > 0$.

More precisely, for every $n > 0$, the extension property fails for the pair $m = n + 1$. Take $A \subset S^{n+1}$ to be an equatorial copy of S^n , and let

$$f : A \longrightarrow S^n$$

be the natural identification. This map preserves spherical distances. If it extended to a distance-nonincreasing map $F : S^{n+1} \rightarrow S^n$, then its restriction to either closed hemisphere would be a continuous retraction of the $(n + 1)$ -dimensional disk onto its boundary S^n . Such a retraction does not exist.

Thus the extension property fails in general for maps between spheres of arbitrary dimensions. This does not assert failure for every individual pair (m, n) : for example, $m = 0$ poses no extension problem, since every subset of S^0 is already either empty, a singleton, or all of S^0 .

The phrase “for all $m \geq 0$ and $n > 0$ ” must therefore mean that one theorem covering all those pairs is false; it cannot mean that extension fails for each pair.

Let S^m and S^n be unit spheres equipped with their spherical distances. The assertion under discussion is:

For every subset $A \subset S^m$, every map $f : A \rightarrow S^n$ satisfying

$$d_{S^n}(f(x), f(y)) \leq d_{S^m}(x, y) \quad (x, y \in A)$$

extends to a map $F : S^m \rightarrow S^n$ satisfying the same inequality.

This assertion holds if $m \leq n$ and fails if $m > n$.

If $m \leq n$, identify S^m with an equatorial subsphere of S^n . This identification preserves spherical distances. Apply the extension theorem for maps from subsets of S^n to S^n , and then restrict the resulting map to the equatorial S^m .

If $m > n$, choose an equatorial copy $A \cong S^n$ in S^m and let $f : A \rightarrow S^n$ be the natural distance-preserving identification. Suppose that an extension $F : S^m \rightarrow S^n$ existed. There is a copy of the closed $(n+1)$ -dimensional hemisphere in S^m whose boundary is A . The restriction of F to this hemisphere would be a continuous retraction of the closed $(n+1)$ -dimensional disk onto its boundary S^n . This is impossible.

Consequently, the precise failure range is

$$m > n \geq 0,$$

including $n = 0$. In that case S^0 consists of two points, and a connected hemisphere cannot be retracted onto its two-point boundary.

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Extension theorem for the circle. Give the unit circle S^1 its spherical distance: $d(x, y)$ is the length of the shorter of the two circular arcs joining x and y . Let $A \subset S^1$, and suppose that $f : A \rightarrow S^1$ satisfies

$$d(f(x), f(y)) \leq d(x, y) \quad (x, y \in A).$$

Then f extends to a map $F : S^1 \rightarrow S^1$ satisfying

$$d(F(x), F(y)) \leq d(x, y) \quad (x, y \in S^1).$$

Proof. If A is empty or consists of one point, a constant map gives the required extension. Otherwise, first extend f to the closure of A : if $a_j \in A$ and $a_j \rightarrow x$, then

$$d(f(a_i), f(a_j)) \leq d(a_i, a_j),$$

so $f(a_j)$ converges. The limit does not depend on the sequence a_j , and the extended map satisfies the same distance inequality. We may therefore assume that A is closed.

Each connected component I of $S^1 \setminus A$ is an open circular arc with endpoints $a, b \in A$. Write L for the length of this arc, and put

$$\delta = d(f(a), f(b)).$$

Since the circular distance between a and b is at most L , the hypothesis gives $\delta \leq L$. Choose a shorter circular arc from $f(a)$ to $f(b)$; if they are antipodal, choose either one. Extend f over the closed arc $\bar{I}$ by traversing this chosen arc at constant speed δ/L . Thus, for $u, v \in \bar{I}$,

$$d(F(u), F(v)) \leq \frac{\delta}{L} (\text{length along } \bar{I} \text{ from } u \text{ to } v) \leq \text{length along } \bar{I} \text{ from } u \text{ to } v.$$

Doing this on every component defines F on all of S^1 .

It remains to check pairs of points in different components. Let $x, y \in S^1$, and choose a shorter arc J from x to y ; its length is $d(x, y)$. If the interior of J contains no point of A , then J lies in the closure of one component of $S^1 \setminus A$, and the preceding inequality applies.

Otherwise, let a and b be respectively the first and last points of A encountered while traversing J from x to y . The portion from x to a lies in the closure of one complementary component, as does the portion from b to y . Hence

$$\begin{aligned} d(F(x), F(y)) &\leq d(F(x), f(a)) + d(f(a), f(b)) + d(f(b), F(y)) \\ &\leq \text{length}(x \text{ to } a \text{ along } J) + d(a, b) + \text{length}(b \text{ to } y \text{ along } J) \\ &\leq \text{length}(J) = d(x, y). \end{aligned}$$

Thus F is the required extension. □

Give the definition of 1-LEP(1-Lipschitz Extension property) define the sectional curvature for Riemannian manifolds $\kappa(X)$ via extension properties for locally defined maps $X \rightarrow X_\kappa$ and $X_\kappa \rightarrow X$.

The extension tests give lower and upper bounds on sectional curvature. A general Riemannian manifold does not have one number $\kappa(X)$ unless its sectional curvature is constant. The local tests use three prescribed points and ask whether the map can be extended to a fourth; this avoids the global obstruction we encountered for spheres. arxiv.org

The 1-Lipschitz extension property. Let Y and Z be metric spaces. We say that the pair (Y, Z) has the 1-Lipschitz extension property, abbreviated 1-LEP(Y, Z), if, for every subset $A \subset Y$, every map $f : A \rightarrow Z$ satisfying

$$d_Z(f(a), f(b)) \leq d_Y(a, b) \quad (a, b \in A)$$

extends to a map $F : Y \rightarrow Z$ satisfying

$$F|_A = f, \quad d_Z(F(y), F(z)) \leq d_Y(y, z) \quad (y, z \in Y).$$

The comparison space. For $\kappa \in \mathbb{R}$, let X_κ denote the complete, simply connected, two-dimensional Riemannian manifold of constant sectional curvature κ : a sphere of radius $1/\sqrt{\kappa}$ when $\kappa > 0$, the Euclidean plane when $\kappa = 0$, and a hyperbolic plane of curvature κ when $\kappa < 0$. Distances on X_κ are its intrinsic Riemannian distances.

Local extension tests. Let X be a Riemannian manifold. In the following tests all points are taken in sufficiently small neighborhoods, so that only local geometry is involved. Write $V_3 = \{x_1, x_2, x_3\}$, and let $V_4 = V_3 \cup \{x_4\}$.

The local extension property in the direction

$$X \longrightarrow X_\kappa$$

means that every distance-nonincreasing map $f : V_3 \rightarrow X_\kappa$, with V_4 a sufficiently small four-point configuration in X , admits a value at x_4 such that the resulting map $F : V_4 \rightarrow X_\kappa$ remains distance-nonincreasing.

The local extension property in the opposite direction

$$X_\kappa \longrightarrow X$$

means the analogous assertion for a sufficiently small four-point configuration $V_4 \subset X_\kappa$ and a map $f : V_3 \rightarrow X$ whose values lie in a sufficiently small neighborhood in X .

These two directions characterize opposite curvature bounds:

$$\boxed{X \longrightarrow X_\kappa \iff \sec_X \geq \kappa,} \quad \boxed{X_\kappa \longrightarrow X \iff \sec_X \leq \kappa.}$$

Here $\sec_X \geq \kappa$ means that the sectional curvature $K_X(p, \sigma)$ of every two-dimensional subspace $\sigma \subset T_p X$, at every $p \in X$, is at least κ ; the upper bound has the analogous meaning.

Consequently, if both local extension properties hold for the same κ , then

$$K_X(p, \sigma) = \kappa \quad \text{for every } p \in X \text{ and every two-plane } \sigma \subset T_p X.$$

For $\kappa > 0$, “sufficiently small” matters: these are local extension tests, not an assertion that every map between the whole spheres extends. The three-to-four-point characterizations are the relevant precise results.

<https://arxiv.org/pdf/1012.5636>

Is it true in the known examples that 1-LEP for maps $X \rightarrow Y$ and $Y \rightarrow Z$ implies 1-LEP for $X \rightarrow Z$?

The 1-LEP is not transitive.

The implications

$$1\text{-LEP}(X, Y) \quad \text{and} \quad 1\text{-LEP}(Y, Z)$$

do not, in general, imply $1\text{-LEP}(X, Z)$.

Take

$$X = S^2, \quad Y = \mathbb{R}, \quad Z = S^1,$$

where the spheres carry their spherical distances.

First, $1\text{-LEP}(S^2, \mathbb{R})$ holds. Indeed, for any metric space X and any 1-Lipschitz map $f : A \subset X \rightarrow \mathbb{R}$, the formula

$$F(x) = \inf_{a \in A} (f(a) + d_X(x, a))$$

defines a 1-Lipschitz extension when $A \neq \emptyset$. If $A = \emptyset$, take F to be constant.

Second, $1\text{-LEP}(\mathbb{R}, S^1)$ holds. Given a 1-Lipschitz map $g : A \subset \mathbb{R} \rightarrow S^1$, extend it by continuity to $\bar{A}$. On each bounded component (a, b) of $\mathbb{R} \setminus \bar{A}$, join $g(a)$ to $g(b)$ along a shortest arc of S^1 at constant speed. This speed is at most 1, because

$$d_{S^1}(g(a), g(b)) \leq b - a.$$

Extend constantly on any unbounded component. The resulting map $G : \mathbb{R} \rightarrow S^1$ is 1-Lipschitz: between any two points, the distance between their images is at most the sum of the lengths of the intervening intervals.

However, $1\text{-LEP}(S^2, S^1)$ fails. Let $A \subset S^2$ be an equator, and identify it isometrically with S^1 . If this identification extended to a 1-Lipschitz map $S^2 \rightarrow S^1$, its restriction to a closed hemisphere would be a continuous retraction of a disk onto its boundary. No such retraction exists.

There is no contradiction with composition: a prescribed map $A \subset X \rightarrow Z$ need not factor as a 1-Lipschitz map $A \rightarrow Y$ followed by a 1-Lipschitz map $Y \rightarrow Z$.

Are there additional conditions to make 1LEP transitive? For instance by requiring all balls to be (strictly) convex.

Strict convexity of balls is not sufficient. The counterexample from the previous answer can be made with spaces whose balls are all strictly convex:

$$X = \mathbb{H}^2, \quad Y = \mathbb{R}, \quad Z = \mathbb{R}^2.$$

We have $1\text{-LEP}(\mathbb{H}^2, \mathbb{R})$ by real-valued Lipschitz extension, and $1\text{-LEP}(\mathbb{R}, \mathbb{R}^2)$ by joining the values at the endpoints of each missing interval with a straight segment. But $1\text{-LEP}(\mathbb{H}^2, \mathbb{R}^2)$ fails. Choose three points x_1, x_2, x_3 at distance $r > 0$ from $p \in \mathbb{H}^2$, with angles 120° at p . Their mutual distance is some $s > \sqrt{3}r$, by the hyperbolic cosine rule. Map them to the vertices of a Euclidean equilateral triangle of side s . This map preserves their pairwise distances. An extension would have to send p to a point within distance r of all three Euclidean vertices. That is impossible: the smallest disk containing those vertices has radius $s/\sqrt{3} > r$. This is also the curvature direction detected by the local extension test. arxiv.org A sufficient extra condition is factorization of each map being extended. Suppose that, whenever $A \subset X$ and $f : A \rightarrow Z$ is 1-Lipschitz, there are 1-Lipschitz maps

$$g : A \rightarrow Y, \quad h : Y \rightarrow Z, \quad f = h \circ g.$$

Then extend g to $G : X \rightarrow Y$ using $1\text{-LEP}(X, Y)$. The map $h \circ G : X \rightarrow Z$ extends f and is 1-Lipschitz. Strict convexity concerns the shape of each ball. Extension at a new point requires a whole family of balls to have a common point. In the hyperbolic example, the three required Euclidean balls are individually strictly convex, yet their intersection is empty.

[Give references to papers for 1-Lipschitz to \$\lambda\$ -Lipschitz maps](#)

For this question, the usual notation is an extension constant:

$$e(X, Y) = \inf \left\{ \lambda \geq 1 : \begin{array}{l} \text{every 1-Lipschitz } f : A \subset X \rightarrow Y \text{ extends to } F : X \rightarrow Y \\ \text{with } \text{Lip}(F) \leq \lambda \end{array} \right\}.$$

The following papers are good starting points, in order of relevance to our discussion. 1. U. Lang and T. Schlichenmaier, Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions, *International Mathematics Research Notices* 2005, 3625–3655.

General extension theorems with controlled loss; the setting includes several classes of negatively curved spaces. arxiv.org

2. M. Mendel and A. Naor, Spectral calculus and Lipschitz extension for barycentric metric spaces.

Develops quantitative extension results for nonlinear targets, including $\text{CAT}(0)$ spaces. arxiv.org

3. G. Basso, Lipschitz extension theorems with explicit constants.

A useful route to actual bounds on λ , including explicit versions of results of Lang–Schlichenmaier and Lee–Naor. arxiv.org

4. A. Naor and Y. Rabani, On Lipschitz extension from finite subsets.

Gives lower bounds: some 1-Lipschitz maps on finite sets require a substantially larger constant in every extension. arxiv.org

5. U. Lang and V. Schroeder, Kirszbraun’s theorem and metric spaces of bounded curvature, *Geometric and Functional Analysis* 7 (1997), 535–560, DOI: 10.1007/s000390050018.

This is the curvature comparison starting point for the exact case $\lambda = 1$. Person For the sphere-to-lower-dimensional-sphere equator example, no finite λ helps: every Lipschitz extension would be continuous and would still give the impossible disk retraction. Thus quantitative extension theorems require hypotheses that also address this topological obstruction.

[more papers by Naor?](#)

Yes. Beyond the Naor–Rabani and Mendel–Naor papers already mentioned, these are particularly relevant to extension with a constant $\lambda > 1$:

1. J. R. Lee and A. Naor, Extending Lipschitz functions via random metric partitions, *Inventiones Mathematicae* 160 (2005), 59–95.

A central method for obtaining quantitative extension bounds. link.springer.com

2. A. Naor, Extension, separation and isomorphic reverse isoperimetry.

Studies the extension modulus $e(X)$, including upper and lower bounds for finite-dimensional normed spaces. This is probably the closest match to your question about the possible size of λ . arxiv.org

3. M. Mendel and A. Naor, A relation between finitary Lipschitz extension moduli.

Relates the cost of adding n points to the cost of extending a map initially defined on n points. arxiv.org

4. A. Naor, Uniform nonextendability from nets.

Gives a stronger negative phenomenon: in its example, an extension from a net cannot even be uniformly continuous, hence no finite λ works. arxiv.org A related but different question is treated in A. Naor, Impossibility of almost extension: it allows an extension to differ slightly from the original map on its initial set, then asks how small that error can be. arxiv.org