John's Ellipsoid and Its Contact Points

0. John's Ellipsoid theorem

A convex body $K \subset \mathbb{R}^n$ contains a **unique** *ellipsoid $E \subset K$ of maximal volume.*

To see that, use simultaneous diagonalization of pairs of quadratic forms.

This ellipsoid doesn't depend on a choice of the Euclidean metric g in the ambient $\mathbb{R}^n$ and one may use the metric where E equals the unit g-ball.

1. John's contact-point theorem

Let $K \subset \mathbb{R}^n$ be a convex body, and let

$$B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

denote the Euclidean unit ball.

An ellipsoid of maximal volume contained in K is called the John ellipsoid of K.

Theorem 1 (John's theorem: general form)**.** *Suppose that $B_2^n \subset K$. Then B_2^n is the John ellipsoid of K if and only if there exist contact points*

$$u_1, \ldots, u_m \in \partial K \cap S^{n-1}$$

and positive numbers $c_1, \ldots, c_m$ such that

$$\boxed{\sum_{i=1}^m c_i u_i = 0} \tag{1}$$

and

$$\boxed{\sum_{i=1}^m c_i \, u_i \otimes u_i = I_n.} \tag{2}$$

Here $u_i \otimes u_i$ denotes the rank-one operator

$$(u_i \otimes u_i)x = \langle x, u_i \rangle u_i.$$

Taking traces in (2) yields

$$\boxed{\sum_{i=1}^m c_i = n.}$$

The first identity says that the weighted contact points have their center of mass at the origin. The second says that their weighted second moment is the identity.

2. Keith Ball's proof in the symmetric case

We now assume that K is centrally symmetric, so that

$$x \in K \Longrightarrow -x \in K.$$

In this case the center-of-mass condition is redundant: contact points may be paired as $u_i, -u_i$ with equal weights.

Theorem 2 (Ball's symmetric characterization)**.** *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body containing B_2^n. Then B_2^n is the John ellipsoid of K if and only if there exist contact points $u_1, \dots, u_m \in \partial K \cap S^{n-1}$ and positive coefficients c_i such that*

$$\boxed{\sum_{i=1}^{m} c_i \, u_i \otimes u_i = I_n.} \tag{3}$$

Proof. We prove both directions.

Part I: Sufficiency.

Suppose that contact points u_i and positive coefficients c_i satisfy (3). Taking traces gives

$$\sum_{i=1}^{m} c_i = n.$$

Let $E \subset K$ be any ellipsoid. Write

$$E = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{\langle x, e_j \rangle^2}{\alpha_j^2} \leq 1 \right\},$$

where $e_1, \dots, e_n$ is an orthonormal basis and $\alpha_j > 0$ are the semiaxis lengths.

Since u_i is a contact point, the hyperplane

$$\{x : \langle x, u_i \rangle = 1\}$$

supports K at u_i. Since $E \subset K$, the ellipsoid E lies in the corresponding half-space. Equivalently, u_i belongs to the polar ellipsoid of E, and hence

$$\sum_{j=1}^{n} \alpha_j^2 \langle u_i, e_j \rangle^2 \leq 1.$$

Multiply by c_i and sum over i:

$$\sum_{i=1}^{m} c_i \sum_{j=1}^{n} \alpha_j^2 \langle u_i, e_j \rangle^2 \leq \sum_{i=1}^{m} c_i = n.$$

Interchanging the sums gives

$$\sum_{j=1}^{n} \alpha_j^2 \left(\sum_{i=1}^{m} c_i \langle u_i, e_j \rangle^2 \right) \leq n.$$

By (3), for every unit vector e_j,

$$\sum_{i=1}^{m} c_i \langle u_i, e_j \rangle^2 = \langle I_n e_j, e_j \rangle = 1.$$

Consequently,

$$\sum_{j=1}^{n} \alpha_j^2 \leq n.$$

The arithmetic-geometric mean inequality now yields

$$\left(\prod_{j=1}^{n} \alpha_j^2 \right)^{1/n} \leq \frac{1}{n} \sum_{j=1}^{n} \alpha_j^2 \leq 1.$$

Therefore,

$$\prod_{j=1}^{n} \alpha_j \leq 1.$$

Since the volume of E is

$$\mathrm{vol}(E) = \mathrm{vol}(B_2^n) \prod_{j=1}^{n} \alpha_j,$$

we conclude that

$$\mathrm{vol}(E) \leq \mathrm{vol}(B_2^n).$$

Equality is possible only when every $\alpha_j = 1$. Thus B_2^n is the unique ellipsoid of maximal volume contained in K.

Part II: Necessity.

Suppose now that B_2^n is the John ellipsoid of K. Let

$$\mathcal{C} = \partial K \cap S^{n-1}$$

be the set of contact points, and define

$$\mathcal{T} = \{u \otimes u : u \in \mathcal{C}\}.$$

We claim that

$$\frac{1}{n} I_n \in \mathrm{conv}(\mathcal{T}).$$

Suppose, to the contrary, that

$$\frac{1}{n} I_n \notin \mathrm{conv}(\mathcal{T}).$$

By the separating hyperplane theorem in the finite-dimensional space of symmetric matrices, there exists a symmetric matrix H such that

$$\mathrm{tr}\left(H \frac{I_n}{n} \right) < \mathrm{tr}\big(H(u \otimes u)\big) \qquad (u \in \mathcal{C}).$$

Because $\mathrm{tr}(u \otimes u) = 1$, we may subtract a scalar multiple of I_n from H without changing the strict separation. We may therefore assume that

$$\mathrm{tr}(H) = 0.$$

The separation inequality becomes

$$\boxed{\langle Hu, u\rangle > 0 \qquad \text{for every } u \in \mathcal{C}.} \tag{4}$$

For sufficiently small $\delta > 0$, the matrix $I_n + \delta H$ is positive definite. Define the ellipsoid

$$E_\delta = \{x \in \mathbb{R}^n : \langle (I_n + \delta H)x, x\rangle \le 1\}.$$

For each contact point $u \in \mathcal{C}$, (4) gives

$$\langle (I_n + \delta H)u, u\rangle = 1 + \delta\langle Hu, u\rangle > 1.$$

Hence every contact point lies outside E_δ.

Because $\mathcal{C}$ is compact, the inequality is uniformly strict on $\mathcal{C}$. Since E_δ converges to B_2^n as $\delta \to 0$, for sufficiently small $\delta > 0$ the ellipsoid E_δ is contained in the interior of K.

It remains to compare volumes. Let $\mu_1, \dots, \mu_n$ be the eigenvalues of $I_n + \delta H$. Since $\operatorname{tr}(H) = 0$,

$$\sum_{j=1}^{n} \mu_j = \operatorname{tr}(I_n + \delta H) = n.$$

The arithmetic-geometric mean inequality gives

$$\prod_{j=1}^{n} \mu_j \le \left(\frac{1}{n}\sum_{j=1}^{n} \mu_j\right)^n = 1.$$

The matrix H is nonzero and has trace zero, so its eigenvalues cannot all be equal. Consequently, the inequality is strict:

$$\det(I_n + \delta H) = \prod_{j=1}^{n} \mu_j < 1.$$

The volume of E_δ is

$$\operatorname{vol}(E_\delta) = \frac{\operatorname{vol}(B_2^n)}{\sqrt{\det(I_n + \delta H)}} > \operatorname{vol}(B_2^n).$$

This contradicts the maximality of B_2^n.

Therefore,

$$\frac{1}{n} I_n \in \operatorname{conv}(\mathcal{T}).$$

By the definition of the convex hull, there exist contact points $u_1, \dots, u_m$ and nonnegative coefficients λ_i with $\sum_i \lambda_i = 1$ such that

$$\frac{1}{n} I_n = \sum_{i=1}^{m} \lambda_i u_i \otimes u_i.$$

Setting $c_i = n\lambda_i$ gives

$$I_n = \sum_{i=1}^{m} c_i\, u_i \otimes u_i,$$

as required. Zero coefficients, if any, can be discarded.

This completes the proof. □

3. The general, nonsymmetric case

For a general convex body, the complete characterization requires both identities

$$\sum_{i=1}^{m} c_i u_i = 0, \qquad \sum_{i=1}^{m} c_i \, u_i \otimes u_i = I_n.$$

The second identity prevents the contact points from being concentrated near a proper subspace. The first prevents them from lying entirely on one side of the origin. Together, they characterize the maximal-volume inscribed ellipsoid.

Remark 1. *Keith Ball's proof above is presented for centrally symmetric bodies. The nonsymmetric necessity argument requires a separation argument that also accounts for translations of the ellipsoid.*

Reference

Keith Ball, *An Elementary Introduction to Modern Convex Geometry*, in *Flavors of Geometry*, MSRI Publications, vol. 31, Cambridge University Press, 1997, Lecture 3, "Fritz John's Theorem," pp. 13–19.
Available at: `https://math.uchicago.edu/~shmuel/AAT-readings/Combinatorial%20Geometry,%20Concentr`