Hilbert Volume in Metric Spaces, Part 1.

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Contents

Abstract

We introduce a notion of *Hilbertian n-volume* in metric spaces with Besicovitch-type inequalities built-in into the definitions. The present Part 1 of the article is, for the most part, dedicated to reformulation of known results in our terms with proofs being reduced to (almost) pure tautologies. If there is any novelty in the paper, this is in forging certain terminology which, ultimately, may turn useful in Alexandrov kind of approach to singular spaces with positive scalar curvature [**?**].

1 Partitions of Unity $\mu = d_\mu p$ in Banach Spaces, $\tilde{L}_q$-Dilation $\|min.\widetilde{dil}^*\|_{L_q}$ and Hilbertian $\widetilde{Hilb}_{/id}$.

We recall in this section a few (more or less) standard definitions and introduce notations used throughout the paper.

The dilation of a map between metric spaces, $f : X \to Y$, is a function in two variables $x_1 \neq x_2 \in X$, that is

$$dil_f(x_1, x_2) = dist_Y(f(x_1), f(x_2)/dist_X(x_1, x_2),\ x_1 \neq x_2 \in X.$$

The Lipschitz constant of f is

$$Lip(f) = \sup_{x_1 \neq x_2 \in X} dil_f(x_1, x_2).$$

Equivalently,

$$Lip(f) = \sup_{B \subset X} diam_Y(f(B))/diam_X(B),$$

where the supremum is taken over all bounded subsets in X.

Let $\mathcal{L} = \mathcal{L}(Y)$ be the space of (usually Lipschitz) functions $l : Y \to \mathbb{R}$, let $\mathcal{L}/const$ be the space of functions modulo additive constants, and let $\mu = d_\mu l$ be a Borel measure on $\mathcal{L}/const$. Define the L_q*-dilation* of a map $f : X \to Y$, $1 \leq q \leq \infty$, as

$$\|dil^* f\|_{L_q(\mu)} = \left(\int_{\mathcal{L}/const} Lip^q(l \circ f) d_\mu l \right)^{1/q}.$$

Example. Let Y be the Euclidean space $\mathbb{R}^n$ and let μ be supported on the n orthogonal projections (modulo constants) of $\mathbb{R}^n$ onto the coordinate axes with equal weights 1 assigned to all projections. Then every isometric map $f : \mathbb{R}^m \to \mathbb{R}^n$ satisfies

$$||dil^* f||_{L_2(\mu)} = \sqrt{m}.$$

In general, *an axis* or a *(parametrized) straight line* in a metric space Y, is an isometric embedding of the real line $\mathbb{R}$ into X, say $a : \mathbb{R} \to Y$. The image of an axis $\mathbb{R} = \mathbb{R}_a \subset Y$ always admits a 1-*Lipschitz projector* of Y onto it, i.e. a map $p : Y \to \mathbb{R}_a \subset Y$ with $Lip(p) = 1$ and such that $p^2 = p$, i.e. $p_{|\mathbb{R}_a} : \mathbb{R}_a \to \mathbb{R}_a = id_{\mathbb{R}_a}$.

Such a projector is non-unique but in certain cases there are preferred ones. For example if Y is a Banach space and $\mathbb{R}_a \subset Y$ is a 1-dimensional linear subspace, then it admits a a *linear* projection $p : Y \to \mathbb{R}_a$ by the *Hahn-Banach* theorem, where this p is unique if the Banach norm $||...||_Y$ is smooth.

There is a similar projector on an axis $\mathbb{R}_a \subset Y$ in an arbitrary metric space Y called *the Busemann function* $b_a : Y \to \mathbb{R} = \mathbb{R}_a \subset Y$ defined with $r \in \mathbb{R}_a$ tending to $+\infty$ by

$$b_a(y) = \lim_{r \to +\infty} \big(dist(a(r), y) - r\big).$$

For instance, if Y is a Banach space and the norm is smooth at the points $y \in \mathbb{R}_a$ then b_a coincides with the Hahn-Banach projection.

An axial projector in a Banach space Y is a pair (a, p), where $a : \mathbb{R} \to Y$ is a linear isometric imbedding and $p : Y \to Y$ is a linear projector (projector means $p^2 = p$), such $Lip(p) = 1$ and the image im_p of p equal that of a where im_a, when identified with $\mathbb{R}$ via a, is denoted $\mathbb{R}_a \subset Y$.

Observe that a is uniquely determined by p up to $\pm$-sign; it will be sometimes denoted a_p or suppressed from the notation.

Denote by $\mathcal{P} = \mathcal{P}(Y)$ the space of axial projectors (a, p) in Y and, given a Borel measure $\mu = d_\mu p$ on $\mathcal{P}$, define *the L_q-dilation* of a map $f : X \to Y$, where X is a metric space, by

$$||dil^* f||_{L_q(\mu)} = \left(\int_{\mathcal{P}} Lip^q(p \circ f) d_\mu p \right)^{1/q}, \ 1 \le q \le \infty.$$

Identity Example. The L_q-dilation of the identity map $id_Y : Y \to Y$ is obviously expressed by the total mass of μ,

$$||dil^* f||_{L_q(\mu)} = \mu(\mathcal{P})^{1/q}.$$

Maps τ, $\mathcal{I}_\mu$ and the Norm $||\widetilde{dil}^* f||_{L_q}$. The space Y tautologically embeds into the space $\Phi(\mathcal{P})$ of functions $\mathcal{P} \to \mathbb{R}$ for $\tau : y \to \phi(p) = p(y) \in \mathbb{R}_p = \mathbb{R}$. A Borel measure μ on $\mathcal{P}$ gives the L_q-norm to the space $\Phi(\mathcal{P})$ that is then denoted $L_q(\mathcal{P}, \mu)$, where the L_p-norm of τ is bounded by $\mu(\mathcal{P})^{1/q}$.

A measure μ on $\mathcal{P}$ also delivers a *linear* 1-*Lipschitz* map $\mathcal{I}_\mu : L_1(\mathcal{P}, \mu) \to Y$ that sends functions $\phi : (a, p) \mapsto \mathbb{R} = \mathbb{R}_p \subset Y$ to their μ-integrals in Y,

$$\mathcal{I}_\mu : \phi(p) \mapsto \int_{\mathcal{P}} a(\phi(p)) d_\mu p.$$

Define $\tilde{L}_q$-*dilation*

$$||\widetilde{dil}^* f||_{L_q(\mu)} \le ||dil^* f||_{L_q(\mu)}$$

via Lipschitz maps $\tilde{f} : X \to L_q(\mathcal{P}, \mu)$, that are $\mathcal{P}$-families of $\mathbb{R}$-valued functions $\tilde{f}_p : X \to \mathbb{R} = \mathbb{R}_a \subset Y$, $(a, p) \in \mathcal{P}$, as

$$\|\widetilde{dil}^* f\|_{L_q(\mu)} = \inf_{\tilde{f}} \left(\int_{\mathcal{P}} Lip^q(\tilde{f}_p) d_\mu p \right)^{1/q}$$

where the infimum is taken over all $\tilde{f}$ such that $\mathcal{I}_\mu \circ \tilde{f} = f$.

An axial partitions of unity in Y is a measure μ on $\mathcal{P}$, such that

$$\mathcal{I}_\mu \circ \tau = \int_{\mathcal{P}} p d_\mu p = id_Y \text{ for the identity operator } id_Y : Y \to Y.$$

Minimal L_q-Dilations. Let

$$\|min.dil^* f\|_{L_q} = \inf_\mu \|dil^* f\|_{L_q(\mu)} \text{ and } \|min.\widetilde{dil}^* f\|_{L_q} = \inf_\mu \|\widetilde{dil}^* f\|_{L_q(\mu)}$$

where the infima are taken over all partitions of unity $\mu = dp_\mu$ in Y.

Notice that if $dim(Y) < \infty$ then both minima are assumed by some partition of unity, say $\mu_{min}(f)$ and $\mu_{\widetilde{min}}(f)$, such that

$$\|min.dil^* f\|_{L_q} = \|dil^* f\|_{L_q(\mu_{min})} \text{ and } \|min.\widetilde{dil}^* f\|_{L_q} = \|\widetilde{dil}^* f\|_{L_q(\mu_{\widetilde{min}})}.$$

The Maps τ and $\mathcal{I}_\mu$ in Hilbert Spaces. If Y is a *Hilbert space*, then the tautological map $\tau : Y \to \Phi(\mathcal{P})$ is *linear* and if μ is a *partition of unity*, then $\tau : Y \to L_2(\mathcal{P}, \mu)$ is *isometric* by the Pythagorean theorem.

The map $\mathcal{I}_\mu : Ł_1(\mathcal{P}, \mu) \to Y$ extends to $\mathcal{I}_\mu : Ł_2(\mathcal{P}, \mu) \to Y$ that is, in the Hilbertian case, equals the adjoint to τ, since the operators $a : \mathbb{R} \to Y$ and $p : Y \to \mathbb{R}_a = \mathbb{R}$ are mutually adjoint. Explicitly,

$$\langle \mathcal{I}_\mu(\phi(p)), y \rangle_Y = \left\langle \int_{\mathcal{P}} a(\phi(p)) d_\mu p, y \right\rangle_Y =$$

$$= \int_{\mathcal{P}} a(\phi(p)) \cdot p(y) d_\mu p = \int_{\mathcal{P}} \phi(p) \cdot \tau(y) d_\mu p = \langle \phi(p), \tau(y) \rangle_{L_2(\mathcal{P}, \mu)}.$$

In other words, $\mathcal{I}_\mu$ equals the orthogonal projection of $L_2(\mathcal{P}, \mu)$ onto $Y = \tau(Y) \subset L_2(\mathcal{P}, \mu)$. Consequently,

$$Lip(\mathcal{I}_\mu) = 1 \textit{ for all partitions of unity } \mu \textit{ in Hilbert spaces.}$$

It follows, that every Lipschitz map $\tilde{f}$ from a metric space X to $L_2(\mathcal{P}, \mu)$, that is a $\mathcal{P}$-family of $\mathbb{R}$-valued functions $\tilde{f}_p : X \to \mathbb{R} = \mathbb{R}_a \subset Y$, $(a, p) \in \mathcal{P}$, satisfies

$$\|dil^*(\mathcal{I}_\mu \circ \tilde{f})\|_{L_2(\mu)} \leq \left(\int Lip^2(\tilde{f}_p) d_\mu p \right)^{1/2}.$$

Partitions of Hilbertian Forms h into Squares l^2. Let h be the Hilbertian quadratic form in Y. Then the partition of unity condition on μ can be equivalently (and obviously) expressed in term of the integral of the squares of linear functions (forms):

$$\int_{\mathcal{P}} p d\mu_p = id_Y \Leftrightarrow \int_{\mathcal{L}} l^2 d\mu_l = h,$$

where $l = l_p : Y \to \mathbb{R} = \mathbb{R}_p$ are linear functions corresponing to the projectors $p : Y \to \mathbb{R}_p$ and $d\mu_l$ is the pushforward of μ from $\mathcal{P}$ to to $\mathcal{L}$ under the map $p \mapsto l_p$.

If X and Y are Hilbert spaces and $f : X \to Y$ is a linear map, then $\|dil^* f\|_{L_2(\mu)}$ *does not depend* on the partition of unity μ in Y: it equals the trace of the induced quadratic form on X

$$\|dil^* f\|_{L_2(\mu)} = \int_{\mathcal{L}} trace_Y (l \circ f)^2 d\mu_l = trace_X \int_{\mathcal{L}} (l \circ f)^2 d\mu_l = trace_X f^*(h),$$

and it is called the *Hilbert -Schmidt* norm $\|f\|_{L_2} = \|dil^* f\|_{L_2(\mu)}$ of f.

Furthermore, since the map $\mathcal{I}_\mu : L_2(\mathcal{P}, \mu) \to Y$ is 1-Lipschitz,

$$\|\widetilde{dil}^* f\|_{L_2(\mu)} = \|dil^* f\|_{L_2(\mu)}.$$

For instance, isometric linear maps $f : X \to Y$ have

$$\|\widetilde{dil}^* f\|_{L_2(\mu)} = \|dil^* f\|_{L_2(\mu)} = \|f\|_{L_2} = \sqrt{dim(X)}$$

for all partitions of unity μ; in particular

$$\|min.\widetilde{dil}^* id_Y\|_{L_2} = \|min.dil^* id_Y\|_{L_2} = \|id_Y\|_{L_2} = \sqrt{dim(Y)}.$$

In general, Lipschitz maps f from metric spaces X to Hilbertian Y satisfy

$$Lip(f) \le \|min.\widetilde{dil}^* f\|_{L_2} \le \|min.dil^* f\|_{L_2} \le Lip(f) \cdot \sqrt{dim(Y)}.$$

This suggests the notation

$$Hilb(f) = Hilb_{/id}(f) = Hilb_{/id_Y}(f) = \|min.dil^* f\|_{L_2} / \|min.dil^* id_Y\|_{L_2}$$

and

$$\widetilde{Hilb}(f) = \widetilde{Hilb}_{/id}(f) = \widetilde{Hilb}_{/id_Y}(f) = \|min.\widetilde{dil}^* f\|_{L_2} / \|min.\widetilde{dil}^* id_Y\|_{L_2}.$$

These definitions make sense for all arbitrary metric spaces X and Banach spaces Y, where, if Y is Hilbertian,

$$\widetilde{Hilb}(f) \le Hilb(f) \le Lip(f) \le \sqrt{dim(Y)} \cdot \widetilde{Hilb}(f).$$

Up to some point, we formulate and prove obvious general properties of the "norms" $\|min.dil^* f\|_{L_q}$ and $\|min.\widetilde{dil}^* f\|_{L_q}$ for all Banach spaces Y and all q – this does not cost us anything, but only maps into Hilbert spaces and $q = 2$ will be essential for the present day geometric applications.

Local Dilations $\|dil^* f(x)\|$, $Hilb(f(x))$ *etc. in Finsler Spaces.* All of the above dilation invariants depends only on the metric but not on the linear structure in Y. This allows the following local versions of these dilations for maps into Finsler, e.g Riemannian, manifolds Y.

Namely, let $Y = (Y, dist)$ be a metric space, where each point $y \in Y$ admits a local Banach metric in a neighbourhood $U_y \subset Y$ of y, say $dist_y$ for all $y \in Y$, such that

$$dist_y(y_1, y_2)/dist(y_1, y_2) \to 1 \text{ for } y_1, y_2 \to y.$$

Define

$$\|dil^* f(x)\|_{L_q(\mu)},\ \|min.dil^* f(x)\|_{L_q},\ \|\widetilde{dil}^* f(x)\|_{L_q(\mu)},\ \text{Hilb(f(x))},\ \text{etc.}$$

for maps $f : X \to Y$ by restricting f to small neighbourhoods $U_x \subset X$ for all $x \in X$, such that $f(U_x) \subset U_{f(x)} \subset Y$, evaluating the corresponding dilations with respect to the local Banach metrics $dist_y$ and then taking the infimum over all neighbourhoods U_x of a point $x \in X$ for all $x \in X$.

Equivalently, in the case Y is a Riemannian manifold, one can, by Nash' theorem, isometrically immerse $Y \subset \mathbb{R}^N$ and define the local dilations via axial partitions of unity in $\mathbb{R}^N$.

Notice, that unlike the Lipschitz constant, where $Lip(f) = \sup_{x\in X} Lip(f(x))$ for maps from *length* spaces X, there is no(?) apparent passage from local to global L_q-dilations.

2 Hadamard's $Jac^{[n]} \le \widetilde{Hilb}^n$ and Inverse Lipschitz Maps.

Hadamard's Inequality for $Jac^{[n]}$. Given a map between Hilbert spaces, say $D : A \to B$, denote by $Jac^{[n]}(D)$ the norm of the n-th exterior power of D, that is the supremum of the *absolute values* of the Jacobians of D on all n-dimensional subspaces $A' \subset A$,

$$Jac^{[n]}(D) = \sup_{dim(A')=n} Jac(D|A') \text{ for } D|A' : A' \to im_{A'} \subset B.$$

If $f : X \to Y$ is a locally Lipschitz, e.g. C^1-smooth, map between Riemannian manifolds then the differential $Df(x) : T_x(X) \to T_{f(x)}(Y)$ exists for almost all $x \in X$ by *Rademacher-Stepanov theorem* and the Jacobian of f is defined as

$$Jac^{[n]}(f(x)) = Jac^{[n]}(Df(x)) \text{ and } Jac^{[n]}(f) = \sup_x Jac^{[n]}(Df(x)),$$

where $\sup_x$ refers to *almost all* $x \in X$.

Hadamard's Inequality. Let X and Y be Riemannian manifolds and $f : X \to Y$ be a Lipschitz map.

Then

$$Jac^{[n]}(f(x)) \le \|n^{-1/2} min.\widetilde{dil}^* f(x)\|^n_{L_2} \text{ for } n = dim(Y) \text{ and almost all } x \in X.$$

In particular, Lipschitz maps f from n-manifolds to $\mathbb{R}^n$ satisfy

$$Jac^{[n]}(f) \le \widetilde{Hilb}^n(f).$$

Proof. In fact, if

$$f = \mathcal{I}_\mu \circ \tilde{f} = \int_{\mathcal{P}} a_p(\tilde{f}) d_\mu p \text{ for } \tilde{f} : X \to L_2(\mathcal{P}, \mu),$$

then

$$\|Df(x)\|_{L_2} \le \|D\tilde{f}(x)\|_{L_2} \le \|min.\widetilde{dil}^* f(x)\|_{L_2} \text{ for almost all } x \in X,$$

while

$$Jac^{[n]}(D) \le n^{-1/2}||D||^n_{L_2},$$

since the discriminants of quadratic forms (induced by D from the Hilbert form in target Hilbert spaces) are bounded by their traces via the arithmetic/geometric mean inequality for the eigenvalues of these forms.

However trivial, the inequality $Jac^{[n]}(f) \le Hilb^n(f)$ is significantly sharper than mere $|Jac^{[n]}|(f) \le Lip^n(f)$.

For instance, let a smooth map f from a Riemannian n-manifold X to $Y = \mathbb{R}^n$ be given by n functions $f_1, ...f_n$. Then $Lip(f)$ may be as big as $(\sum_i Lip^2(f_i))^{1/2}$, while the Hadamard's inequality, applied to the Hilbert form $\sum_{i=1}^n \lambda_i b_i^2$ for $\lambda_i = (Lip(f_i))^{-1/2}$ on Y, says that

$$|Jac^{[n]}(f)| \le \prod_{i=1,...,n} Lip(f_i).$$

(This is, of course, obvious anyway.)

Sharpness of Hadamard. Hadamard's inequality bounds the volume of the image of a Lipschitz map between n-dimensional Riemannian manifolds, $f : X \to Y$, by

$$vol_n(f(X)) \le \int_X Hilb^n(Df(x))dx \le vol_n(X) \cdot \sup_x Hilb^n(Df(x)),$$

since

$$vol_n(f(X)) \le \int_X Jac^{[n]}(f(x))dx \le vol_n(X) \cdot \sup_x Jac^{[n]}(Df(x)),$$

where the two suprema are taken over *almost all* $x \in X$.

Since the algebraic inequality $Jac^{[n]}(D) \le Hilb^n(D)$ is sharp for linear maps $D : \mathbb{R}^n \to \mathbb{R}^n$, i.e. the equality $Jac^{[n]}(D) = Hilb^n(D)$ implies that D is a homothety, the corresponding integral inequalities are also sharp:

A. *If*

$$vol_n(f(X)) = \int_X Hilb^n(Df(x))dx < \infty \text{ and } dim(X) = dim(Y) \ge 2,$$

then the map f is a conformal diffeomorphism on its image.

(This is not used in the sequel.)

B. *If*

$$\sup_x Hilb^n(Df(x)) = 1 \text{ and } vol_n(f(X)) = vol_n(X) < \infty,$$

then f is an isometric diffeomorphism onto its image.

Proof. The equality $vol_n(f(X)) = \int_X Jac^{[n]}(Df(x))dx$ implies that the local topological degree of f, either equals +1 at almost all $x \in X$ or it is almost everywhere −1. If such an f is conformal *almost everywhere* and $n \ge 2$, then it is conformal *everywhere*. For example, if $n = 2$ this is seen with the Cauchy integral formula, while the case $n \ge 3$ is (essentially) trivial because of the Liouville theorem.

The remaining case of **B** for $n = 1$ is, of course, obvious but it points out to the following extension to more general Lipschitz maps.

C. *If a Lipschitz map $f : X \to Y$ between n-dimensional Riemannian manifolds without boundaries satisfies*

$$vol_n(f(X)) = \int_X Jac^{[n]}(f(x))dx < \infty$$

and

$$vol_n(f^{-1}(U)) \le C \cdot vol(U) \text{ for all open } U \subset Y \text{ and a constant } C < \infty.$$

Then f is a homeomorphism onto its image $im_f = f(X) \subset Y$ and the inverse map $f^{-1} : im_f \to X$ satisfies

$$\sup_{y \in im_f} Lip_y(f^{-1}) \le C \cdot Jac^{[n-1]}(f) \le C \cdot Lip^{n-1}(f).$$

This is well known with the proof, probably, buried somewhere in [**?**]. Here is how it goes.

Since $vol_n(f^{-1}(U)) \le C \cdot vol(U)$, the pullbacks $f^{-1}(y) \subset X$ are *zero dimensional* subsets in X for all $y \in im_f$; in fact, this follows from the inequality $vol_n(f^{-1}(B_y(\varepsilon))) = o(\varepsilon^{n-1})$ for small ε-balls in Y.

Since $dim(f^{-1}(y)) \le 0$, every point $x \in f^{-1}(y)$ admits an arbitrarily small neighbourhood, say $B_x \subset X$, such that the the boundary $\partial B_x \subset X$ does not intersect $f^{-1}(y)$; hence, the local topological degree of f is defined at x.

This degree must be *non-zero*, otherwise, all points x' close to x would have "partners" x'' such that $f(x') = f(x)$ that is incompatible with $vol_n(f(X)) = \int_X Jac^{[n]}(f(x))dx$.

Therefore, the image of every neighbourhood of $x \in X$ contains a neighbourhood of $y = f(x) \in Y$ for all $x \in X$. Thus, f is what is called an *open map*.

Besides, again because of $vol_n(f(X)) = \int_X Jac^{[n]}(f(x))dx$, the map f is one-to-one on a dense subset (of full measure) in X, and, obviously,

densely one to one + open $\Rightarrow$ one-to-one.

Granted one-to-one, take an ε-narrow cylinder $V_\varepsilon \subset im_f$ around a distance minimizing geodesic segment $[y_1, y_2] \subset im_f$, denote by $\partial_1 V_\varepsilon \ni y_1$ and $\partial_2 V_\varepsilon \ni y_2$. the top and the bottom of this cylinder and let $\partial_{lat} V_\varepsilon$ be the remaining (lateral) part of its bounadry,

$$\partial_{lat} V_\varepsilon = \partial V_\varepsilon \setminus (\partial_1 V_\varepsilon \bigcup \partial_2 V_\varepsilon).$$

Observe that the relative homology group $H_{n-1}(V_\varepsilon, \partial_{lat} V_\varepsilon)$ is free cyclic and let $\kappa_{n-1}\varepsilon^{n-1}$ denote the minimum of the $(n-1)$-volumes of relative $(n-1)$-cycles representing *non-zero* classes in $H_{n-1}(V_\varepsilon, \partial_{lat} V_\varepsilon)$.

Observe that κ_{n-1} is close to the volume of the Euclidean $(n-1)$-ball and

$$vol_n(V_\varepsilon) = \kappa_{n-1}\varepsilon^{n-1} \cdot dist(y_1, y_2) + o(\varepsilon^{n-1}).$$

Take the pull-backs $\tilde{V}_\varepsilon = f^{-1}(V_\varepsilon) \subset X$ and $\partial_{lat}\tilde{V}_\varepsilon = f^{-1}(\partial_{lat} V_\varepsilon)$ and let $\tilde{\kappa}_{n-1}\varepsilon^{n-1}$ be the minimum of the $(n-1)$-volumes of relative $(n-1)$-cycles representing *non-zero* classes in $H_{n-1}(\tilde{V}_\varepsilon, \partial_{lat}\tilde{V}_\varepsilon)$.

Clearly,

$$\kappa_{n-1} \le Jac^{[n-1]}(f) \cdot \tilde{\kappa}_{n-1}.$$

On the other hand,

$$vol_n(\tilde{V}_\varepsilon)) \geq dist_X(\partial_1\tilde{V}_\varepsilon, \partial_2\tilde{V}_\varepsilon) \cdot \tilde{\kappa}_{n-1}\varepsilon^{n-1}$$

for $\partial_1\tilde{V}_\varepsilon, \partial_2\tilde{V}_\varepsilon \subset X$ being the pullbacks of the top and the bottom of the cylinder $V_\varepsilon \subset Y$.

The proof of this is standard: map $\tilde{V}_\varepsilon$ to $\mathbb{R}_+$ by $d : \tilde{v} \mapsto dist_X(\tilde{v}, \partial_1\tilde{V}_\varepsilon)$, observe that the pullbacks $d^{-1}(r) \subset \tilde{V}_\varepsilon$ support non non-zero classes in $H_n(\tilde{V}_\varepsilon, \partial_{lat}\tilde{V}_\varepsilon)$ for $0 \leq r \leq dist_X(\partial_1\tilde{V}_\varepsilon, \partial_2\tilde{V}_\varepsilon)$ and apply the classical (and obvious) *coarea inequality* (that happens to be equality in the present case and that is extended in section 8 to more general setting of Hilbert volumes),

$$vol_n(\tilde{V}_\varepsilon) \geq \int_{\mathbb{R}_+} vol_{n-1}(d^{-1}(r))dr.$$

Then the proof follows with $\varepsilon \to 0$.

D. *Remark.* The statement of **C** can be obviously reformulated without any reference to differentiability of f and the above argument still applies as it does not use the Rademacher-Stepanov theorem.

Besides, one does not truly need f to be Lipschitz: what is essential is a bound on the $(n-1)$-Jacobian $Jac^{[n-1]}(f)$ that can be defined as

$$Jac^{[n-1]}(f) = \sup_H vol_{n-1}(f(H))/vol_{n-1}(H)$$

for all rectifiable hypersurfaces $H \subset X$.

Moreover, X and Y do not have to carry any metrics, what is needed are measures on X and Y and the notion of "$(n-1)$-volume" for hypersurfaces. In fact, the distance can be *derived* from these data via the *Almgren-Besicovitch-Derrick inequality* (see section 9).

Warnings. (a) A Lipschitz map $f : X \to Y$ where $Df(x) : T_x(X) \to T_{f(x)}(Y)$ is an isometry almost everywhere does not have to be one-to-one. In fact, paradoxically, every Riemannian n-manifold admits a 1-Lipschitz map into $\mathbb{R}^n$, that preserves the lengths of *all* rectifiable curves in X, see 2.4.11 in [**?**].

(b) There are lots of C^∞-smooth maps $f : X \to Y$ such that $vol_n(f(X)) = \int_X Jac^{[n]}(f(x))dx$, where $Jac^{[n]}(f) \neq 0$ almost everywhere; yet, where the unions of those pullbacks $f^{-1}(y) \subset X$ that are diffeomorphic to the $(n-1)$-ball, are *dense* in X.

3 Controlled $\tilde{L}_q$-Dilation Extensions, Banach Straightening and Hilbertian Volume Domination.

If $X \supset X_0 \overset{f_0}{\to} Y$ is a partially defined λ-Lipschitz map between metric spaces, then it admits an extension to a λ-Lipschitz map $f : X \to Y$ only for rather exceptional spaces Y. For instance, this is, obviously, possible if Y is a tree, e.g. $Y = \mathbb{R}$, a (finite or infinite measurable) Cartesian product of trees with the *sup*-metric, e.g. the L_∞-space over a measure space, such as $L_\infty(\mathcal{P}, \mu)$, or else, an *Uryson's universal space.*

But such extensions do not exist, in general, not even for $Y = \mathbb{R}^n$ if $n \geq 2$, except for particular domains X, such as *Alexandrov's spaces* with curvature ≥ 0 [**?**], while for general X, only an extension $f : X \to Y = \mathbb{R}^n$ with $Lip(f) \leq n^{1/2} Lip(f_0)$ is (obviously) possible.

However, Lipschitz extensions of maps $\tilde{f}_0 : X_0 \to L_\infty(\mathcal{P}, \mu)$ to $\tilde{f} : X \to L_\infty(\mathcal{P}, \mu)$ trivially yield a control over $\tilde{L}_q$-dilation of maps $f = \mathcal{I}_\mu \circ \tilde{f} : X \to Y$ as follows.

Let X be a metric space and Y be a Banach space with an axial partition of unity μ.

Then every partial map, $X \supset X_0 \overset{f_0}{\to} Y$, extends to a map $f : X \to Y$, such that

$$||\widetilde{dil}^* f||_{L_q(\mu)} \leq ||\widetilde{dil}^* f_0||_{L_q(\mu)} \leq ||dil^* f_0||_{L_q(\mu)} \leq$$
$$\leq ||dil^* f_0||_{L_\infty(\mu)} \cdot \big(\mu(\mathcal{P})\big)^{1/q} \leq Lip(f_0) \cdot \big(\mu(\mathcal{P})\big)^{1/q}$$
$$\textit{for all } 1 \leq q \leq \infty.$$

Remark. The only point here which needs a minor attentions (and which is unneeded for our applications) is to make sure that the same extension $\tilde{f}$ of $\tilde{f}_0$ serves the spaces $L_q(\mathcal{P}, \mu)$ simultaneously for all q.

This is achieved by making Lipschitz extensions $\tilde{f}_p : X \to \mathbb{R} = \mathbb{R}_p \subset Y$, $p \in \mathcal{P}$, of the corresponding functions $(\tilde{f}_0)_p : X_0 \to \mathbb{R} = \mathbb{R}_p \subset Y$ depend *measurably* on p, e.g. by taking, for each $p \in \mathcal{P}$, a *minimal* such $\tilde{f}_p$. Namely, in general, given a partial defined function $X \supset X_0 \overset{\phi_0}{\to} \mathbb{R}$, its minimal Lipschitz extension is

$$\phi(x) = \sup_{x_0 \in X_0} (\phi_0(x_0) - dist_X(x, x_0)).$$

For example, if X_0 is isometric to an axis $\mathbb{R} \subset X$ and $X_0 \to \mathbb{R} = X_0$ is the identity map, this minimal extension equals the Busemann function $X \to \mathbb{R}$.

Banach Straightening at Infinity. Define the *Lipschitz constant at infinity* of a map $f : X \to Y$ between metric spaces by

$$Lip_\infty(f) = \limsup_{dist_X(x_1, x_2) \to \infty} dil_f(x_1, x_2).$$

Equivalently, this can be defined with restrictions of f to *D-separated nets* $X_D \subset X$, that are subsets in X with distances between all pairs of points $\geq D$, as follows,

$$Lip_\infty(f) = \limsup_{X_D \to \infty} Lip(f_{|X_D}).$$

Similarly, define the $\tilde{L}_q$-dilation at infinity, denoted $||\widetilde{dil}^*_\infty f||_{L_q(\mu)}$, by restricting f to D-separated nets and taking *limsup* of $||\widetilde{dil}^* f_{|X_D}||_{L_q(\mu)}$ over X_D with $D \to \infty$.

Then the above extension of maps $f|X_D : X_D \to Y$ from $X_D \subset X$ to all of X implies the following

Banach ε-Straightening. Let $f : X \to Y$ be a map from a metric space X to a Banach space Y with a partition of unity μ in Y.

Then, for every $\varepsilon > 0$, there exists a map $f_\varepsilon : X \to Y$ within finite distance from f, i.e.

$$||f_\varepsilon(x) - f(x)||_Y \leq r = r_\varepsilon < \infty \textit{ for all } x \in X,$$

and such that

$$\|\widetilde{dil}^{*} f\|_{L_q(\mu)} \le (\|dil^{*}_{\infty} f\|_{L_\infty(\mu)} + \varepsilon) \cdot \mu(\mathcal{P})^{1/q} \text{ for all } q \in [1, \infty].$$

On $\varepsilon \to 0$. One can, in many cases, pass to the limit map $f_{\varepsilon\to 0} : X \to Y$ but this $f_{\varepsilon\to 0}$ may be far away from f, e.g. it may be constant for an f *isometric at infinity*, i.e. where $\lim_{D\to\infty} dil_{f|X_D} = 1$.

Hilbert-Hadamard Volume Domination. Let X be an n-dimensional Riemannian manifold, let μ be a partition of unity in the Euclidean/Hilbertian space $Y = \mathbb{R}^n$ and let $f : X \to Y$ a proper (pullbacks of compact sets are compact) continuous map.

Let $\Omega_R \subset \mathbb{R}^n$, $R \to \infty$, be bounded domains, such that $vol_n(\Omega_R) = const \cdot R^n$ and such that for every r_0 the volumes of the r_0-neighbourhoods of their boundaries satisfy

$$vol_n(U_{r_0}(\partial\Omega_R)) = o(R^n) \text{ for } R \to \infty,$$

e.g. Ω_R are Euclidean R-balls.

If f has non-zero topological degree, then the Euclidean volumes of the domains $\Omega_R \subset \mathbb{R}^n$ are "asymptotically sharply bounded" by the Riemannian volumes of their f-pullbacks as follows

$$vol_n\big(f^{-1}(\Omega_R)\big) \ge \|dil^{*}_{\infty}\|^{n}_{L_\infty(\mu)} \cdot vol_n(\Omega_R) - o(R^n) \text{ for } R \to \infty.$$

Proof. Since $deg(f) \ne 0$ and $dist(f, f_\varepsilon) < r_\varepsilon$, the images $f_\varepsilon(f^{-1}(\Omega_R)) \subset \mathbb{R}^n$ contain $[\Omega_R - r_\varepsilon] =_{def} \Omega_R \setminus U_{r_\varepsilon}(\partial\Omega_R)$; hence,

$$vol_n(f^{-1}(\Omega_R)) \ge vol_n(f_\varepsilon^{-1}[\Omega_R - r_\varepsilon]).$$

On the other hand, by the above,

$$\widetilde{Hilb}(f_\varepsilon) = n^{-1/2}\|\widetilde{dil}^{*} f_\varepsilon\|_{L_2(\mu)} \le \|dil^{*}_{\infty} f\|_{L_\infty(\mu)} + \varepsilon$$

while the Jacobian of f_ε is bounded by Hadanmard's inequality $Jac^{[n]}(f_\varepsilon) \le \widetilde{Hilb}^{n}(f_\varepsilon)$; hence,

$$vol_n(f_\varepsilon^{-1}[\Omega_R - r_\varepsilon]) \ge \widetilde{Hilb}^{-n}(f_\varepsilon) \cdot vol_n[\Omega_R - r_\varepsilon]$$

$$\ge \|dil^{*}_{\infty} f\|^{-n}_{L_\infty(\mu)} \cdot vol_n(\Omega_R) - \varepsilon \cdot const \cdot R^n - o(R^n) \text{ for all } \varepsilon > 0.$$

QED.

Local $\tilde{L}_q$-Extensions. Let Y be a smooth manifold with a continuous Riemannian metric and $X \supset X_0 \overset{f_0}{\to} Y$ a partial Lipschitz map. Suppose that that

$$\|min\widetilde{dil}^{*} f_0(x)\|_{L_q} < \phi(x) \text{ for all } x \in X_0$$

where ϕ is a continuous function on X.

Then f_0 extends to a neighbourhood $X_1 \supset X_0$ by a map $X \supset X_1 \overset{f}{\to} Y$, such that

$$\|min\widetilde{dil}^{*} f(x)\|_{L_q} < \phi(x) \text{ for all } x \in X_0.$$

Proof. Combine the above extension with the ordinary Lipschitz partition of unity in X by the following trivial $\arg\sum_{i\in I}=1$ment.

First, let $Y=\mathbb{R}^N$ and let $U_i\subset X$, $i\in I$, be a locally finite covering of X such that

$$\|\widetilde{mindil}^*\, f_{0|U_i\cap X_0}\|_{L_q}<\phi(x) \text{ for all } x\in U_i.$$

Extend the maps $f_{0|U_i\cap X_0}: U_i\cap X_0\to\mathbb{R}^N$ to $f_i: U_i\to\mathbb{R}^N$, such that

$$\|\widetilde{mindil}^*\, f_i\|_{L_q}\le\|\widetilde{mindil}^*\, f_{0|U_i\cap X_0}\|_{L_q}<\phi(x) \text{ for all } x\in U_i.$$

Let $\phi_i: U_i\to\mathbb{R}_+$ be Lipschitz functions with supports strictly inside U_i and such that $\sum_{i\in I}=1$. Then the map

$$F=\sum_{i\in I}\phi_i\cdot f_i \text{ satisfies } \|\widetilde{mindil}^*\, F(x)\|_{L_q}<\phi(x)$$

in some neighbourhood $X_1\subset X$ of X_0 by an obvious computation.

Finally, if Y is C^0-Riemannian, isometrically C^1-embed $Y\subset\mathbb{R}^N$ and observe that some neighbourhood $Y_1\subset\mathbb{R}^N$ of Y admits a Lipschitz projection $\pi: Y_1\to Y$ such that $Lip(\pi(y))=1$ for all $y\in Y$. (If $Y\subset\mathbb{R}^N$ is C^2-smooth, one can use the normal projection to Y, that does not always exist for C^1-submanifolds.)

Compose the above F with this π and thus, obtain the required extension $f=\pi\circ F: X\to Y$ of f_0 from X_0 to $X_1\supset X_0$. QED.

Question on L_q-Dilation at Infinity. Is there a counterpart to the above with $\|dil^*f\|_{L_q}$ instead of $\|\widetilde{dil}^*f\|_{L_q}$?

Namely, what is the minimal (or, rather, infimal) constant $C=C(Y,q,q')$, such that every $f: X\to Y$ admits an $f': X\to Y$ within bounded distance from f, such that $\|dil^*f'\|_{L_{q'}(\mu)}\le C\cdot\|dil^*_\infty f\|_{L_q(\mu)}$?

4 John's $h_\diamond$ and John's Ellipsoid in Banach Spaces with a Riemannian Corollary.

So far, everything was compiled of definitions + trivial generalities. Now comes something deceptively simple but more substantial.

Fritz John's Ellipsoid Theorem. *An n-dimensional Banach space $Y=(Y,\|...\|_Y)$ admits a unique Hilbert quadratic form $h_\diamond$, such that*

$$\|...\|_{h_\diamond}\ge\|...\|_Y,$$

and such that the identity map $id_\diamond=id: Y=(Y,\|...\|_Y)\to Y=Y_\diamond=(Y,\|...\|_{h_\diamond})$ satisfies

$$Hilb_{h_\diamond}=Hilb_{/Y_\diamond}(id_\diamond)\le 1,$$

for $Hilb_{/Y_\diamond}(id_\diamond)=n^{-1/2}\|min.dil^*id_\diamond\|_{L_2}$.

Proof. Let $H_+(1)$ be the subset in the space of positive semidefinite (Hilbertian) quadratic forms h on Y such that $Hilb_h\le 1$, i.e such that the identity map $id=id_h: (Y,\|...\|_Y)\to(Y,h)$. satisfies $n^{-1/2}\|min.dil^*id_h\|_{L_2}\le 1$

Observe that this $H_+(1)$ is a *convex* susbset in the linear space of all quadratic forms on Y by the definition of $\|min.dil^*\|_{L_2}$ via partitions of Hilbertian forms h into squares l^2 (see section 1).

($\diamond$) Let $h_\diamond \in H_+(1)$ *maximize* the Hilbertian (Euclidean) Haar measure of Y associated to it among all $h \in H_+(1)$, i.e. the Jacobian of the identity map $id : (Y, h_\diamond) \to (Y, h)$, denoted $Jac^{[n]}_{h_\diamond \to h}$, satisfies $Jac^{[n]}_{h_\diamond \to h} \le 1$ for all $h \in H_+(1)$. Then

$$\|...\|_{h_\diamond} \ge \|...\|_Y.$$

Indeed, assume otherwise, let l be a linear form on Y such that $\|l\|^2_Y = n = dim(Y)$ and $\|l\|^2_{h_\diamond} = n/c$ for $c > 1$ and let $h_\varepsilon = (1-\varepsilon)h_\diamond + \varepsilon l^2$, $0 \le \varepsilon \le 1$.

Clearly,

$$Jac^{[n]}_{h_\diamond \to h_\varepsilon} = \left((1-\varepsilon)^{(n-1)} \cdot ((1-\varepsilon) + cn\varepsilon)\right)^{1/2},$$

and

$$\log Jac^{[n]}_{h_\diamond \to h_\varepsilon} = \frac{1}{2}\left(-(n-1)\varepsilon + c(n-1)\varepsilon\right) - o(\varepsilon) > 0 \text{ for } \varepsilon \to 0;$$

hence, the form h_ε has greater Haar measure than $h_\diamond$ for small $\varepsilon > 0$.

Since $n \cdot l^2 \in H_+(1)$, the form h_ε also lies in $H_+(1)$ by convexity of $H_+(1)$; this contradicts the extremality of $h_\diamond$ and uniqueness of $h_\diamond$ follows by the same argument. QED.

John's $\mu_\diamond$. The partition of unity $\mu_\diamond = \mu_{min}$ in the Hilbert space $(Y, \|...\|_{h_\diamond})$ for which $\|dil^* id_\diamond\|_{L_2} = \sqrt{n}$ is not, in general unique. However, the inequalities $\|...\|_{h_\diamond} \ge \|...\|_Y$ and $Hilb_{h_\diamond} \le 1$ trivially imply that

every projector $p \in \mathcal{P}(Y, \|...\|_{h_\diamond})$ from the support of some $\mu_\diamond$ lies in $\mathcal{P}(Y, \|...\|_Y)$, i.e. $\|p\|_Y = 1$, and the norm $\|...\|_{h_\diamond}$ equals $\|...\|_Y$ on the axes $\mathbb{R}_p = im_p \subset Y$ of all $p \in supp(\mu_\diamond)$.

Remark. The above shows that the total masses of John's partitions of unity $\mu_\diamond$ in Y satisfy $\mu_\diamond(\mathcal{P}) = n = \dim Y$. In fact, the existence of a partition of unity in Y with the total mass $\le n$ is equivalent (by a trivial argument) to the full John's theorem.

Riemannian Lower Volume Bound. Let Y be an n-dimensional Banach space, let $\Omega_R \subset Y$, $R \to \infty$, be bounded domains, as in the Hilbert-Hadamard volume domination from section 3, i.e. such that their n-volumes with respect to John's Hilbert/Euclidean metric $h_\diamond$ satisfy $vol_\diamond(\Omega_R) = const \cdot R^n$ and such that for every r_0 the volumes of the r_0-neighbourhoods of their boundaries satisfy

$$vol_\diamond(U_{r_0}(\partial\Omega_R)) = o(R^n) \text{ for } R \to \infty,$$

e.g. Ω_R are the R-balls $\{\|y\| \le R\} \subset Y$.

Asymptotic Banach-John Volume Inequality of Burago-Ivanov. Let X be an n-dimensional Riemannian manifold, let $f : X \to Y$ be a proper (pullbacks of compact set are compact) map of non-zero degree and let $Lip_\infty(f)$ be the Lipschitz constant of f at infinity with respect to the metric $\|y_1 - y_2\|_Y$ in Y.

Then the (John's Euclidean) $h_\diamond$-volumes of $\Omega_R \subset Y$ are "asymptotically sharply bounded" by the Riemannian volumes of their f-pullbacks by

$$vol_n(f^{-1}(\Omega_R)) \ge Lip^{-n}_\infty(f) \cdot vol_\diamond(\Omega_R) - o(R^n) \text{ for } R \to \infty.$$

Consequently, since $\|...\|_{h_\diamond} \geq \|...\|_Y$*, the asymptotic volume growth of* R*-balls in* X *around any given point* $x_0 \in X$ *is asymptotically sharply bounded by the growth of the Euclidean balls,*

$$\liminf_{R\to\infty} vol(B_X(R))/vol(B_{\mathbb{R}^n}(R)) \geq 1.$$

Proof. Since John's $\mu_\diamond$ serves as partition of unity in $(Y, \|...\|_Y)$ as well as in $(Y, \|...\|_{h_\diamond})$, the map $f : X \to \mathbb{R}^n = (Y, \|...\|_{h_\diamond})$ satisfies

$$\|dil^*_{h_\diamond} f\|_{L_\infty(\mu)} \leq Lip_\infty(f)_{\|...\|_Y},$$

where – this is the main point – the dilation $\|dil^* f\|$ is measured with the (Euclidean) $h_\diamond$-metric in Y, while $Lip_\infty(f)$ is evaluated with the original Banach metric. Hence the Hilbert-Hadamard volume domination applies and the proof follows.

5 Co-Lipschitz at Infinity, Federer-Whitney Metric Descendants and Abelian Volume Growth Inequality of Burago-Ivanov.

Federer-Whitney Theorem. Define the *co-Lipschitz constant at infinity* of a map between metric spaces, $f : X \to Y$, by

$$coLip_\infty(f) = \limsup_{diam_Y(B)\to\infty} diam_X(f^{-1}(B))/diam_Y(B),$$

where $B \subset Y$ run over all *bounded* subsets in Y.

Observe that if

$$\sup_{y\in Y} diam_X(f^{-1}(y)) < \infty,$$

(which is weaker than $coLip_\infty(f) < \infty$) then

$$coLip_\infty(f) = \Big(\liminf_{dist(x_1,x_2)\to\infty} dil_f(x_1, x_2)\Big)^{-1}.$$

A map between metric spaces, $f : X \to Y$, is called an *isometry at infinity* if the image of f intersect all R-balls in Y for $R \geq R_0 < \infty$ and

$$Lip_\infty(f) = coLip_\infty(f) = 1;$$

this is equivalent to

$$dil_f(x_1, x_2) = \frac{dist_X(x_1, x_2)}{dist_Y(f(x_1)f(x_2))} \to 1 \text{ for } dist_X(x_1, x_2) \to \infty.$$

Let X be a locally compact metric space, Y an $\mathbb{R}$-linear n-space, where both spaces are acted upon by a locally compact group Γ, such that the action of Γ on X is *isometric* and the action on Y is *affine*.

Let this affine action be *co-compact quasi-translational*, i.e. Y/Γ is compact and Γ admits a *co-compact* subgroup that acts on Y by *parallel translations*.

Federer-Whitney f-Descendant Metric. Let $f : X \to Y$ be a proper continuous Γ-equivariant map.

Then there exists a unique Banach distance on Y, denoted $dist_Y(y_1, y_2) = \|y_1 - y_2\|_Y$ and called Federer-Whitney f-Descendant of the metric $dist_X$, such that the map f is isometric at infinity with respect to $dist_X$ and $dist_Y$.

Proof. Clearly, there exists a unique *maximal* Banach distance $dist_Y$ in Y with respect to which $Lip_\infty(f) \le 1$.

To see that $coLip_\infty(f) = 1$ as well, observe that every two disjoint r-balls in Y satisfy

$$dist_Y(B_{y_1}(r), B_{y_2}(r)) = \lim_{\lambda\to\infty} \lambda^{-1} \cdot dist_X\big(f^{-1}(B_{\lambda y_1}(\lambda r)), f^{-1}(B_{\lambda y_2}(\lambda r))\big),$$

where the scaling $y \to \lambda y$ is understood with some point in Y taken for zero and where the existence of the limit, as well the convexity and homogeneity of the limit distance function, trivially follow from

- the triangle inequality for $dist_X$,
- elementary properties of *asymptotically sub-additive* positive functions $d(\rho)$, $\rho > 0$, i.e. such that $d(\rho_1 + \rho_2) \le d(\rho_1) + d(\rho_2) + o(\rho_1)$; the essential property of such functions d is the existence of the limit $\lim_{\rho\to\infty} d(\rho)/\rho$.

Remarks. (a) The above is the special (trivial) case of the full *Federer-Whitney duality theorem* between the *stable volume norm on homology* $H_k(V)$ and the *comass norm* on cohomology $H^k(V;\mathbb{R})$. (See sections 4.C and 4.D in [**?**] and 7.4 in [**?**] for the statement and further applications of this theorem.)

(b) The Federer-Whitney theorem generalizes [**?**] to maps $X \to Y$, where Y is a simply connected *nilpotent* Lie group with an *expanding* automorphism λ and where the existence of the limit

$$\lim_{N\to\infty} \lambda^{-N} \cdot dist_X\big(f^{-1}(\lambda^N(B_{y_1})), f^{-1}(\lambda^N(B_{y_2}))\big)$$

is not so obvious.

(c) The property of being isometric at infinity for equivariant maps f sometimes implies a much stronger (and non-obvious) one:

$$|dist_X(f^{-1}(y_1), f^{-1}(y_1)) - dist_Y(y_1, y_2)| \le const < \infty.$$

This was proven by D. Burago [**?**] for surjective maps from length metric spaces (where the distance is given by infima of length of curves between points) into linear spaces Y and, later on, for maps into some nilpotent Lie groups and all hyperbolic Γ-spaces by S. Krat [**?**], but also there are counterexamples [**?**]. Yet, the full geometry of isometric at infinity equivariant maps has not been fully clarified at the present day.

Abelian Volume Growth Inequality of Burago-Ivanov . Let $\tilde V$ be the universal covering of a compact Riemannian n-manifold V that admits a continuous map of non-zero degree onto the n-torus, $V \to \mathbb{T}^n$, e.g. V is homeomorphic to $\mathbb{T}^n$.

Then the asymptotic volume growth of R-balls $B(R)$ in $\tilde V$ for $R \to \infty$ is minorized by that in $\mathbb{R}^n$,

$$\liminf_{R\to\infty} vol(B(R))/vol(B_{\mathbb{R}^n}(R)) \ge 1.$$

Proof. Let Y be the universal covering of the torus $\mathbb{T}^n$, let X be the corresponding $\mathbb{Z}^n$-covering of V and $f : X \to Y$ the $\mathbb{Z}^n$-equivariant map induced by our $V \to \mathbb{T}^n$.

Let $\|...\|_Y$ be the Federer-Whitney norm on Y and $h_\diamond$ John's quadratic Hilbertian (Euclidean) form.

Since f is isometric at infinity with respect to the Federer-Whitney metric $\|y_1 - y_2\|_Y$, the f-pullbacks of the (Euclidean) $h_\diamond$-balls $B_\diamond(R - o(R)) \subset \mathbb{R}^n = (Y, h_\diamond)$ of radii $R - o(R^n)$ are contained in the Riemannian R-balls in $X = \tilde{V}$, and since $Lip_\infty(f) = 1$, the asymptotic Banach-John volume inequality from the previous section applies.

Remarks. (a) This proof:

Federer-Whitney + John's $h_\diamond$ + Hadamard's inequality

is similar to the original one in [**?**], except that the authors of [**?**] use at some point a rather subtle Burago's theorem (stated in the above Remark (c)) instead of the (almost obvious) Federer-Whitney theorem.

(b) *Volume Rigidity.* The above argument combined with "sharpness of Hadamard" (see **B** in section 2) implies by a simple argument another theorem from [**?**].

Let X be a Riemannian Γ-manifold of dimension n, where the asymptotic volume growth of balls equals that in $\mathbb{R}^n$,

$$\liminf_{R\to\infty} vol(B_X(R))/vol(B_{\mathbb{R}^n}(R)) = 1.$$

If X admits a Γ-equivariant map $f_0 : X \to \mathbb{R}^n$ of a non-zero degree, where the action of Γ on $\mathbb{R}^n$ is affine co-compact quasi-translational, then X is isometric to $\mathbb{R}^n$.

This, however, leaves open the following

Question. What is the minimal volume growth of an X with the corresponding Federer-Whitney Banach space being in a given isometry class?

(We shall address this in Part 2 of this paper.)

(c) An attractive feature of the volume growth theorem, is that a general convexity argument (John's e theorem) yields, rather unexpectedly, a *sharp purely Riemannian* inequality.

(A non-sharp lower bound on asymptotic volumes of balls follows by the above argument, see [**?**] and 4.C in [**?**], from Federer-Whitney and the inequality $\|...\|_A \geq \sqrt{n} \cdot \|...\|_{h_\diamond}$ that is the standard corollary to John's $Hilb_{h_\diamond} \leq 1$.)

But we shall see with our definition of *Hilbertian volume* in section 9 that, in truth, there is not so much "Riemannian" in this inequality after all: the Abelian volume growth inequality and the volume rigidity hold for (almost) arbitrary metric Γ-spaces X.

6 Γ-Spaces, Averaging and Hilbert's Straightening Map $f_\diamond$.

The Federer-Whitney Banach metric is, obviously, invariant under actions of a group Γ acting on our spaces. Let us explain how to make other constructions Γ-invariant as well.

Recall, that *a* Γ*-space* in a given (topological) category is a space X with an action of a group Γ, denoted $\gamma_X : X \to X$, $\gamma \in \Gamma$, where these maps γ_X are morphisms in our category. We are mainly concerned with *metric* Γ*-spaces* where the maps γ_X are isometries.

Equivariant Hilbert Straightening Theorem. Let X and Y be topological Γ-spaces as in the Federer-Whitney theorem, i.e. X is a locally compact metric Γ-space, and Y an $\mathbb{R}$-affine n-space isomorphic to $\mathbb{R}^n$, where the action of Γ is co-compact quasi-translational. We denote by $Y_\bullet$ the linear space associated to Y that is the space of parallel translations of Y and that can be (non-functorially) thought of as Y with a point in it distinguished for 0.

Let $f_0 : X \to Y$ be a proper continuous Γ-equivariant map.

Then there exist a unique Γ*-invariant (Federer-Whitney-John's) Euclidean (Hilbertian) metric* $dist_\diamond = ||...||_{h_\diamond}$ *on* Y*, and a (non-unique)* Γ*-equivariant Lipschitz map* $f_\diamond : X \to Y$*, such that*

$$dist_{h_\diamond}(f_\diamond, f_0) \leq \infty,$$

the Hilbert constant of $f_\diamond$ *with respect to the Hilbertian form* $h_\diamond$ *on* Y *satisfies*

$$\widetilde{Hilb}(f_\diamond) = 1,$$

and, at the same time,

$$coLip_\infty(f_\diamond) \leq 1.$$

Proof. The distance $||...||_{h_\diamond}$ in Y for John's $h_\diamond$ associated to the Federer-Whitney Banach metric $dist_Y$ is Γ-invariant due to uniqueness of $h_\diamond$ but the underlying measure $\mu_\diamond$ – partition of unity $(d_{\mu_\diamond} p)$ or equivalently, partition of $h_\diamond$ (that is $d_{\mu_\diamond} l$), is not necessarily invariant.

But since the linear isometry group G of $(Y_\bullet, ||...||_Y)$, where $Y_\bullet$ is the linear space associated to Y, is compact, one can average partitions of unity $\mu_\diamond$ (or rather partitons of $h_\diamond$ into squares of linear functions) over G and have John's partition of unity $\mu_\diamond$ invariant under the action of Γ.

Let us make the ε-straightened maps $f_\varepsilon : X \to Y$ from section 3 equivariant by the following standard averaging over Γ.

Observe that Γ acts on the space F of maps $f : X \to Y$ by $\gamma_F : f \mapsto \gamma_Y^{-1} \circ f \circ \gamma_X$ with fixed points of this action corresponding to equivariant maps f.

If we replace maps $f \in F$ by $b_f = f - f_0$ for our (equivariant!) $f_0 : X \to Y$, this actions becomes the obvious shift action on the space of maps $b : X \to Y_\bullet$ for $\gamma : b(x) \mapsto b(\gamma_X(x))$.

Since the full isometry group $iso(Y) \supset \Gamma$ is *amenable* one can average *bounded* functions to Γ-invariant ones, where "bounded for b_f" corresponds to the "finite distance from f_0."

$$dist_Y(f, f_0) = \sup_{x \in X} dist_\diamond(f(x), f_0(x)) < \infty.$$

Since the straightening maps $f_\varepsilon : X \to Y$ from section 3 do lie within finite distance from f_0, they can be averaged to equivaruant one, say $\overline{f}_\varepsilon : X \to Y$, and since $||\widetilde{dil}^* f_\varepsilon||_{L_2(\mu)} \leq 1 + \varepsilon$ and the function(al) $f \mapsto ||\widetilde{dil}^*||^2_{L_2(\mu)}$ is (obviously) *convex*, these averaged maps also have their $\tilde{L}_2$-dilations bounded by

$$||\widetilde{dil}^* \overline{f}_\varepsilon||_{L_2(\mu)} \leq 1 + \varepsilon$$

Finally, since the maps $\overline{f}_\varepsilon$ are uniformly Lipschitz, some sequence of them converges with $\varepsilon \to 0$ to the desired equivariant map $f_\diamond : X \to Y$. QED.

Remark. The Hilbert straightening map $f_\diamond$ can be regarded as a metric counterpart to the *Abel-Jacobi-Albanese maps* from Kähler manifolds to their Jacobians. The latter maps, being holomorphic are (pluri)harmonic; they minimize Dirichlet's $\int_{X/\Gamma} \|Df(x)\|^2_{L_2} dx$ over all Γ-equivariant maps f, while the map $f_\diamond$ minimizes, in a way, the norm $\sup_x \|Df(x)\|_{L_2}$.

Besides $f_\diamond$, there is also a *dynamical* counterpart to the Abel-Jacobi-Albanese, called *the Shub-Franks map* that is associated with *hyperbolic endomorphisms* of tori and infra-nil-manifolds in general. The three maps and their generalizations intricately intertwine in their applications to the geometric rigidity theory (See [**?**] for a related discussion on A-J-A – S-F connection.)

6.1 On Infinite Dimensional Γ-Spaces and Codiffusion Spaces.

The above suggests the following Γ-version of the notion of the L_q-dilation from section 1.

Let Y be a Banach space with an affine isometric action of a locally compact (e.g. discrete) group Γ. Let $\Pi \subset \Gamma$ be the subgroup of parallel translations in Γ and let $\Gamma_0 = \Gamma/\Pi$.

Thus, Γ_0 and, hence, Γ itself, act on Y by *linear* isometric transformations.

A proper axial Γ-partition of unity in Y is a Γ_0-*invariant* measure $\mu = d_\mu p$ on the spaces $\mathcal{P}$ of axial projectors $p : Y \mapsto \mathbb{R}_p = im_p \subset Y$, such that the action of Γ_0 on the measure space $(\mathcal{P}, \mu)$ is *proper*:

there exists a measurable subset $U \subset \mathcal{P}$ such that the Γ_0-orbit of U equals almost all $\mathcal{P}$ and such that the subset of those $\gamma \in \Gamma_0$ for which $\mu(\gamma(U) \cap U) > 0$ is precompact in Γ_0.

If the action is proper, the quotient space $\underline{\mathcal{P}} = \mathcal{P}/\Gamma_0$ carry a Lebesgue-Rochlin measure, say $\underline{\mu}$, and one can integrate invariant functions on $\mathcal{P}$ over $(\underline{\mathcal{P}}, \underline{\mu})$.

Thus, one can define the L_q-dilation of a Γ-equivariant map f from a metric Γ-space X into Y as

$$\|dil^* f\|_{L_q(\underline{\mu}=d_{\underline{\mu}}p)} = \left(\int_{\underline{\mathcal{P}}} Lip^q(f) d_{\underline{\mu}} p \right)^{1/q}.$$

Several constructions we met earlier, namely those which do not involve integration apart from that over $\mathcal{P}$ generalize to this setting: that are

• Equivariant minimal "norms" $\|min.dil^* f\|_{L_q}$ and $\|min.\widetilde{dil}^* f\|_{L_q}$,

• Banach straightening at infinity (section 3) and equivariant Hilbert straightening,

• John's $h_\diamond$, where one needs to(?), regretfully, assume that the Banach space Y is "essentially Hilbertian" to start with: it admits a Γ_0-invariant Hilbert form h, such that $\|...\|_Y \leq \|...\|_h \leq const \cdot \|...\|_Y$.

(This assumption rules out, for example, the only natural candidate for John's $h_\diamond$ on the spaces $l_q(\Gamma)$, that is Hilbert's $h = \|...\|^2_{l_2(\Gamma)}$ for countable groups Γ.)

Where are we to go from this point? Are there unexplored domains populated by interestingly structured infinite dimensional Γ-spaces ?

One may start a search for them among *symbolic* Γ*-spaces*, [**?**], [**?**], and/or *concentrated spaces* [**?**]).

Axes, Non-Linear Partitions of Unity and Codiffusion. A distance minimizing geodesic in a metric space Y that is the image of an isometric embedding $\mathbb{R} \mapsto \mathbb{R}_\bullet = im_p \subset Y$ admits a 1-Lipschitz projection p onto itself. Thus "many" spaces Y admit "many" axial projectors $p : Y \to \mathbb{R}_p \subset Y$.

Notice that an axis $\mathbb{R}_\bullet \subset Y$ may have several projections $Y \to \mathbb{R}_\bullet$. Two, apparently, most natural ones correspond to the two *Busemann functions* $b_\pm$ on Y associated to this axes with two possible orientations: $y \mapsto b_\pm(y) \in \mathbb{R} = \mathbb{R}_\bullet \mapsto Y$.

There are two alternative ways to define the L_2-dilation of a map $f : X \to Y$.

(1) If Y is a Riemannian/Hilbertian manifold with the Riemannian quadratic differential form g, one may forfeit projectors and use measures $\mu = d_\mu b$ on the space $\mathcal{B}$ of 1-Lipschitz functions $b : Y \to \mathbb{R}$.

The normalization condition of μ being a partition of unity may be formulated by requiring the integral of the squares of the differentials of our functions to be equal to g,

$$\int_{\mathcal{B}} db^2 d_\mu b = g.$$

Here it seems reasonable to limit the support of relevant measures μ to the space of Busemann functions or, more generally of all *horofunctions*. Besides, if Y is a proper Γ-space, it may be to one's advantage to restrict to Γ invariant measures μ on $\mathcal{B}$ and integrate over $\mathcal{B}/\Gamma$ rather than over $\mathcal{B}$.

(2) In order to sum/integrate projectors with a measure on the space of projectors one may use an additional structure in X, called

Codiffusion. Let $\mathcal{N} = \mathcal{N}(Y)$ be the space of probability measures ν on Y where Y is embedded to $\mathcal{N}$ by assigning Dirac's δ_y to all $y \in Y$.

A codiffusion in Y is a projection $R : \mathcal{N} \to Y \subset \mathcal{N}$, where "projection" means $R^2 = R$.

For example, every complete convex (locally) affine topological space, e.g. a closed convex subset in a Banach space, comes with a natural codiffusion that sends every measure ν on X to its *center of mass*

The center of mass construction generalizes to those geodesic spaces, in particular to Riemannian manifolds, Y where the square distance functions $y' \mapsto dist^2(y', y)$ are strictly convex on all geodesics: the center of mass of a ν is defined as the minimum point $y'_{min} \in Y$ of the (strictly convex!) function $y' \to \int_Y dist^2(y', y) d_\nu y$ (see [**?**], [**?**] and also [**?**] for the history of this concept and new applications.)

However, all this does not seem to help in answering the following

Questions (a) What are "interesting" instances of equivariant maps $f : X \to Y$ between n-dimensional Riemannian (Finsler?) Γ-spaces such that

$$coLip_\infty(f) \cdot \widetilde{Hilb}(f(x)) = 1 \text{ for all } x \in X$$

where the group Γ is *not* virtually Abelian?

(b) Let X be a metric Γ-space, $Y = (Y, h)$ a Finsler (e.g. Riemannian) Γ-manifold and $f : X \to Y$ be a proper equivariant map. Suppose that all geodesics in Y are distance minimizing.

What is the minimal $C = C(Y) > 0$ for which there exists a Γ-equivariant map $f_\bullet : X \to Y$ within bounded distance from f such that $\widetilde{Hilb}(f_\bullet(x)) \le C \cdot Lip_\infty(f)$ for all $x \in X$?

(c) Let Y be a contractible metric Γ-space. Consider all, possibly singular, (oriented?) Riemannian Γ-spaces X of dimension n that admit equivariant maps $f : X \to Y$ that have $Lip_\infty(f) \le 1$ and such that the induced homomorphisms on their invariant n-dimensional real cohomologies do not vanish, e.g. $dim(Y) = dim(X) = n$ and $deg(f) \neq 0$.

When does there exist an X_{min} among them that minimizes $vol_n(X/\Gamma)$?

How does X_{min} change (if at all) with passing to subgroups of finite indices in Γ?

Can this X_{min} be explicitly described for particular "simple" (possibly, infinite dimensional, e.g. isometric to Hilbert's $\mathbb{R}^\infty$) spaces Y?

Given subgroups $\Gamma_i \subset \Gamma$ of finite indices $ind_i = card(\Gamma/\Gamma_i)$, let $f_i : X_i \to Y$ be an $ind_i^{-1} vol_n(X/\Gamma_i)$-minimizing sequence of Γ_i-equivariant maps $f_i : X_i \to Y$ with $Lip_\infty(f_i) \le 1$. When and how does such a sequence converge?

7 Burago-Ivanov's Solution of Hopf Conjecture.

Let us show (essentially) following the original argument in [**?**] how Hilbert's straightening implies *Hopf conjecture*:

Let $V = (V, g)$ be a compact Riemannian manifold without conjugate points and $\underline{f} : V \to \mathbb{T}^n$ be a continous map such that the induced homomorphism of the fundamental groups, $\pi_1(X) \to \pi_1(\mathbb{T}^n) = \mathbb{Z}^n$, is an isomorphism. Then the universal covering $X = (X, g)$ of V is isometric to $\mathbb{R}^n$.

In fact, if all geodesic segments in X are distance minimizing, then

Hilbert's straightening map $f_\diamond : (X, g) \to (Y, h_\diamond) = \mathbb{R}^n$ is an isometry,

where Y is the universal covering of the torus $\mathbb{T}^n$ and $h_\diamond$ is John's Hilbertian form associated to the Federer-Whitney f-descendant $||...||_Y$ of the metric $dist_g$ on X for $f : X \to Y$ being the lift of $\underline{f}$ to X.
This is immediate with the following simple classical formulas **(LS1)** and **(LS2)**.

Liouville-Santalo Integral Identities. Let $X = (X, g)$ be a complete Riemannian manifold.

Given a measure $\lambda = d_\lambda \tau$ on the tangent bundle $T(X)$, denote by $\underline{\lambda} = d_{\underline{\lambda}} x$ the push-forward of λ under the projection $\pi : T(X) \to X$, and assume that $\underline{\lambda}(W) < \infty$ for all compact subsets $W \subset X$.

Call a measure λ *balanced* if, for every continuous quadratic form h on $T(X)$,

$$\int_X trace_{g_x}(h_x) d_{\underline{\lambda}} x = dim(X) \int_{T(X)} h(\tau) d_\lambda \tau.$$

For example, the *Liouville measure*, that is supported on the unit tangent bundle $S(X) \subset T(X)$, is balanced, since the trace of a quadratic form h on $\mathbb{R}^n$ equals n-times the average values of h on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Let $\Sigma(R) = \Sigma(X; R)$ be the space of geodesic segments of length R in X that are locally isometric (geodesic) maps $\sigma : [0, R] \to X$ and denote by $\partial_\sigma \in S_{\sigma(0)}(X) \subset T(X)$ the unit tangent vector to our geodesic σ at the point $\sigma(0) \in X$.

The correspondence $\sigma \leftrightarrow s = \partial_\sigma$ identifies $\Sigma(R)$ with the unit tangent bundle $S(X)$; accordingly, the measures on $\Sigma(R)$ corresponding to $\lambda = d_\lambda s$ on $S(R)$ are denoted $d_\lambda \sigma$.

Let X be acted upon discretely and isometrically by a group Γ, and let λ be a measure on $S(X)$ that is *invariant under the action of* Γ and that is *normal* in the sense the total λ-mass of $S(X)/\Gamma$ equals one,

$$\lambda(S(X)/\Gamma) = \int_{S(X)/\Gamma} d_\lambda \tau = 1.$$

Let $Y = (Y, h)$ be another complete Riemannian Γ-manifold and let $f : X \to Y$ be a proper Γ-equivariant Lipschitz map.

Average/integrate the dilation of f at the two ends of our geodesic segments, or, rather, the dilation of the composed maps $f \circ \sigma : [0, R] \to Y$,

$$dil_{f\circ\sigma}(0, R) = R^{-1} dist_Y(f \circ \sigma(0), f \circ \sigma(R)),$$

over $(\Sigma(R), d_\lambda \sigma)$ and denote this averaged dilation by

$$dil_{av}(f \circ \sigma, R) = \int_{S(X)/\Gamma = \Sigma(R)/\Gamma} dil_{f\circ\sigma}(0, R) d_\lambda \sigma.$$

Clearly, this dil_{av} is bounded by the *average length* of the parametized curves $f \circ \sigma : [0, R] \to Y$ divided by R,

$$dil_{av}(f \circ \sigma, R) \le R^{-1} length_{av}(f \circ \sigma[0, R]) = R^{-1} \int_{\Sigma(R)/\Gamma} length(f \circ \sigma[0, R]) d_\lambda \sigma.$$

If the measure λ is *invariant under the geodesic flow*, then, by Fubini's theorem, this integral equals the integrated norm of $\partial_\sigma f \in T_{f(x)}(Y)$, for $x = \sigma(0) = \pi(\partial_\sigma)$, that is the derivative (differential) of f in the direction of the unit tangent vector $\partial_\sigma \in S_x \subset T(X)$,

(LS1) $$R^{-1} length_{av}(f \circ \sigma[0, R]) = \int_{\Sigma(R)/\Gamma} \|\partial_\sigma f\| d_\lambda \sigma,$$

where, by *Schwartz inequality,*

$$\int_{\Sigma(R)/\Gamma} \|\partial\sigma(0)\| d_\lambda \sigma \le \left(\int_{\Sigma(R)/\Gamma} \|\partial\sigma(0)\|^2 d_\lambda \sigma \right)^{1/2}.$$

If λ is *balanced*, then

(LS2) $$\int_{\Sigma(R)/\Gamma} \|\partial\sigma(0)\|^2 d_\lambda \sigma = Hilb^2_{av}(f),$$

where $Hilb_{av}(f)$ is the normalized *Dirichlet-Hilbert* energy of f that is

$$Hilb^2_{av}(f) =_{def} n^{-1} \int_{X/\Gamma} \|Df\|^2_{L_2} d_\lambda x = n^{-1} \int_{X/\Gamma} trace_{g(x)}(Df(x))^*(h) d_\lambda$$

for the pullback $(Df(x))^*(h)$ of the Riemannian form h from $T(Y)$ to $T(X)$ by the differential of f.

In sum, **(LS1)**+**(LS2)**+(Schwartz inequality) imply:

if λ is a normal Γ-invariant measure on $T(X)$ that is balanced and invariant under the geodesic flow and $f : X \to Y$ is a Γ-equivariant map, then the λ-average dilation of f at the ends of geodesic segments in X of length R satisfies

$$dil_{av}(f \circ \sigma, R) \leq Hilb_{av}(f) = n^{-1/2} \left(\int_{X/\Gamma} \|Df\|^2_{L_2} d_{\underline{\lambda}} x \right)^{1/2}.$$

This integral inequality is sharp: the equality is possible only if all quantities involved are equal point-wise and this sharpness is independent of R, since the integral $\int_{\Sigma(R)/\Gamma} \|\partial_\sigma f\| d_\lambda \sigma = R^{-1} length_{av}(f \circ \sigma[0, R])$ does not depend on R.

It follows, in particular, that

if

$$\limsup_{R \to \infty} dil_{av}(f \circ \sigma, R) \geq 1 \text{ and } Hilb_{av}(f) \leq 1,$$

then the map f is isometric on every geodesic $\sigma : (-\infty, +\infty) \to X$ that is contained in the support of λ.

The Hopf conjecture follows by applying the above to the universal covering X of a torus without conjugate points with the Liouville measure λ on $T(X)$ and to the Hilbert straightening map $f_\diamond : X \to \mathbb{R}^n = (Y, h_\diamond)$.

In fact, "no conjugate points" says that all geodesic segments in X, that are $\sigma : [0, R] \to X$, have $dist_X(\sigma(0), \sigma(R)) = R$. Therefore, the average dilation of $f_\diamond$ satisfies

$$\left(\limsup_{R \to \infty} dil_{av}(f_\diamond \circ \sigma, R) \right)^{-1} \leq coLip_\infty(f_\diamond),$$

where, recall,

$$coLip_\infty(f) = \liminf_{dist(x_1, x_2) \to \infty} dist(f(x_1), f(x_2))/dist(x_1, x_2)$$

for proper maps f.

On the other hand the inequality $Hilb(f_\diamond) \leq 1$ implies that $\|Df_\diamond\|^2_{L_2} \leq n$ for almost all $x \in X$. Hence, $f_\diamond$ is an isometry. QED.

Questions. Let X be a complete Riemannian Γ-manifold of dimension n and $f : X \to \mathbb{R}^n$ a proper continuous Γ-equivariant map for some discrete cocompact isometric action of the group Γ on $\mathbb{R}^n$. For example, X may be equal the universal covering of a compact manifold homeomorphic to the n-torus.

What is a possible geometry of X if

$$dil_{av}(f \circ \sigma, R) \geq Hilb_{av}(f) - \varepsilon, \text{ for all } R \geq R_0,$$

where the averages are taken with the Liouville measure?

What can be geometry of an X if the geodesic segments $\sigma : [0, R] \to X$, satisfy $dist_X(\sigma(0), \sigma(R)) \geq \phi(R) \cdot R$ for a given function $0 < \phi(R) < 1$ and all sufficiently large R?

In particular, does the inequality $dist_X(\sigma(0), \sigma(R)) \geq (1 - \varepsilon)R$ for a small (any<1?) ε and all σ imply that X is isometric to $\mathbb{R}^n$?

Do "majority" of X have $dist_X(\sigma(0), \sigma(R)) \sim R^{1/2}$ on the average?

8 Hilbert Jacobian, Homological Volume, Semicontinuity, Coarea Inequality and John's Identity.

Let $V = (V^n, \chi)$ be a smooth n-dimensional manifold with a smooth measure χ. Assume that V is diffeomorphic to an open subset in $\mathbb{R}^n$ and, given a Lipschitz map f from a metric space X to Y, consider all diffeomorphisms $\phi : V \to \mathbb{R}^n$ for which $Jac^{[n]}(\phi(v)) \geq 1$ for all $v \in V$ (such $\phi(v)$ always exist and can be even taken smooth).

Define the *global Hilbert Jacobian* of f restricted to a subset $U \subset X$ as

$$\widetilde{Hil}Jac^{[n]}_{glb}(f|_U) = \inf_{\phi} \widetilde{Hilb}^n(\phi \circ f|_U),\ \phi \circ f|_U : U \to \mathbb{R}^n$$

and

$$\widetilde{Hil}Jac^{[n]} f_{/v} = \inf_{V_v \ni v} \widetilde{Hil}Jac^{[n]}_{glb}(f|_{f^{-1}(V)}),\ y \in Y,$$

where the infimum is taken over all neighbourhoods $V_y \subset V$ of v.

Since every smooth n-manifold is locally diffeomorphic $\mathbb{R}^n$, this definition makes sense for *all* smooth, including closed ones, manifolds $V = (V^n, \chi)$.

Next let

$$\widetilde{Hil}Jac^{[n]} f(x) = \inf_{U \ni x} \widetilde{Hil}Jac^{[n]}_{glb}(f|_U),\ x \in X,$$

where the infimum is taken over all neighbourhoods $U \subset X$ of x and where, again, this is defined for all V.

Finally, define the point-wise Hilbert Jacobian $\widetilde{Hil}Jac^{[n]} f(x)$ of a Lipschitz map f between arbitrary metric spaces, $f : X \to Y$, as the infimum of the numbers J such that

$$\widetilde{Hil}Jac^{[n]}(\tau \circ f)(x) \leq J \cdot \widetilde{Hil}Jac^{[n]} f(x)$$

for all Lipschitz maps $\tau : Y \to \mathbb{R}^n$.

It is immediate that

$$\widetilde{Hil}Jac^{[n]}(f) \leq Lip^n(f)$$

and that composed maps $X \underset{f}{\to} Y \underset{g}{\to} Z$ satisfy

$$\widetilde{Hil}Jac^{[n]} g \circ f(x) \leq \widetilde{Hil}Jac^{[n]} f(x) \cdot \widetilde{Hil}Jac^{[n]} g(y) \text{ for } y = f(x)$$

Also observe that $\widetilde{Hil}Jac^{[n]}(f)$ equals the (absolure value of the) ordinary Jacobian $Jac^{[n]}(f)$ for smoth maps f between Riemannian manifolds.

Homologically Stable Hilbertian Volume. We want to define n-volumes in metric spaces X with a usual behaviour under (Jacobians of) Lipschitz maps $f : X \to Y = (Y^n, \chi)$, e.g. decreasing under maps f with $\widetilde{Hil}Jac^{[n]}(f(x)) \leq 1$, where we are concerned with *stability of the images* under small continuous perturbations maps that is defined as follows.

Fix a coefficient group (e.g. a field) $\mathbb{F}$ and call a point $y \in Y$ *cohomologically ($H^n_{\mathbb{F}}$-image) stable* for a continuous map $f : X \to Y$ if for all sufficiently small

neighbourhoods $V = V_y \subset Y$ of y the induced homomorphism on the relative cohomology

$$f^{*n} : H^n(V, \partial V; \mathbb{F}) \to H^n(f^{-1}(V), f^{-1}(\partial V); \mathbb{F})$$

does not vanish. The set of these stable points is denoted by $stbl_{\mathbb{F}}im_f(X) \subset Y$.

As we shall be dealing with locally compact spaces, the cohomology will be *Čech;* to be specific, we stick to $\mathbb{F} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

The hyper-Hilbertian global n-volume $glb_{\mathbb{F}}vol^{[n]}(\mathbb{R}^n \backslash X)$ *of* X *over* $\mathbb{R}^n$ is the supremum of the numbers H such that X admits a Lipschitz map $f : X \to \mathbb{R}^n$ with $\widehat{Hil}Jac_x^{[n]}(f) \leq 1$, $x \in X$, and such that the Lebesgue-Haar measure of the set $stbl_{\mathbb{F}}im_f(X) \subset \mathbb{R}^n$ is $\geq H$.

For example, $glb_{\mathbb{F}}vol^{[n]}(\mathbb{R}^n \backslash X) = vol_n(X)$, for smooth open Riemannian manifolds X, while closed manifolds X satisfy $glb_{\mathbb{F}}vol^{[n]}(\mathbb{R}^n \backslash X) = vol_n(X)/2$. (To get rid of this silly $1/2$ one has to map X to the n-spheres S^n rather than to $\mathbb{R}^n$.)

A more interesting example is provided by closed minimal hypersurfaces $X = X^n$ in compact Riemannian manifolds $X^{n+1} \supset X$ representing homology classes in $H_n(X^{n+1})$: such an X admits a λ-Lipschitz map, $f : X \to S^n$ of non-zero degree, where $\lambda \leq \lambda_0(X^{n-1}, vol_n(X)) < \infty$ as it follows from the *compactness* of the space of minimal subvarieties in X^{n+1} with bounded volumes and the *maximum principle* for minimal hypersurfaces.

Then one sees, by looking at tangent cones, that every point x in such a minimal $X \subset X^{n+1}$ admits an arbitrarily small neighbourhood $U_x \subset X$ such that

$$glb_{\mathbb{F}}vol^{[n]}(\mathbb{R}^n \backslash U_x) \geq \varepsilon \cdot vol_n(U_x) \text{ for some } \varepsilon \geq \varepsilon(X) > 0.$$

Homology Domination and Semicontinuity of $glb_{\mathbb{F}}vol^{[n]}$. Let Z be a metric space, $X \subset Z$ a locally compact locally closed (open $\cap$ closed) subset, and let X_i, i=1,2,..., be a sequence of locally closed subsets that converges to X in the sense that for every neighbourhood $Z_X \subset Z$ of X all but finitely many of X_i are contained in Z_X.

Say that X_i *(co)homologically dominate* X *in dimension* n *with a given coefficient domain* $\mathbb{F}$ if for every pair of open subsets $Z_1 \subset Z_2 \subset Z$ every relative cohomology class in $H^n(Z_2, Z_1; \mathbb{F})$ that restricts to a *non-zero* class in $H^n(Z_2 \cap X, Z_1 \cap X; \mathbb{F})$ also restricts to *non-zero* classes in $H^n(Z_2 \cap X_i, Z_1 \cap X_i; \mathbb{F})$ *for all but finitely many* i.

By invoking the local $\tilde{L}_q$-extension with $q = 2$ from section 1, one obtains:

If subsets $X_i \subset Z$ *homologically dominate a subset* X *in a metric space* Z *then*

$$\liminf_{i \to \infty} glb_{\mathbb{F}}vol^{[n]}(X_i) \geq glb_{\mathbb{F}}vol^{[n]}(X).$$

(Semi)Local Coarea Inequality. Let $Y = Y^m$, $m \leq n$, be a topological manifold and $f : X \to Y$ a continuous map. Call the map f *(co)mologically (co)stable over a point* y *in dimension* n *with coefficients* $\mathbb{F}$ if the fiber $X_y = f^{-1}(y) \subset X$ and f near X_y have the following property.

[STB] Let $f_y : X \to \mathbb{R}^{n-m}$ be a continuous map and $V \subset \mathbb{R}^{n-m}$ be an open subset such that the induced cohomology homomorphism

$$H^{n-m}(V, \partial V; \mathbb{F}) \to H^{n-m}(f_y^{-1}(V) \cap X_y, f_y^{-1}(\partial V) \cap X_y; \mathbb{F})$$

does not vanish. Then for all sufficiently small closed neighbourhoods $W_y \subset Y$ of y the map

$$F = (f, f_y) : f^{-1}(W_y) \to W_y \times \mathbb{R}^{n-m}$$

induces *non-zero* homomorphism

$$F^* : H^n(W_y \times V, \partial(W_y \times V); \mathbb{F}) \to H^n(F^{-1}(W_y \times V), F^{-1}(\partial(W_y \times V); \mathbb{F})).$$

For example an $\mathbb{R}$-valued function f on a topological n manifold X is stable over a point $y \in \mathbb{R}$ if the level $f^{-1}(y)$ contains no local maxima and/or minima points of f.

Now let Y be smooth m-manifold with a smooth measure $\chi^{[m]}$ on it, let $f : X \to Y$ be a Lipschitz map and let $W_y(\varepsilon) \subset Y$ be ε-balls around y for some metric in Y.

If f satisfies **[STB]** *then*

$$\liminf_{\varepsilon \to 0} glb_{\mathbb{F}} vol^{[n]}(f^{-1}(W_y(\varepsilon)) \geq \frac{\chi^{[m]}(W_y(\varepsilon)) \cdot glb_{\mathbb{F}} vol^{[n-m]}(f^{-1}(y))}{\widetilde{Hil} Jac^{[m]} f_{/y}}.$$

Proof. The condition **[STB]**, albeit unpleasantly restrictive, matches the local $\tilde{L}_q$-extension property and the coarea inequality trivially follows.

Let us partially localize the above "volume" with countable systems $\mathcal{U}$ of open subsets $U_i \in X$, $i \in I$ that have bounded intersection multiplicity, denoted $mult(\mathcal{U}) < \infty$, and let

$$Hil.vol^{[n]}_{\mathbb{F}}(\mathbb{R}^n \backslash X) = \sup_{\mathcal{U}} (mult(\mathcal{U}))^{-1} \sum_{i \in I} glb_{\mathbb{F}} vol^{[n]}(\mathbb{R}^n \backslash U_i),$$

where this will be often abbreviated as follows

$$\widetilde{Hil}.vol^{[n]}(X) \text{ instead of } \widetilde{Hil}.vol^{[n]}_{\mathbb{F}}(\mathbb{R}^n \backslash X)$$

with $\mathbb{F} = \mathbb{F}_2$, unless otherwise indicated.

The homological volume $\widetilde{Hil}.vol^{[n]}(X)$ of smooth and piece-wise smooth Riemannian spaces X equals the ordinary volume $vol_n(X)$.

But, in general, $U \mapsto \widetilde{Hil}.vol^{[n]}(U)$, $U \subset X$, is not, a priori, an additive set function. This can be amended but using countable systems $\mathcal{U}_j$ of open sets $U_{ij} \subset X$ where $diam(U_{ij}) \to 0$ for $j \to \infty$ and localizing with

$$loc.vol(X) = \limsup_{j \to \infty} \mathcal{U}_j (mult(\mathcal{U}_j))^{-1} \sum_{i \in I} glb_{\mathbb{F}} vol^{[n]}(\mathbb{R}^n \backslash U_{ij}).$$

This "local volume" equals $\widetilde{Hil}.vol^{[n]}(X)$ if small ε-balls in X have volumes $\geq V_\varepsilon$ where $V_\varepsilon / \varepsilon^{n+1} \underset{\varepsilon \to 0}{\to} \infty$; in general, however, the local volume, probably, may be (I have not worked out a convincing example) smaller than $\widetilde{Hil}.vol^{[n]}$ that is defined with unrestricted U_i. To compensate for this, we shall introduce another "volume" in Part 2 of the paper by means of maps $X \to \mathbb{R}^\infty$ with controlled filling volumes of the boundaries of images of subsets (or rather of n-cycles) similarly in certain respects to [**?**].

Jonn's-Hilbert Volume of Banach Balls. The obvious corollary to John's Ellipsoid Theorem (pointed out in slightly different terms in [**?**]) reads:

the Hilbertian volume of the unit ball $B_X(1)$ in an n-dimensional Banach space X equals its Euclidean volume with respect to John's Euclidean (Hilbertian) metric $h_\diamond$,

$$Hil.vol^{[n]}(B_X(1)) = vol_{h_\diamond}(B_X(1)),$$

Consequently, since $B_{h_\diamond}(1) \subset B_X(1)$,

the Hilbertian volume $vol_{h_\diamond}(B_X(1))$ is greater than the (ordinary) volume of the unit Euclidean ball, where the inequality is strict unless X is isometric to $\mathbb{R}^n$.

9 Lower Volume Bounds.

Let us reformulate and reprove a few standard geometric inequalities in the Hilbert volume setting where we abbreviate: $vol^{[n]}(X) = Hil.vol^{[n]}(X)$ for general metric spaces X.

$\square$-*Spaces and their Faces.* A topological space P with a given collection of n pairs of disjoint closed subsets, $\pm Q_i = \pm Q_i(P) \subset P$, $i = 1, .., n$, is called an *n-cubical space* where $\pm Q_i$ are called *faces* and their union $\partial P = \bigcup_i \pm Q_i$ is regarded as *the boundary* of P.

Every such P, assuming it is a normal space, admits a continuous map π into the n-cube, $\pi : P \to \square = [0,1]^n$, such that the faces $\pm Q_i(P)$ go to the $(n-1)$-faces $\pm Q_i(\square)$ of the cube, where such a map is given by n-functions $\pi_i : P \to [0,1]$ where $-Q_i$ goes to 0 and $+Q_i$ goes to 1. Clearly, such a map π is unique up-to homotopy.

$\square$-*Straightening Lemma.* Let P be an n-cubical metric space, and $\pi : P \to \square = [0,1]^n$ be a continuous map that sends faces of P to the corresponding faces of the cube $\square$.

If

$$dist_P(Q_i, -Q_i) \geq \lambda_i, \ \ i = 1, .., n,$$

then π is homotopic to a Lipschitz map $F : P \to \square = [0,1]^n$ given by n-functions $f_1, ..., f_n : P \to [0,1]$, such that

$$Lip(f_i) \leq \lambda_i^{-1}.$$

Proof. Take $f_i(x) = \lambda^{-1} dist(x, Q_i)$ for all $x \in P$,where $dist(x, Q_i) \leq \lambda_i$ and let $f_i(x) = 1$ for $dist(x, Q_i) \leq \lambda_i$.

Corollary: Besikovich-Derrick-Almgren $\square$-Inequality. If the map $\pi : P \to \square = [0,1]^n$ has non-zero degree, i.e. the induced cohomology homomorphism $\pi^* : H^n(\square, \partial\square; \mathbb{F}) \to H^n(P, \partial P; \mathbb{F})$ doe not vanish for some coefficient domain $\mathbb{F}$, then

the $\mathbb{F}$-homological Hilbertian volume of P is bounded from below by the product of the distances between opposite faces in P,

$$vol^{[n]}(P) \geq \prod_{i=1,...,n} dist_P(Q_i, -Q_i).$$

Proof. Since F is homotopic to π and $deg(\pi) \neq 0$, the homomorphism $F^* : H^n(\square, \partial\square; \mathbb{F}) \to H^n(P, \partial P; \mathbb{F})$ does not vanish; hence, all points $y \in Y$ are cohomologically stable and

$$1 = vol_n(\square) \leq \|Jac^{[n]}(F)\|_{sup} \cdot vol^{[n]}(P) \leq vol^{[n]}(P) \cdot \prod_i \lambda_i^{-1}.$$

Example. Let X a Riemannian n-manifold and $\pm V_i \subset X$, $i = 1, 2, ..., n$, be smooth closed domains that are n-submanifolds with smooth boundaries $\partial(\pm V_i) \subset V_i$, e.g. $X = \mathbb{R}^n$ and $\pm V_i \subset \mathbb{R}^n$ are half-spaces.

Let P equal the intersection of $\pm V_i$,

$$P = \bigcap_i \pm V_i \text{ and } \pm Q_i = \pm Q_i(P) = P \cap \partial(\pm V_i).$$

If P is compact, if the n hypersurfaces $\partial(+V_i)$, $i = 1, 2, ..., n$, intersect *transversally* and if the intersection

$$\bigcap_i +Q_i = P \cap \partial(+V_1) \cap \partial(+V_2) \cap ... \cap \partial(+V_n)$$

(that is necessarily finite) consists of *odd* number of points, then

$$vol_n(P) \geq \prod_i dist(+Q_i, -Q_i).$$

Indeed, our map $F : P \to \square$ has non-zero $\mathbb{F}_2$-degree because the F-pullback of every point $y \in \square$ close to the corner $\bigcap_i +Q_i(\square)$ of $\square$ is a finite set of odd cardinality.

Recall that the map $\pi : P \to \square$, for every n-cubical space P sends $\partial P \to \partial\square$ and the essential property of π used in the construction of F, say for $dist_P(-Q_i.+Q_i) \geq 1$, is that π is 1-*Lipschitz on the boundary* ∂P with respect to the *sup-norm* on $\mathbb{R}^n \supset [0,1]^n = \square$, that is $\|(y_1, ..., y_i, ..., y_n)\|_{sup} = max_i|y_i|$.

Seen from this angle, the $\square$-inequality appears as a special case of the following corollary (stated slightly differently in [**?**]) of Jonn's theorem.

Filling Extremality of Banach spaces. Let $A = (A\|...\|_A)$ be an n-dimensional Banach space and let $U \subset A$ be a bounded open subset with connected boundary ∂U.

Let X be a compact metric space, let $Y \subset X$ be a closed subset and let $F : Y \to \partial U \subset A$ be a 1-Lipschitz map.

If the inclusion/restriction homomorphism $H^{n-1}(X; \mathbb{F}) \to H^{n-1}(Y; \mathbb{F})$ vanishes for some coefficient group $\mathbb{F}$, while the cohomology homomorphism $f^* : H^{n-1}(\partial U; \mathbb{F}) \to H^{n-1}(Y; \mathbb{F})$ does not vanish, then

$$vol^{[n]}(X) \geq vol^{[n]}(U) = vol_{h_\diamond}(U)$$

for John's Euclidean (Hilbertian) quadratic form $h_\diamond$ on A.

Proof. Extend f to a Lipschitz map $F : X \to A$ with $Hilb_{h_\diamond}(F) \leq 1$ (see section 1) and observe that the cohomology assumptions imply that this map is homologically image stable at all points $u \in U$.

Since the Hilbertian Jacobian of $F : X \to (A, h_\diamond)$ is bounded by

$$Hil.Jac^{[n]}(F) \leq Hilb^n(F) \leq 1,$$

the proof follows from John's volume identity in section 8.

$\triangle$-*Inequalities.* There is another generalization of the $\square$-nequality, besides the above Banach filling extremality, where the cube $[0,1]^n$ is replaced by the Cartesian product of regular Euclidean n_j-simplices, denoted $\times_j \triangle_{n_j}$, $j = 1, ..., k,$.

Start with the case $k = 1$ and define a $\triangle_n$*-space* as a topological space P with a distinguished collection of subsets, called *faces* $Q_i = Q_i(P)$, $i = 0, 1, ...n$, indexed by the $n+1$ codimension one faces in $\triangle_n$, such that the intersection $\bigcap_i Q_i$ is empty and where the union $\bigcup_i Q_i \subset P$ is regarded as the boundary $\partial P \subset P$

If P is a normal space, it admits a continuous map $\pi : P \to \triangle_n$, such that every face of P goes to the corresponding face of the simplex $\triangle_n$, where this π is unique up to homotopy.

Let P me a metric space and denote by $\Sigma = \Sigma_P : P \to \mathbb{R}_+$ the sum of the distance functions to the faces,

$$\Sigma(p) = \sum_i dist_P(p, Q_i),$$

let

$$\Sigma_\partial = \Sigma_\partial(P) = \inf_{p \in \partial P} \Sigma(p),$$

and let

$$S = \Sigma_\partial(P)/\Sigma_\partial(\triangle_n).$$

If the map $\pi : P \to \triangle_n$ has non-zero degree, then

$$Hil.vol^{[n]}(P) \geq S^n vol_n(\triangle_n).$$

Proof. There obviously exits an axial partition of unity $\mathcal{P} = \{p_i\}$ $i = 0, 1, ..., n$ in $\mathbb{R}^n \supset \triangle_n$ with $n+1$ axes $\mathbb{R}_i \subset \mathbb{R}^n$ that are normal to the $(n-1)$-faces of $\triangle_n$ and such that the quaisi-projectores p_i on $\triangle_n$, that are $p_i : \triangle_n : \mathbb{R}_i = \mathbb{R}$, are given by the functions $y \mapsto \lambda \cdot dist(y)$ for $\lambda = \sqrt{n/(n+1)}$.

Let $f_i : P \to \mathbb{R} = \mathbb{R}_i$ be given by $p_i \mapsto \lambda S^{-1} \cdot dist(p, Q_i)$ and observe that the map

$$F = \sum_i f_i : P \to \mathbb{R}^n \supset \triangle_n$$

sends the boundary of P *outside the interior of* $\triangle_n$ where, moreover, this map

$$F : \partial P \to \mathbb{R}^n \smallsetminus int(\triangle_n)$$

is *homotopic to*

$$\pi : \partial P \to \partial \triangle_n \subset \mathbb{R}^n \smallsetminus int(\triangle_n).$$

Since $deg(F) = deg(\pi) \neq 0$, the image of F homologically stably covers $\triangle_n$ and the proof follows for

$$Hil.Jac^{[n]}(F) \leq Hilb^n(F) \leq S^{-n}.$$

Now we turn to metric $\times_j \triangle_{n_j}$*-spaces* P, where $j = 1, 2, ..., k$, and where a $\times_j \triangle_{n_j}$-structure is given by a collection of "faces" $Q_{ji} \subset P$, $i = i_j = 0, ...n_j$ and

where Q_{ji} correspond to those faces of codimension 1 in the Cartesian product that equal the pullbacks of the faces of $\triangle_{n_j}$ under the coordinate projection $\times_j \triangle_{n_j} \to \triangle_{n_j}$.

The condition we impose on the set of faces Q_{ji} reads: a subset of faces has a common point in P only if the corresponding faces in $\times_j \triangle_{n_j}$ have a common point.

This is equivalent to the existence of a continuous map $\pi : P \to \times_j \triangle_{n_j}$ that sends faces to faces, where such a π is unique up-to a homotopy.

Let

$$\Sigma_j(p) = \sum_{i=1,...,n_j} dist_P(p, Q_{ji}),\ \Sigma_{\partial_j} = \Sigma_{\partial_j}(P) = \inf_{p \in \partial P} \Sigma_j(p),$$

$$S_j = \Sigma_{\partial_j}(P)/\Sigma_{\partial}(\triangle_{n_j}).$$

If the map $\pi : P \to \triangle_n$ has non-zero degree, then

$$Hil.vol^{[n_1+...+n_j]}(P) \geq \prod_{j=1,...,k} S_j^{n_j} vol_{n_j}(\triangle_{n_j}).$$

Proof. Take a partition of unity in $\mathbb{R}^{n_1+...+n_k} \supset \times_j \triangle_{n_j}$ with the axes normal to the faces of $\times_j \triangle_{n_j}$and argue as earlier.

9.1 Digression: Acute Polyhedra.

What should be a generalization of the above for *non-regular* simplices $\triangle_{n_j}$? Namely, what is the sharp lower bound on the distance function $dist_P(p_1, p_2)$ for $p_1, p_2 \in \partial P$ that would imply $Hil.vol^{[n]}(P) \geq \times_j \triangle_{n_j}$?

Notice in this respect, if P is a convex *acute* Euclidean polyhedron, i.e. all dihedral angles are $\leq \pi/2$ then the distances to its codimension 1 faces $d_i : p \mapsto dist(p, Q_i(P))$ satisfy

$$\sum_i c_i d_i = const = vol(P) \text{ for } c_i = n^{-1} vol_{n-1}(Q_i(P))$$

but the mere $\sum_i c_i d_i \geq const$ does not non-sufficient even for P (homologically) over (i.e. mapped with non-zero degree to) *non-regular* Euclidean triangles $\triangle_2$.

What is the minimal set of (preferably linear and/or log-linear) inequalities between d_i that would imply $Hil.vol^{[n]}(P) \geq \times_j \triangle_{n_j}$?

This question make sense for all acute Euclidean polyhedra besides $\times_j \triangle_{n_j}$ (some "non-acute" lower volume bounds can be derived from "acute" ones , see 7.3 in [**?**]) but, in fact, there aren't any other by *Steinitz*(?) *theorem*:

every acute polyhedron $P \subset \mathbb{R}^n$ is a Cartesian product of simplices.

Proof. Start by observing that an acute convex *spherical polyhedron* P (that is an intersection of hemispheres in S^n) is a either a *simplex*, or, in the degenerate case, the *spherical suspension* over a simplex in $S^{n-i} \subset S^n$.

Indeed, the dual polyhedron, say $P_p^\perp \subset S^n$, has all its edges longer than $\pi/2$. Consequently, the distance between *every two* vertices in $P_p^\perp$ is $\geq \pi/2$; hence, there are at most $n+1$ vertices in $P_p^\perp$.

Also note that (non-strictly) acute spherical triangles $\triangle \subset S^2$ have all their edges bounded in length by $\pi/2$. It follows that all m-faces, $m = 2, 3, ..., n-1$, of acute spherical n-simplices are acute.

Now, let P be *Euclidean* acute take an $(n-1)$-face $Q \subset P$, move its supporting hyperplane $H^{n-1} = H_Q \supset Q$, inward parallel to itself until it hits a vertex, say $p \in P$ and denote by $H_p \subset \mathbb{R}^n$ the so moved hyperplane.

Since the normal projections from P onto the hyperplanes supporting $(n-1)$-faces adjacent to Q send P *into* (hence, onto) these faces, all of P is contained in the "band" $[H_Q, H_p] \subset \mathbb{R}^n$ between the parallel hyperplanes $H_Q \subset \mathbb{R}^n$ and $H_p \subset \mathbb{R}^n$. Moreover,

• if the opposite face $-Q = H_p \cap P$ has $dim(-Q) = n-1$ then P is orthogonally splits into the Cartesian product of Q with a segment; it is seen with the two orthogonal projections $P \to Q, -Q$.

•• If Q is an $(n-1)$-simplex and $dim(-Q) < n-1$ then, obviously, P is an n-simplex.

Since the spherical polyhedra underlying the tangent cones at all vertices of P are acute, • and •• apply to the faces of P.

Namely, let $\Delta^m \subset P$ be a *simplex*-face of *maximal* dimension m. Then every $(m+1)$-face containing this Δ^m orthogonally splits according to • and ••. It follows by induction on m that all of P splits, $P = \Delta^m \times P'$, and the proof is concluded by induction on n.

Conclude by noticing that the number of $(n-1)$-faces of a *convex* polyhedron $P^n \subset \mathbb{R}^n$ with all dihedral angles $\leq \pi - \varepsilon$, is (obviously) universally bounded, say by $100^{n/\varepsilon}$; a classical problem in convexity is to " effectively enumerate" these P.

10 Volume Stability In The Riemannian Category.

The Burago-Ivanov Volume rigidity theorem (see section 5), like any other sharp inequality between two geometric invariants,

$$inv_1(X)/inv_2(X) \geq 1,$$

where the equality implies that X is isometric to a particular space X_0 (or a member of a "small explicit" class of manifolds) raises the following

Stability Problems. (a) Describe spaces X where $inv_1(X)/inv_2 \leq 1+\varepsilon$;

(b) find a metric in the space $\mathcal{X}$ of all X, such that the subspace $\mathcal{X}_\varepsilon \subset \mathcal{X}$ of those X where $inv_1(X)/inv_2 \leq 1+\varepsilon$ is compact.

Prior to engaging into general discussion on relevant metrics in the space of metric spaces (we shall come to this in Part 2 of the paper and in [**?**]) it is instructive to look at a "metric" in the space of Riemannian manifolds, which (almost) adequately reflects the volume rigidity picture.

10.1 *Directed Lipschitz Metric Normalized by Volume.*

To simplify/normalize, let X and X' be closed connected oriented Riemannian n-manifolds with $vol(X) = vol(X') = V$.

Define $\overrightarrow{dist}_{Lip/vol}(X', X)$ as the infimum of $\varepsilon \geq 0$, such that X' admits an e^ε-Lipschitz map $L : X' \to X$ of degree ± 1, where, observe, $e^\varepsilon = 1 + \varepsilon + o(\varepsilon)$.

This "metric" is *non-symmetric*, it may be equal $+\infty$, but it satisfies the *triangle inequality* and it equals zero if and only if X and X' are isometric.

There is nothing wrong with being non-symmetric. Limits make perfect sense for such "metrics".

For example, the "true", in the category theoretic sense, Hausdorff metric on subsets $Y \subset X$, call it $\overrightarrow{dist}_{Hau}(Y_1, Y_2)$ – *the minimal* ε *such that* Y_1 *is contained in the* ε*-neighbourhood of* Y_2, is also non-symmetric.

Let us show that suitable bounds on the diameters and curvatures guaranty an "almost isometric" diffeomorphism between *Lip/vol*-close manifolds.

Denote the minimium of the diameters of X and X' by

$$diam_{min} = min(diam(X), diam(X'))$$

and the maximum of the absolute values of their sectional curvatures at all tangent 2-planes by

$$|curv|_{max} = max\Big(\sup_{\tau_2 \subset T(X)} |sect.curv_{\tau_2}|(X), \sup_{\tau_2' \subset T(X')} |sect.curv_{\tau_2'}|(X')\Big).$$

$\overrightarrow{\mathbf{D}ist} \leq \varepsilon \Rightarrow \mathbf{D}iff$. *Given numbers* $n = 1, 2, ...,$ *and* $c, D \geq 0$, *there exists an* $\varepsilon = \varepsilon(n, cD^2) > 0$, *such that the Riemannian* n*-manifolds* X' *and* X *satisfying the inequalities*

$$diam_{min} \leq D,\ |curv|_{max} \leq c \text{ and } \overrightarrow{dist}_{Lip/vol}(X', X) \leq \varepsilon$$

are diffeomorphic.

Moreover, the implied diffeomorphism $X' \to X$ *is* $(1+\delta)$*-bi-Lipschitz with* $\delta \to 0$ *for* $\varepsilon \to 0$.

To show this, rescale the metrics in X and X' in order to have $|curv|_{max} = 1$ and then approximate the implied e^{ε}-Lipschitz map $L : X' \to X$ by a $(1+\delta)$-bi-Lipschitz diffeomorphism $\bar{L} : X' \to X$ as follows.

Denote by $\tilde{B}_x(\rho) \underset{\exp}{\to} X$ the ρ ball in the tangent space $T_x(X)$ with the metric induced from X by the exponential map and observe that if ρ is sufficiently small, $\rho \leq \rho_0(n, D) > 0$, and ε is much smaller than ρ, then, for every $x' \in X$,

the map L *from the ball* $B_{x'}(\rho) \subset X'$ *to* $B_x(\rho e^{\varepsilon}) \subset X$, $x = L(x')$, *uniquely lifts to a map* $\tilde{L}_{x'} : \tilde{B}_{x'}(\rho) \to \tilde{B}_x(\rho e^{\varepsilon})$.

Indeed, if otherwise, the map L would send some short geodesic loops from X' to loops contractible in their immediate vicinity in X; this, in turn, would imply that $deg(L) = 0$.

Now we turn L into a diffeomorphism as follow. Take the center of mass $\tilde{x} \in \tilde{B}_x(\rho e^{\varepsilon}) \subset X$ of the Riemannian measure of $\tilde{B}_{x'}(\rho)$ mapped to $\tilde{B}_x(\rho e^{\varepsilon})$ by $\tilde{L}_{x'}$ and let $\bar{L}(x') = \exp_x(\tilde{x}) \in X$. It is easy to see, as in [**?**], [**?**], that for small $\rho > 0$ and $\varepsilon << \rho$,

the so defined map $\bar{L} : X' \to X$ *is a* $(1+\delta)$*-Lipschitz diffeomorphism, where* $\delta \to 0$ *for* $\rho \to 0$ *and* $\varepsilon/\rho \to 0$.

It is significant that the above does not work if you replace the bound on the diameters of X and X' by volume bounds, i.e. if you bound $|curv|_{max} V^{2/n}$ while allowing $|curv|_{max} \cdot diam_{min}^2 \to \infty$.

(Counter)example. Let X_0 and X_0' be complete Riemannian $(n-1)$ manifolds that are cylindrical at infinity and have their sectional curvatures bounded by $|sect.curv| \le 1/2$. Let $L_0 : X_0' \to X_0$ be a 1-Lipschitz map that is isometric at infinity.

Thus, both manifold equal $Y \times [0,\infty)$ on their common cylindrical end for a closed $(n-2)$-manifold Y, where we denote by $t(x) = t(x') \in [0,\infty)$, $x \in X_0$, $x' \in X_0'$, the axial parameter on this end.

(All this can be arranged, for instance, if X is diffeomorphic to $\mathbb{R}^{n-1}$ and X' is diffeomorphic to the complement of a ball in an *arbitrary* closed $(n-1)$-manifold, where X_0 and X_0' can be made isometric to $S^{n-2} \times [0,\infty)$ at infinity.)

Let f be a smooth positive function on X_0 and let $f'(x') = (1-\varepsilon)f(L_0(x'))$, $x' \in X_0'$. (One may take an f which depends only on t on the cylindrical end of X_0, i.e. $f(x) = \phi(t(x))$, and which is constant away from this end.)

Multiply X_1 and X_1' by the unit circle S^1 and modify the product metrics in $X_1 = X_0 \times S^1$ and $X_1' = X_0' \times S^1$ by multiplying the circles over $x_0 \in X_0$ and $x_0' \in X_0'$, that are $x_0 \times S^1$ and $x_0' \times S^1$, by the functions f and f' respectively. Denote by X_f and $X_{f'}'$ the resulting complete Riemannain n-manifolds, where we organize the matter with a suitable f, such that both manifolds X_f and $X_{f'}'$ have $|sect.curv| \le 1$.

Observe that the map $L_1 : X_{f'}' = X_0' \times S^1 \to X_f = X_0 \times S^1$ for $(x', s) \mapsto (L_0(x'), s)$ is $(1+\varepsilon)$-Lipschitz for these metrics.

There obviously exists an axial value $t_\varepsilon \in [0,\infty)$, such that the integrals of the two functions over the cut-off manifolds $X_0(t_\varepsilon) \subset X_0$ and $X_0'(t_\varepsilon) \subset X_0'$, defined by $t(x), t(x') \le t_\varepsilon$, are equal. Then the corresponding compact n-manifolds $X_f(t_\varepsilon) \subset X_f$ and $X_{f'}'(t_\varepsilon) \subset X_{f'}'$, with the boundaries $\partial X_f(t_\varepsilon) = \partial X_{f'}'(t_\varepsilon) = Y \times S^1 \times t_\varepsilon$, have equal n-volumes.

If we replace $f \mapsto \epsilon f$ and $f' \mapsto \epsilon f'$ with an arbitrary $\epsilon > 0$, we do not change the curvatures of the manifolds; thus we can make the volumes of $X_f(t_\varepsilon)$ and $X_{f'}'(t_\varepsilon)$ arbitrarily small, say both equal $\epsilon/2$ (to avoid more letters in the notation) still keeping $|sect.curv| \le 1$.

Finally, we take the doubles of these manifolds and obtain, for all $n \ge 3$ and arbitrarily small $\varepsilon, \epsilon > 0$, lots of

closed Riemannian n-manifolds $X = X(\varepsilon,\epsilon)$ and $X' = X'(\varepsilon,\epsilon)$ both with $|sect.curv| \le 1$ and with the volumes $= \epsilon$, where X admits a $(1+\varepsilon)$-Lipschitz map $L : X' \to X$ of degree 1 that is not a homotopy equivalence. (The diameters of these manifolds are $\approx t_\varepsilon \approx \varepsilon^{-1}$.)

On can recapture, however, the implication $\overrightarrow{\mathbf{D}ist} \le \varepsilon \Rightarrow \mathbf{D}iff$ for complete Riemannian manifolds with $|sect.curv| \le 1$ if, for a given $d = 1, 2, ...$, the condition $vol(X') = vol(X)$ is replaced by

$$[B' \lesssim_\varepsilon d \cdot B] \qquad vol(L^{-1}(B)) \le (d+\varepsilon)vol(B) \text{ for all unit balls } B \subset X.$$

Then, the argument used for $\overrightarrow{\mathbf{D}ist} \le \varepsilon \Rightarrow \mathbf{D}iff$ also delivers

an approximation of every proper $(1+\varepsilon)$-Lipschitz map $L : X \to X'$ of degree d by a locally diffeomorphic locally $(1+\delta)$-bi-Lipschitz map $X' \to X$ for all sufficiently small $\varepsilon \le \varepsilon_0(n) > 0$ and $\delta \le \delta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$. (Such an approximating map $X' \to X$ necessarily is a d-sheeted covering map.)

Questions. Let X and X' be Riemannian n-manifolds (or possibly singular Alexandrov spaces for this matter) with their sectional curvatures bounded from

below by -1 and let $L : X' \to X$ be a $(1+\varepsilon)$-Lipschitz map of degree d that satisfy $[B' \lesssim_\varepsilon d \cdot B]$.

Does then, for small $\varepsilon > 0$, the map L lift to a *homotopy equivalence* between X' and a d-sheeted covering of X? (This is easy in the non-collapsed case.)

Is, moreover, L homotopic to a locally homeomorphic map?

Is there anything meaningful here with the lower bound on the sectional curvatures relaxed to a lower bound on the Ricci curvatures of X and/or X'?

What happens for $dim(X') - dim(X) = k > 0$?

Namely, let $L : X' \to X$ be an $(1+\varepsilon)$-Lipschitz map such that $vol(X') \le \|deg_k(L)\|_{vol_k} \cdot vol(X)$, where $\|deg(L)\|_{vol_k}$ is the $\mathbb{R}$-*mass* of the homology class $deg_k(L) \in H_k(X')$ that is the class represented by a generic pullback $L^{-1}(x) \subset X'$, $x \in X$.

Are there particular topological/homological constrains on such an L for small ε in the case where X and X' satisfy specified bounds on their sizes and the local geometries, say an upper bound on the diameters and on the absolute values of the sectional curvatures?

10.2 Mean Curvature Stability.

Below is another kind of situation where Lipschitz and volume confront one another.

Let X and X' be smooth Riemannian n-manifolds, $L = L_\varepsilon : X' \to X$ be a $(1+\varepsilon)$-bi-Lipschitz homeomorphism and let $\phi : X \to \mathbb{R}$ be a continuous function.

Let $Y \subset X$ be a smooth closed cooriented hypersurface with mean curvature $mn.curv_y(Y) = \phi(y)$ and let positive numbers $\rho, \epsilon > 0$ be given.

If $\varepsilon \le \varepsilon_0 = \varepsilon_0(X, \phi, \rho, \epsilon) > 0$, then there exists a closed hypersurface $Y_{min'} \subset X$ such that the following conditions $[\mathbf{U}_\rho]$, $[\mathbf{mn.curv}\pm\epsilon]$ and $[\mathbf{diff}]$ are satisfied.

$[\mathbf{U}_\rho]$ *$Y_{min'}$ is contained in the ρ-neighbourhood $U_\rho(Y) \subset X$ of Y where it is homologous to Y.*

$[\mathbf{mn.curv}\pm\epsilon]$ *The hypersurface $Y'_{min} = L^{-1}(Y_{min'}) \subset X'$ is C^2-smooth and its mean curvature with respect to the Riemannian metric in X' satisfies*

$$|mn.curv_{y'}(Y'_{min}) - \phi'(y')| \le \epsilon \text{ for } \phi'(y') = \phi \circ L(y') \text{ and all } y' \in Y'_{min}.$$

Notice that even if L is smooth, the hypersurface $Y_{min'} \subset X$ is *not, in general, C^1-close to Y*. Moreover, even for $dim(Y) = 1$, the normal projection $Y_{min'} \to Y$ is *not,* typically, one-to-one. However,

$[\mathbf{diff}]$ *the manifold Y'_{min} is diffeomorphic to Y. In fact the composition of $L : Y'_{min} \to U_\rho(Y)$ with the normal projection $U_\rho(Y) \to Y$ can be approximated by a diffeomorphism $Y'_{min} \to Y$.*

Proof. We may assume that Y is connected and $X \supset Y$ equals a small normal neighbourhood of Y. Thus, Y divides X into two halves, call them *inside and outsides* of Y, written $in(Y) \subset X$ and $out(Y) \subset X$, with common boundary $\partial(in(Y)) = \partial(out(Y)) = Y$.

μ-Area and μ-Bubbles. Given a measure μ on X, let

$$area_{-\mu}(Y) =_{def} vol_{n-1}(Y) - \mu(in(Y))$$

and call a hypersurface $Y \subset X$ a *stable μ-bubble* if it *locally minimizes* the function $Y \mapsto area_{-\mu}(Y)$ among all hypersurfaces in X homologous to Y.

If μ is given by a continous density function $\phi(x)$, $x \in X$, i.e. $\mu = \phi \cdot vol_n$ for the Riemannian n-volume (measure) vol_n, then these are called *ϕ-bubles*. Clearly, the mean curvature of a ϕ-bubble $Y \subset X$ satisfies $mn.curv(y) = \phi(y)$.

In particular, *ε-bubbles*, where μ equals the Riemannian n-volume vol_n in X times $\varepsilon \in \mathbb{R}$, have constant mean curvature ε.

Local Traps. Let $Y \subset X$ be a closed smooth cooriented hypersurface and $\phi(x)$ be a C^1-smooth function such that $\phi(y) = mn.curv_y(Y)$ for all $y \in Y$.

If *the inward normal derivative* $\frac{d\phi(y)}{d\nu^{in}}$ *on Y, is sufficiently large*, namely

$$\frac{d\phi(y)}{d\nu^{in}_y} > curv^2_y(Y) + Ricci_X(\nu^{in}_y, \nu^{in}_y) \text{ for all } y \in Y,$$

where $curv^2$ denotes the sum of squares of the principal curvatures of Y and where, observe, $Ricci(\nu^{in}, \nu^{in}) = Ricci(\nu^{out}, \nu^{out})$, then

Y is a stable ϕ-bubble; moreover, there is a (small) neighbouthood $U_0 \subset X$ of Y, such that every hypersurface $Y' \neq Y \subset U_0$ homologous to Y has strictly greater ϕ-area than Y.

This trivially follows from the *second variation formula* for $vol_{n-1}(Y)$.

Now, given a smooth hypesurface $Y \subset X$, let $\phi : X \to \mathbb{R}$ be equal the mean curvature of Y on Y and have large inward normal derivative. Then (almost) obviously, the domain $L^{-1}_\varepsilon(U_0) \subset X'$ contains a locally minimal ϕ'-bubble, say $Y'_{min} \subset L^{-1}_\varepsilon(U_0)$, homologous to $L^{-1}_\varepsilon(Y) \subset X'$ for all $\varepsilon \leq \varepsilon(X, Y, \phi) > 0$

The hypersurface Y'_{min} can be, a priori, singular. However, if ε is sufficiently small it is smooth by the following standard argument/

Let $r > 0$ be a small number that is, however, much bigger then ε. Then the volume of the intersection of Y'_{min} with the r-ball $B'_y(r) \subset X'$ is about ε-close to the volume of the Eucliden ball $B^{n-1}_{Eucl}(r)$ for all $y \in Y'_{min}$. Since X' on the r-scale has almost the same filling inequalities as $\mathbb{R}^{n-1}$, it follows, by the standard monotonicity argument, that the ratio

$$\frac{vol_{n-1}(B'_y(r) \cap Y'_{min})}{vol_{n-1}(B^{n-1}_{Eucl}(r))}$$

is close to one for all *arbitrarily small* balls.

Hence, by Almgren-Allard regularity theory (that is the only non-elementary ingredient of our argument) the hypersurface Y'_{min} is C^2-smooth with the mean curvature equal ϕ'_ϵ on it.

It remains to show that Y'_{min} is diffeomorphic to Y.

You may assume L is smooth, this does not cost you anything for small ε, but the geometric measure theory tells you nothing, a priori, about the topology and geometry of $Y'_{min} \subset X'$ and of the corresponding $Y_{min'} = L(Y'_{min}) \subset X$.

Yet, since $Y'_{\min}$ is a *minimal* bubble, and L is $(1+\varepsilon)$-bi-Lipshitz, the hypersurface is $Y_{min'}$ is *ε'-quasi-minimal* for vol_n and $\varepsilon' \approx \varepsilon$ in the sense of [**?**] on *all sufficiently small scales*. Then the standard blow-up rescaling/limit argument yields the following.

Weak Distortion Property. If the distance function from a point $x \in X$ to $Y_{min'}$ has two minima $y_1, y_2 \in Y_{min'}$,

$$dist(x, y_1) = dist(x, y_2) = \delta = dist(x, Y_{min'}),$$

then the angle between the corresponding minimal geodesic segments $[x, y_1]$ and $[x, y_2]$ at x must be small, roughly of order ε, for all not very large δ.

It follows that the distance function $x \to dist(x, Y_{min'})$, say outward $Y_{min'}$, can be "bi-Lipschitz" approximated in the δ-neighbourhood $U_\delta(Y_{min})$ for, say $\delta = 100^n\rho$ for a sufficiently small ρ and ϵ. by a smooth function $d(x)$ *without critical points* which vanishes on $Y_{min'}$. Similarly one sees that the normal projection of the $\delta/2$-level of $d(x)$ to Y is a diffeomorphism. QED.

Concluding Remarks. The above $(1+\varepsilon)$-bi-Lipschitz assumption, as well as the e^ε-Lipschitz constrain on maps in the previous section, are unduly restrictive. These will be relaxed in the Hilbert volume framework, in the spirit of Stephan Wenger's metric in the space of manifolds, [**?**] [**?**], in Part 2 of our paper.

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