The Burago–Ivanov Asymptotic Volume Inequality

The Chat GPT rendition of the proof from the Hilbert-volume paper

The basic ingredients are the asymptotic norm of a periodic metric, the contact-point decomposition for John's ellipsoid, Lipschitz straightening, and the area formula. It is an expanded reconstruction of the volume-inequality argument, not a verbatim transcription. The equality/flatness rigidity assertion is not proved here.

1 Statement and covering space

Write $\omega_n = \mathrm{vol}(B^n)$ for the volume of the Euclidean unit ball in $\mathbb{R}^n$.

Theorem 1.1 (Burago–Ivanov volume inequality)**.** *Let V be a closed connected oriented Riemannian n-manifold admitting a continuous map $\phi : V \to T^n = \mathbb{R}^n/\mathbb{Z}^n$ of nonzero degree. If $\widetilde{V}$ is its universal Riemannian covering and $\widetilde{v} \in \widetilde{V}$, then*

$$\liminf_{R\to\infty} \frac{\mathrm{vol}(B_{\widetilde{V}}(\widetilde{v}, R))}{\omega_n R^n} \geq 1.$$

In particular, the inequality holds for arbitrary Riemannian metrics on T^n.

The induced map on fundamental groups has image $\Gamma = \phi_*\pi_1(V) \subset \mathbb{Z}^n$. This image has finite index: otherwise the induced map on top-dimensional homology is zero, contradicting $\deg \phi \neq 0$. Thus Γ is a lattice of rank n. Let $X \to V$ be the regular covering corresponding to $\ker \phi_*$. Its deck group is Γ, acting freely, properly and cocompactly by isometries. The map ϕ lifts to a proper continuous map

$$f : X \longrightarrow Y = \mathbb{R}^n, \qquad f(\gamma x) = f(x) + \gamma \quad (\gamma \in \Gamma).$$

The map f is proper because both X/Γ and Y/Γ are compact and f is equivariant. Its proper degree is nonzero. Indeed it descends to $V \to Y/\Gamma$, whose composition with the finite covering $Y/\Gamma \to T^n$ equals ϕ, and hence has nonzero degree. We work on X until the final step.

2 The asymptotic norm of the periodic metric

Fix $x_0 \in X$, and define $L(\gamma) = d_X(x_0, \gamma x_0)$ for $\gamma \in \Gamma$. Because Γ is abelian and acts by isometries,

$$L(\gamma + \delta) \leq L(\gamma) + L(\delta), \qquad L(-\gamma) = L(\gamma).$$

Lemma 2.1 (Stable/Federer–Whitney norm)**.** *The limits*

$$\|\gamma\| = \lim_{k\to\infty} \frac{L(k\gamma)}{k} \qquad (\gamma \in \Gamma)$$

exist and extend uniquely to a norm $\|\cdot\|$ on Y. For every $\eta > 0$ there is $C_\eta < \infty$ such that for all $x, x' \in X$,

$$\|f(x) - f(x')\| \leq (1+\eta) d_X(x, x') + C_\eta, \tag{1}$$

$$d_X(x, x') \leq (1+\eta)\|f(x) - f(x')\| + C_\eta. \tag{2}$$

Proof. Existence of the stable-length limit follows from subadditivity. If $|\gamma|_S$ is word length for a finite set of generators of Γ, cocompactness of the isometric action gives constants $a, b > 0$ with

$$a|\gamma|_S - b \le L(\gamma) \le a^{-1}|\gamma|_S + b.$$

The upper bound follows by concatenating paths joining x_0 to translates by generators. For the lower bound, subdivide a minimizing path into pieces of uniformly bounded size and record the successive translates of a compact fundamental set meeting these pieces. There are only finitely many possible increments, producing a word of length $O(L(\gamma) + 1)$.

Subadditivity makes $\|\gamma\|$ subadditive and homogeneous under integral multiplication. Since $|k\gamma|_S$ grows linearly in k for $\gamma \neq 0$ in the lattice, the word-length bound shows $\|\gamma\| > 0$. Rational homogeneity and continuity now give a norm on Y.

We prove the uniform estimate

$$L(\gamma) = \|\gamma\| + o(\|\gamma\|) \quad \text{as } \|\gamma\| \to \infty, \quad \gamma \in \Gamma. \tag{3}$$

The inequality $L(\gamma) \ge \|\gamma\|$ follows from subadditivity. For the converse, fix $\delta > 0$. Choose finitely many rational directions whose positive cones cover Y in narrow cones, so that in each cone a vector y can be written as a positive combination of its generators v_j with $\sum_j t_j \|v_j\| \le (1+\delta)\|y\|$. This is possible by continuity of the norm and a sufficiently fine rational polyhedral approximation of its unit ball. Replace each v_j by a sufficiently large lattice multiple so that $L(v_j) \le (1+\delta)\|v_j\|$, which follows from the definition of stable length. Writing any lattice vector in a cone as $\gamma = \sum_j k_j v_j + r$, with nonnegative integers k_j and r in a bounded set (rounding down the positive coefficients), gives

$$L(\gamma) \le (1+\delta)^2 \|\gamma\| + O_\delta(1).$$

Letting $\delta \downarrow 0$ establishes (3).

Every point of X lies at bounded distance from a translate of x_0; on a compact fundamental set f has bounded image. Thus replacing x, x' by orbit points changes $d_X(x, x')$ and $\|f(x) - f(x')\|$ by bounded quantities. Equation (3) proves (1)–(2). □

3 John's ellipsoid and its contact points

Let $K = \{y \in Y : \|y\| \le 1\}$. Its unique maximal-volume inscribed ellipsoid E is centered at the origin because K is symmetric. Choose a scalar product h on Y such that

$$E = \{y : |y|_h \le 1\}, \qquad |y|_h^2 = h(y, y).$$

Since $E \subset K$, we have

$$\|y\| \le |y|_h \qquad (y \in Y). \tag{4}$$

Lemma 3.1 (John contact-point decomposition; see Ball [1])**.** *There exist unit vectors $u_1, \dots, u_N \in \partial E \cap \partial K$ and positive numbers $c_1, \dots, c_N$ such that*

$$\sum_{i=1}^{N} c_i \, u_i \otimes_h u_i = \mathrm{Id}_Y, \qquad \sum_{i=1}^{N} c_i = n,$$

where $(u \otimes_h u)y = h(y, u)u$. If $\ell_i(y) = h(y, u_i)$, then

$$y = \sum_i c_i \ell_i(y) u_i, \qquad |y|_h^2 = \sum_i c_i \ell_i(y)^2, \qquad |\ell_i(y)| \le \|y\|. \tag{5}$$

Proof. The contact-point decomposition is the standard optimality characterization of a maximal-volume inscribed ellipsoid (John's theorem in the formulation emphasized by Ball). Taking traces gives $\sum c_i = n$. The first two formulas in (5) are the operator identity and its scalar-product pairing with y. At a contact point u_i, the tangent hyperplane to E supports K, hence $\ell_i(y) \le 1$ for $y \in K$; symmetry of K gives $|\ell_i(y)| \le 1$ on K, which by homogeneity is the last inequality. □

4 Straightening to a sharp Jacobian bound

Lemma 4.1 (Lipschitz straightening)**.** *For every $\eta > 0$ there is a proper locally Lipschitz map $F_\eta : X \to (Y, h)$, boundedly close to f, with*

$$\operatorname{Jac}_n F_\eta(x) \le (1+\eta)^n \quad \text{for almost every } x \in X.$$

The proper degrees of F_η and f agree.

Proof. Put $a_i(x) = \ell_i(f(x))$. Equations (1) and (5) imply

$$|a_i(x) - a_i(z)| \le (1+\eta) d_X(x,z) + C_\eta.$$

Define

$$b_i(x) = \inf_{z \in X} \big(a_i(z) + (1+\eta) d_X(x,z)\big).$$

Taking $z = x$ and applying the preceding inequality shows

$$a_i(x) - C_\eta \le b_i(x) \le a_i(x).$$

The triangle inequality proves $\operatorname{Lip}(b_i) \le 1 + \eta$. Define

$$F_\eta(x) = \sum_i c_i b_i(x) u_i. \tag{6}$$

By (5), $f(x) = \sum_i c_i a_i(x) u_i$, so

$$|F_\eta(x) - f(x)|_h \le C_\eta \sum_i c_i = n C_\eta.$$

Thus F_η is proper. The linear homotopy $(1-t)f + tF_\eta$ stays within bounded distance of the proper map f and is therefore proper, preserving degree.

It remains to prove the Jacobian estimate. Give $\mathbb{R}^N$ the weighted Euclidean norm $|t|_c^2 = \sum_i c_i t_i^2$, and let

$$A : \mathbb{R}^N \to Y, \qquad A(t) = \sum_i c_i t_i u_i.$$

Its adjoint is $A^* y = (\ell_i(y))_i$. By (5), A^* is an isometry and $AA^* = \mathrm{Id}_Y$. Consequently $\|A\|_{\mathrm{op}} = 1$ and

$$\left| \sum_i c_i t_i u_i \right|_h^2 \le \sum_i c_i t_i^2. \tag{7}$$

By Rademacher's theorem, the functions b_i are differentiable almost everywhere. At a common differentiability point and for $v \in T_x X$, (7) yields

$$|dF_\eta(v)|_h^2 \le \sum_i c_i \, db_i(v)^2.$$

Sum over a g-orthonormal basis of T_xX:

$$\|dF_\eta\|_{\mathrm{HS}}^2 \le \sum_i c_i |db_i|_g^2 \le (1+\eta)^2 \sum_i c_i = n(1+\eta)^2.$$

If $\sigma_1, \dots, \sigma_n$ are the singular values of dF_η, the arithmetic–geometric mean inequality gives

$$\mathrm{Jac}_n F_\eta = \prod_{j=1}^n \sigma_j \le \left(\frac{\sum_j \sigma_j^2}{n} \right)^{n/2} \le (1+\eta)^n.$$

□

Exercises. 1. Formulate and prove Rademacher's theorem.
2. Replace Rademacher's theorem by a smooth approximation of b_i.
(Ask Chat GPT to help if neccessary.)

5 Degree, area formula, and large balls

Lemma 5.1 (Volume from degree)**.** *Let $F : X \to \mathbb{R}^n$ be a proper locally Lipschitz map of nonzero degree d. If $\mathrm{Jac}_n F \le J$ almost everywhere, then for every bounded open set $\Omega \subset \mathbb{R}^n$,*

$$\mathrm{vol}_X(F^{-1}(\Omega)) \ge \frac{|d|}{J} \mathrm{vol}_{\mathbb{R}^n}(\Omega).$$

Proof. For almost every target point the multiplicity $N(F, y)$ is at least $|d|$: the sum of signed local multiplicities equals the proper degree. The area formula implies

$$J \,\mathrm{vol}_X(F^{-1}(\Omega)) \ge \int_{F^{-1}(\Omega)} \mathrm{Jac}_n F \, d\,\mathrm{vol}_X = \int_\Omega N(F, y)\, dy \ge |d|\, \mathrm{vol}_{\mathbb{R}^n}(\Omega).$$

□

Proof of Theorem 1.1. Fix $\eta > 0$. Apply Lemma 4.1 and denote its map by $F = F_\eta$. Since $|F - f|_h$ is bounded and (4) holds, the second inequality of Lemma 2.1 yields a constant D_η such that

$$d_X(x, x_0) \le (1+\eta)|F(x) - F(x_0)|_h + D_\eta \qquad (x \in X). \tag{8}$$

Consequently, for $R > D_\eta$,

$$F^{-1}\left(B_h\left(F(x_0), \frac{R - D_\eta}{1+\eta} \right) \right) \subset B_X(x_0, R).$$

The map F has nonzero degree and $\mathrm{Jac}_n F \le (1+\eta)^n$. Lemma 5.1 therefore gives

$$\mathrm{vol}_X(B_X(x_0, R)) \ge \frac{\omega_n (R - D_\eta)^n}{(1+\eta)^{2n}}.$$

Divide by $\omega_n R^n$, let $R \to \infty$, and then let $\eta \downarrow 0$:

$$\liminf_{R\to\infty} \frac{\mathrm{vol}_X(B_X(x_0, R))}{\omega_n R^n} \ge 1.$$

Finally, the universal cover $\widetilde{V} \to X$ is a Riemannian covering. Its radius-R ball maps onto the radius-R ball of X (lift minimizing geodesics from its center). The covering is a local isometry; the area formula therefore gives

$$\mathrm{vol}_{\widetilde{V}}(B_{\widetilde{V}}(\widetilde{v}, R)) \ge \mathrm{vol}_X(B_X(x_0, R)).$$

This proves the theorem. □

Remark 5.2 (Why the numerical constant is 1)**.** *The decisive chain is*

$$\|dF\|_{\mathrm{HS}}^2 \le \sum_i c_i |db_i|^2 \le n(1+\eta)^2, \qquad \mathrm{Jac}_n F \le \left(\frac{\|dF\|_{\mathrm{HS}}^2}{n}\right)^{n/2}.$$

The trace identity $\sum_i c_i = n$ *cancels the factor* n *in the Hadamard/AM–GM inequality. A general bilipschitz Euclidean approximation would only produce a dimension-dependent constant.*

Remark. The GPT version is shorter and it doesn't use the concept of the Hilbert volume from [3], where the Burago-Ivanov theorem is reformulated in a form applicable to non-Riemannian spaces.

References

[1] K. M. Ball, *Ellipsoids of maximal volume in convex bodies*, Geometriae Dedicata **41** (1992), no. 2, 241–250. `https://doi.org/10.1007/BF00182424`. Preprint: `https://arxiv.org/abs/math/9201217`.

[2] D. Burago and S. Ivanov, *On asymptotic volume of tori*, Geom. Funct. Anal. **5** (1995), 800–808.

[3] M. Gromov, *Hilbert volume in metric spaces. Part 1*, Open Mathematics (formerly Central European Journal of Mathematics) **10** (2012), no. 2, 371–400. `https://doi.org/10.2478/s11533-011-0143-7`.

[4] F. John, *Extremum problems with inequalities as subsidiary conditions*, in *Studies and Essays Presented to R. Courant on his 60th Birthday*, Interscience, New York, 1948, 187–204.